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INTEGRABILITY OF THE DISTRIBUTIONS OF GCR-LIGHTLIKE SUBMANIFOLDS OF (ε)-SASAKIAN MANIFOLDS

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Abstract. We study GCR-lightlike submanifolds of (ε)-Sasakian manifolds and derived some important structural characteristics equations for further uses. We also obtain some necessary and sufficient conditions for the integrability of various distributions of GCR-lightlike submanifolds of (ε)-Sasakian manifolds.

1. Introduction

As a generalization of complex and totally real submanifolds, Cauchy-Riemann (CR)-submanifolds of Kaehler manifolds were introduced by Bejancu [1] in 1978 and further studied by many authors on using positive definite metric. In [3], Duggal introduced the geometry of CR-submanifolds with Lorentz metric and showed mutual interplay between the Cauchy-Riemann structure and physical spacetime geometry. In [4], Duggal showed the interaction of Lorentz CR-submanifolds with relativity and also studied a new class of CR-submanifolds. Later on, Duggal and Bejancu [5] introduced the concept of CR-lightlike submanifolds of indefinite Kaehler manifolds but which excluded the complex and totally real subcases, therefore Duggal and Sahin [6] introduced Screen Cauchy-Riemann (SCR)-lightlike submanifolds of indefinite Kaehler manifolds which included complex and screen real subcases but there was no inclusion relation between SCR and CR classes. Thus as an umbrella of complex, real hypersurfaces, screen real and CR-lightlike submanifolds, Duggal and Sahin [7] introduced Generalized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite Kaehler manifolds and further studied by [10–15]. Since there are significant applications of contact geometry in thermodynamics, optics, mechanics and many more. Therefore, Duggal and Sahin [8] introduced the geometry of (GCR)-lightlike submanifolds of indefinite Sasakian manifolds and further studied by [16–18]. Recent developments in the geometry of GCR-lightlike submanifolds motivated us to extend this work. Kumar et al. [9] contributed in the study of

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(ε)-Sasakian manifolds and our aim of this paper is to study GCR-lightlike submanifolds of (ε)-Sasakian manifolds.

2. Preliminaries

2.1. **(ε)-Sasakian Manifolds.** Assume that $\bar{M}$ is a $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure (ϕ, η, V), where ϕ is a $(1,1)$-type tensor field, η is a 1-form and V is a vector field on $\bar{M}$, called the characteristic vector field, satisfying

$$\phi^2 X = -X + \eta(X) V, \quad \eta(V) = 1, \tag{2.1}$$

$$\eta(\phi X) = 0, \quad \phi(V) = 0, \quad rank\phi = 2n, \tag{2.2}$$

then $\bar{M}$, with the triple (ϕ, η, V) is called an almost contact manifold. If there exists a semi-Riemannian metric $\bar{g}$ such that

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \varepsilon \eta(X) \eta(Y), \quad \forall X, Y \in T\bar{M}, \tag{2.3}$$

$$\eta(X) = \varepsilon \bar{g}(X, V), \quad \bar{g}(V, V) = \varepsilon, \quad \forall X \in T\bar{M}, \tag{2.4}$$

for any vector fields X, Y on $\bar{M}$, where $\varepsilon = \mp 1$, then $(\phi, \eta, V, \bar{g})$ is called an (ε)-almost contact metric structure on $\bar{M}$. If $d\eta(X, Y) = \bar{g}(\phi X, Y)$, then (ε)-almost contact metric structure is called an (ε)-contact metric structure and $\bar{M}$ endowed with this structure is called an (ε)- contact metric manifold. Furthermore, if the (ε)-contact metric structure is normal, that is, if satisfying

$$[\phi X, \phi Y] + \phi^2 [X, Y] - \phi [X, \phi Y] - \phi [\phi X, Y] = -2d\eta(X, Y) V, \tag{2.5}$$

then (ε)-contact metric structure is called an (ε)-Sasakian structure and $\bar{M}$ endowed with this structure is called as an (ε)-Sasakian manifold [2].

Remark 1. *From the relations $g(V, V) = \varepsilon$ and $\varepsilon = \mp 1$, it is clear that the vector field V can never be null. If $\varepsilon = -1$ and the index of $\bar{g}$ is odd; then $\bar{M}$ is called a time-like Sasakian manifold. If $\varepsilon = 1$ and the index of $\bar{g}$ is even; then $\bar{M}$ is called a space-like Sasakian manifold. In particular, if $\varepsilon = -1$ and the index of $\bar{g}$ is either zero or one; then $\bar{M}$ is said to be a usual Sasakian manifold or a Lorentz-Sasakian manifold, respectively.*

Theorem 1 ([2, Theorem 3])**.** *The necessary and sufficient conditions for an (ε)-almost contact metric structure $(\phi, \eta, V, \bar{g})$ to be an (ε)-Sasakian structure is*

$$\left(\bar{\nabla}_X \phi\right) Y = \bar{g}(X, Y) V - \varepsilon \eta(Y) X, \quad \forall X, Y \in T\bar{M}, \tag{2.6}$$

for any vector fields X, Y on $\bar{M}$, where $\bar{\nabla}$ denotes the Levi-Civita connection with respect to $\bar{g}$. Moreover, we also have

$$\bar{\nabla}_X V = -\varepsilon \phi X, \tag{2.7}$$

for any $X \in T\bar{M}$.

2.2. **Lightlike Submanifolds.** Suppose that $(\bar{M}^{m+n}, \bar{g})$ is a semi-Riemann manifold and M^m is its immersed submanifold. Then, M^m is called a lightlike submanifold; if the metric g on M induced from $\bar{g}$ has a radical distribution $Rad\,(TM)$ of rank r, for $1 \leq r \leq m$, for details see [5]. Then, its semi-Riemannian complementary distribution in TM, denoted by $S(TM)$, is known as the screen distribution and it follows that $TM = Rad\,(TM) \perp S\,(TM)$. The orthogonal complementary of $Rad\,(TM)$ in $TM^{\perp}$, denoted by $S\left(TM^{\perp}\right)$, is also a semi-Riemannian bundle and known as a screen transversal bundle of M. Since $S(TM)$ is a non-degenerate vector subbundle of $T\bar{M}|_M$; then, we have $T\bar{M}|_M = S(TM) \perp S(TM)^{\perp}$ where $S(TM)^{\perp}$ is the complementary orthogonal vector bundle of $S(TM)$ in $T\bar{M}|_M$. Then, clearly we have $S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}$. If (M, g) is an r-lightlike submanifold of $(\bar{M}, \bar{g})$; then, for the local basis $\{\xi_i\}_{i=1}^{r}$ of $Rad\,(TM)$ on a coordinate neighbourhood $\mathcal{U}$ of M, there exist smooth sections $\{N_i\}_{i=1}^{r}$ of $S(TM^{\perp})^{\perp}|_{\mathcal{U}}$ such that $\bar{g}\,(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, for any $i, j \in \{1, \ldots, r\}$. Then, there exists a vector subbundle of $S(TM^{\perp})^{\perp}$ spanned by $\{N_i\}_{i=1}^{r}$, known as the lightlike transversal vector bundle of M and denoted by $ltr(TM)$. Consider a vector bundle $tr(TM) = ltr(TM) \perp S(TM^{\perp})$, which is a complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$ and known as the transversal vector bundle of M. Thus, we have the following decomposition

$$T\bar{M}|_M = TM \oplus tr(TM) = S(TM) \perp \{Rad(TM) \oplus ltr(TM)\} \perp S(TM^{\perp}).$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$; then using above decomposition, the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h\,(X, Y)\,, \;\; \forall X, Y \in \Gamma\,(TM)\,, \tag{2.8}$$

$$\bar{\nabla}_X U = -A_U X + \nabla_x^t U, \; \forall X \in \Gamma\,(TM)\,, \; U \in \Gamma\,(tr\,(TM))\,, \tag{2.9}$$

where $\{\nabla_X Y, \; A_U X\}$ and $\left\{h\,(X, Y)\,, \; \nabla_X^t U\right\}$ are the elements of $\Gamma\,(TM)$ and $\Gamma\,(tr\,(TM))$, respectively. Here ∇ and ∇^t are the linear connections on TM and $tr\,(TM)$, respectively and the linear operator A_U on M is called the shape operator and the symmetric bilinear form h on TM is called the second fundamental form.

Consider projection morphisms $\mathcal{L}$ and $\mathcal{S}$ of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then particularly Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h^{\ell}\,(X, Y) + h^{s}\,(X, Y)\,, \tag{2.10}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\ell}\,(N) + D^{s}\,(X, N)\,, \tag{2.11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla_X^{s}\,(W) + D^{\ell}\,(X, W)\,, \tag{2.12}$$

for any $X, Y \in \Gamma\,(TM)$, $N \in \Gamma\,(\ell tr\,(TM))$ and $W \in \Gamma\left(S\left(TM^{\perp}\right)\right)$, where $h^{l}(X, Y) = \mathcal{L}(h(X, Y))$ and $h^{s}(X, Y) = \mathcal{S}(h(X, Y))$ are the lightlike second fundamental form and the screen second fundamental form of M, respectively. It should be noted that $D^{l} : \Gamma(TM) \times \Gamma(S(TM^{\perp})) \rightarrow \Gamma(ltr(TM))$ and $D^{s} : \Gamma(TM) \times \Gamma(ltr(TM)) \rightarrow \Gamma(S(TM^{\perp}))$ are $\mathcal{F}(M)$-bilinear mappings. ∇^{ℓ} and ∇^{s} are the lightlike and the screen transversal connection on M, respectively. In the consequence of (2.8), (2.10), (2.11) and (2.12), we have

$$g\,(A_W X, \, Y) = \bar{g}\,(h^{s}\,(X, \, Y)\,, W) + \bar{g}\left(Y, D^{\ell}\,(X, W)\right), \tag{2.13}$$

$$g\left(A_N X, \bar{P}Y\right) = \bar{g}\left(N, \nabla_X \bar{P}Y\right), \tag{2.14}$$

where $\bar{P}$ is the projection of TM on $S(TM)$. Furthermore, we also have

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*\left(X, \bar{P}Y\right), \tag{2.15}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.16}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where ∇^* and ∇^{*t} are the linear connections on $S(TM)$ and $Rad(TM)$, respectively. h^* and A^* are $\Gamma(Rad(TM))$-valued and $\Gamma(S(TM))$-valued bilinear forms and called as second fundamental forms of distributions $S(TM)$ and $Rad(TM)$, respectively. By the virtue of (2.16) and (2.17), we have

$$\bar{g}\left(h^*\left(X, \bar{P}Y\right), N\right) = g\left(A_N X, \bar{P}Y\right) \tag{2.17}$$

3. GCR-lightlike submanifolds of (ε)-Sasakian manifolds

Definition 1. *Suppose $\left(\bar{M}, \bar{g}\right)$ is an (ε)-Sasakian manifold and $(M, g, S(TM))$ is its real lightlike submanifold, where V is tangent to M. Then, M is called a GCR-lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:*

- (A) *There exist two subbundles D_1 and D_2 of $Rad(TM)$; such that $Rad(TM) = D_1 \oplus D_2$, where $\phi(D_1) = D_1$ and $\phi(D_2) \subset S(TM)$.*
- (B) *There exist two subbundles D_0 and $\bar{D}$ of $S(TM)$; such that $S(TM) = \{\phi D_2 \oplus \bar{D}\} \perp D_0 \perp V$ and $\phi(\bar{D}) = L \perp S$,*

where D_0 is invariant non-degenerate distribution on M, $\{V\}$ is one dimensional distribution spanned by V, L and S are vector subbundles of $ltr(TM)$ and $S(TM^\perp)$, respectively. Then, the tangent bundle TM of M is decomposed as $TM = \{D \oplus \bar{D} \oplus \{V\}\}$, where $D = Rad(TM) \oplus D_0 \oplus \phi(D_2)$.

Suppose $(M, g, S(TM))$ is a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, any $X \in TM$ can be written as

$$X = P_1 X + P_2 X + P_0 X + \phi P_2 X + QX + \eta(X) V, \tag{3.1}$$

where P_1X, P_2X, P_0X, $\phi P_2 X$ and QX belong to the distributions D_1, D_2, D_0, ϕD_2 and $\bar{D}$, respectively. Assume that L^1 represents the orthogonal complement of the vector subbundle L in $\ell tr(TM)$; then using the definition of GCR-lightlike submanifold, for any $N \in \Gamma(\ell tr(TM))$, we have

$$\phi N = TN + CN, \tag{3.2}$$

where $TN \in \Gamma(\phi L)$ is the tangential part of ϕN and $CN \in \Gamma\left(L^\perp\right)$ is the transversal part of ϕN. Similarly, suppose that $S^\perp$ represents the orthogonal complement of the vector subbundle S in $S\left(TM^\perp\right)$; then for any $W \in \Gamma\left(S\left(TM^\perp\right)\right)$, we have

$$\phi W = TW + CW, \tag{3.3}$$

where $TW \in \Gamma(\phi S)$ is the tangential part of ϕW and $CW \in \Gamma\left(S^\perp\right)$ is the screen transversal part of ϕW. Using (3.1), we obtain

$$\phi X = \phi(P_1 X) + \phi(P_2 X) + \phi(P_0 X) - P_2 X + \phi QX, \tag{3.4}$$

where $\phi QX \in \Gamma(L\perp S)$ and we can write

$$\phi QX = LX + SY, \tag{3.5}$$

where $LX \in \Gamma(L)$ and $SY \in \Gamma(S)$. So, we have

$$\begin{aligned} U(X,Y) &= \nabla_X(\phi P_1 Y) + \nabla_X(\phi P_2 Y) - \nabla_X(P_2 Y) \\ &\quad + \nabla_X(\phi P_0 Y) - A_{LY}X - A_{SY}X, \end{aligned} \tag{3.6}$$

for any $X, Y \in TM$.

Lemma 1. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, for any $X, Y \in TM$ the following equalities hold*

$$P_1 U(X,Y) - \phi P_1 \nabla_X Y = -\varepsilon\eta(Y) P_1 X, \tag{3.7}$$

$$P_2 U(X,Y) + P_2 \nabla_X Y = -\varepsilon\eta(Y) P_2 X, \tag{3.8}$$

$$P_0 U(X,Y) - \phi P_0 \nabla_X Y = -\varepsilon\eta(Y) P_0 X, \tag{3.9}$$

$$\phi P_2 U(X,Y) - \phi P_2 \nabla_X Y = -\varepsilon\eta(Y) \phi P_2 X, \tag{3.10}$$

$$QU(X,Y) - QTh^\ell(X,Y) - QTh^s(X,Y) = -\varepsilon\eta(Y) QX, \tag{3.11}$$

$$\{\eta(U(X,Y)) - \bar{g}(X,Y)\} V = -\varepsilon\eta(Y)\eta(X) V, \tag{3.12}$$

$$\begin{aligned} &\nabla^\ell_X(LY) + D^\ell(X,SY) - L\nabla_X Y + h^\ell(X,\phi P_1 Y) \\ &\quad + h^\ell(X,\phi P_2 Y) + h^\ell(X,\phi P_0 Y) - h^\ell(X,P_2 Y) - Ch^\ell(X,Y) = 0, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\nabla^s_X(SY) - S\nabla_X Y + D^s(X,LY) + h^S(X,\phi P_1 Y) \\ &\quad + h^S(X,\phi P_2 Y) - h^s(X,P_2 Y) + h^s(X,\phi P_0 Y) - Ch^s(X,Y) = 0. \end{aligned} \tag{3.14}$$

Proof. Let $Y \in \Gamma(TM)$; then using (3.4) and (3.5), it follows that $\phi(P_1 Y) - P_2 Y$, $\phi(P_2 Y) + \phi P_0 Y$, LY and SY belong to $Rad(TM)$, $S(TM)$, $\ell tr(TM)$ and $S(TM^\perp)$, respectively. Also for any $X, Y \in \Gamma(TM)$, it is known that

$$(\bar{\nabla}_X \phi) Y = \bar{\nabla}_X(\phi Y) - \phi(\bar{\nabla}_X Y). \tag{3.15}$$

Using (2.10), (2.11), (2.12) and (3.4) in (3.15) and afterwards applying (3.6), we obtain

$$\begin{aligned} (\bar{\nabla}_X \phi) Y &= (P_1 U(X,Y) - \phi P_1 \nabla_X Y) + (P_2 U(X,Y) + P_2 \nabla_X Y) \\ &\quad + (P_0 U(X,Y) - \phi P_0 \nabla_X Y) + \phi(P_2 U(X,Y) - \phi P_2 \nabla_X Y) \\ &\quad + \left(QU(X,Y) - Th^\ell(X,Y) - Th^s(X,Y)\right) + \eta(U(X,Y)) V \\ &\quad + \Big(\nabla^\ell_X(LY) + D^\ell(X,SY) - L\nabla_X Y + h^\ell(X,\phi P_1 Y) + h^\ell(X,\phi P_2 Y) \\ &\quad + h^\ell(X,\phi P_0 Y) \ - h^\ell(X,P_2 Y) - Ch^\ell(X,Y)\big) \\ &\quad + (\nabla^s_X(SY) - S\nabla_X Y + h^s(X,\phi P_1 Y), \end{aligned} \tag{3.16}$$

for any $X, Y \in \Gamma(TM)$. Also from (2.6) and (3.1), it follows that

$$\begin{aligned} (\bar{\nabla}_X \phi) Y &= -\varepsilon\eta(Y)(P_1 X) - \varepsilon\eta(Y)(P_2 X) - \varepsilon\eta(Y)(P_0 X) - \varepsilon\eta(Y)(\phi P_2 X) \\ &\quad - \varepsilon\eta(Y) QX + (\bar{g}(X,Y) - \varepsilon\eta(X)\eta(Y)) V. \end{aligned} \tag{3.17}$$

By using (3.16) and (3.17), it is easy to obtain

$$\begin{aligned}\left(\bar{\tilde{N}}_X f\right) Y &= \left(P_1 U(X,Y) - f P_1 \tilde{N}_X Y\right) + P_2\left(U(X,Y) + \tilde{N}_X Y\right)\\ &+ \left(P_0 U(X,Y) - f P_0 \tilde{N}_X Y\right) + f P_2\left(U(X,Y) - \tilde{N}_X Y\right)\\ &+ QU(X,Y) - Th^l(X,Y) - Th^s(X,Y) + \eta(U(X,Y))V\\ &+ \left(\tilde{N}^l_X(LY) + D^l(X,SY) - L\tilde{N}_X Y + h^l(X,fP_1Y) + h^l(X,fP_2Y)\right.\\ &+ h^l(X,fP_0Y) - h^l(X,P_2Y) - Ch^l(X,Y))\\ &+ \left(\tilde{N}^s_X(SY) - S\tilde{N}_X Y + h^s(X,fP_1Y\right) + h^s(X,fP_2Y)\\ &- h^s(X,P_2Y) + h^s(X,fP_0Y) + D^s(X,LY) - Ch^s(X,Y)).\end{aligned}$$

Then, (3.7) to (3.14) follow on comparing the components of the vector bundles D_1, D_2, D_0, ϕD_2, $\bar{D}$, $\{V\}$, $\ell tr(TM)$ and $S\left(TM^{\perp}\right)$, respectively. □

Lemma 2. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, for any $X \in \Gamma(TM)$ and $N \in \ell tr(TM)$, the following relations hold*

$$P_1\nabla_X(TN) - P_1A_{CN}X + \phi P_1(A_N X) = 0, \tag{3.18}$$
$$P_2\nabla_X(TN) - P_2A_{CN}X - P_2(A_N X) = 0, \tag{3.19}$$
$$P_0\nabla_X(TN) - P_0A_{CN}X + \phi P_0(A_N X) = 0, \tag{3.20}$$
$$\phi P_2(\nabla_X(TN)) - \phi P_2(A_{CN}X) + \phi(P_2A_N X) = 0, \tag{3.21}$$
$$Q\nabla_X(TN) - QA_{CN}X - T\nabla^\ell_X N - TD^s(X,N) = 0, \tag{3.22}$$
$$\eta(\nabla_X TN - A_{CN}X) = \bar{g}(P_1X, N) + \bar{g}(P_2X, N), \tag{3.23}$$
$$h^\ell(X,TN) + \nabla^\ell_X(CN) - C\nabla^\ell_X N + LA_N X = 0, \tag{3.24}$$
$$h^s(X,TN) + D^s(X,CN) - CD^s(X,N) + SA_N X = 0. \tag{3.25}$$

Proof. Let $X \in \Gamma(TM)$ and $N \in \Gamma(\ell tr(TM))$, then we have

$$\begin{aligned}\left(\bar{\nabla}_X\phi\right)N &= \bar{\nabla}_X(TN) + \bar{\nabla}_X(CN)\\ &+\phi(A_N X) - \phi\left(\nabla^\ell_X(N)\right) - \phi(D^s(X,N)).\end{aligned}\tag{3.26}$$

Further on using the equations (2.10) and (2.11) in (3.26), we get

$$\begin{aligned}\left(\bar{\nabla}_X\phi\right)N &= \nabla_X(TN) + h^\ell(X,TN) + h^S(X,TN) - A_{CN}X\\ &+\nabla^\ell_X(CN) + D^s(X,CN) + \phi(A_N X)\\ &-\phi\left(\nabla^\ell_X(N)\right) - \phi D^s(X,N)\end{aligned}\tag{3.27}$$

Using (3.1) to (3.3), we also have

$$\begin{aligned}\nabla_X(TN) &= P_1\nabla_X(TN) + P_2\nabla_X(TN) + P_0\nabla_X(TN)\\ &+\phi P_2\nabla_X(TN) + Q\nabla_X(TN) + \eta(\nabla_X(TN))V,\end{aligned}\tag{3.28}$$

$$\begin{aligned}A_{CN}X &= P_1A_{CN}X + P_2A_{CN}X + P_0A_{CN}X\\ &+\phi P_2A_{CN}X + QA_{CN}X + \eta(A_{CN}X)V,\end{aligned}\tag{3.29}$$

$$\begin{aligned}\phi\left(A_N X\right) &= \phi\left(P_1 A_N X\right)+\phi\left(P_2 A_N X\right)\\ &\quad +\phi\left(P_0 A_N X\right)-P_2\left(A_N X\right)+\phi Q\left(A_N X\right),\end{aligned} \tag{3.30}$$

$$\phi\left(\nabla_X^{\ell} N\right)=T\left(\nabla_X^{\ell} N\right)+C\left(\nabla_X^{\ell} N\right), \tag{3.31}$$

$$\phi\left(D^s(X, N)\right)=T D^s(X, N)+C D^s(X, N). \tag{3.32}$$

On using the equations from (3.28) to (3.32) in equation (3.27), we get

$$\begin{aligned}\left(\bar{\nabla}_X \phi\right) N &= \left\{P_1 \nabla_X T N-P_1 A_{C N} X+\phi P_1\left(A_N X\right)\right\}\\ &\quad +\left\{P_2\left(\nabla_X T N\right)-P_2\left(A_{C N} X\right)-P_2\left(A_N X\right)\right\}\\ &\quad +\left\{P_0 \nabla_X T N-P_0 A_{C N} X+\phi P_0 A_N X\right\}\\ &\quad +\left\{\phi P_2\left(\nabla_X T N\right)-\phi P_2\left(A_{C N} X\right)+\phi P_2\left(A_N X\right)\right\}\\ &\quad +\left\{Q \nabla_X T N-Q A_{C N} X-T \nabla_X^{\ell} N-T D^s(X, N)\right\}\\ &\quad +\left\{\eta\left(\nabla_X T N\right)-\eta\left(A_{C N} X\right)\right\} V\\ &\quad +\left\{h^{\ell}(X, T N)+\nabla_X^{\ell}(C N)-C \nabla_X^{\ell} N+L A_N X\right\}\\ &\quad +\left\{h^s(X, T N)+D^s(X, C N)-C D^s(X, N)+S A_N X\right\},\end{aligned} \tag{3.33}$$

which implies $\phi Q\left(A_N X\right)=L\left(A_N X\right)+S\left(A_N X\right)$, where $L\left(A_N X\right) \in \Gamma(L)$ and $S\left(A_N X\right) \in \Gamma(S)$. Also using (2.6), we have

$$\left(\bar{\nabla}_X \phi\right) N=\bar{g}\left(P_1 X, N\right) V+\bar{g}\left(P_2 X, N\right) V. \tag{3.34}$$

On using (3.33) in (3.34), we obtain

$$\begin{aligned}\bar{g}\left(P_1 X, N\right) V+\bar{g}\left(P_2 X, N\right) V &= \left\{P_1 \nabla_X T N-P_1 A_{C N} X+\phi P_1\left(A_N X\right)\right\}\\ &\quad +\left\{P_2\left(\nabla_X T N\right)-P_2\left(A_{C N} X\right)-P_2\left(A_N X\right)\right\}\\ &\quad +\left\{P_0 \nabla_X T N-P_0 A_{C N} X+\phi P_0 A_N X\right\}\\ &\quad +\left\{\phi P_2\left(\nabla_X T N\right)-\phi P_2\left(A_{C N} X\right)+\phi P_2\left(A_N X\right)\right\}\\ &\quad +\left\{Q \nabla_X T N-Q A_{C N} X-T \nabla_X^{\ell} N-T D^s(X, N)\right\}\\ &\quad +\left\{\eta\left(\nabla_X T N\right)-\eta\left(A_{C N} X\right)\right\} V\\ &\quad +\left\{h^{\ell}(X, T N)+\nabla_X^{\ell}(C N)-C \nabla_X^{\ell} N+L A_N X\right\}\\ &\quad +\left\{h^s(X, T N)+D^s(X, C N)-C D^s(X, N)+S A_N X\right\}.\end{aligned}$$

Then, the relations from (3.18) to (3.25) are obtained on comparing the components of the vector bundles D_1, D_2, D_0, ϕD_2, $\bar{D}$, $\{V\}$, $\ell tr(TM)$ and $S\left(TM^{\perp}\right)$, respectively. $\square$

Lemma 3. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, for any $X \in \Gamma(TM)$ and $W \in \Gamma(S\left(TM^{\perp}\right))$, the following relations hold*

$$P_1\left\{\nabla_X T W-A_{C W} X+\phi\left(A_N X\right)\right\}=0, \tag{3.35}$$

$$P_2\left\{\nabla_X T W-A_{C W} X-A_W X\right\}=0, \tag{3.36}$$

$$P_0\left\{\nabla_X T W-A_{C W} X+\phi\left(A_W X\right)\right\}=0, \tag{3.37}$$

$$\phi P_2 \{\nabla_X TW - A_{CW}X + A_W X\} = 0, \tag{3.38}$$

$$Q\nabla_X TW - QA_{CW}X - T\nabla^s_X W - TD^\ell (X, W) = 0, \tag{3.39}$$

$$\eta (\nabla_X TW - A_{CW}X) = 0, \tag{3.40}$$

$$h^\ell (X, TW) - CD^\ell (X, W) + D^\ell (X, CW) + LA_W X = 0, \tag{3.41}$$

$$h^s (X, TW) + \nabla^s_X (CW) - C (\nabla^s_X W) + SA_W X = 0. \tag{3.42}$$

Proof. Let $X \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$; then using (2.12) and (3.3), it follows that

$$\left(\bar{\nabla}_X \phi\right) W = \bar{\nabla}_X (TW) + \bar{\nabla}_X (CW) + \phi (A_W X) - \phi (\nabla^s_X W) - \phi \left(D^\ell (X, W)\right). \tag{3.43}$$

Then, further using (2.10), (2.12), (3.1), (3.3) and (3.4) in equation (3.43), we obtain

$$\begin{aligned} \left(\bar{\nabla}_X \phi\right) W \quad = \quad & P_1 \{\nabla_X (TW) - (A_{CW}X) + \phi (A_W X)\} \\ & + P_2 \{\nabla_X (TW) - A_{CW}X - A_W X\} \\ & + P_0 \{\nabla_X (TW) - A_{CW}X + \phi (A_W X)\} \\ & + \phi P_2 \{\nabla_X TW - A_{CW}X + A_W X\} \\ & + \left\{Q\nabla_X TW - QA_{CW}X - T\nabla^s_X W - TD^\ell (X, W)\right\} \\ & + \{\eta (\nabla_X TW) - \eta (A_{CW}X)\} V \\ & + \left\{h^\ell (X, TW) + D^\ell (X, CW) - CD^\ell (X, W) + LA_N X\right\} \\ & + \{h^s (X, TW) - C\nabla^s_X W + \nabla^s_X (CW) + S (A_N X)\}. \end{aligned} \tag{3.44}$$

In consequence of (2.6), we know that $\left(\bar{\nabla}_X \phi\right) W = 0$; then the relations from (3.35) to (3.42) follow immediately on comparing the components of the vector bundles D_1, D_2, D_0, ϕD_2, $\bar{D}$, $\{V\}$, $\ell tr (TM)$ and $S\left(TM^\perp\right)$, respectively. □

Lemma 4. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, for any $X \in D$ and $Y \in \bar{D}$, we have the following relations*

$$\nabla_X V = -\varepsilon\phi X, \;\; h^\ell (X, V) = 0, \;\; h^s (X, V) = 0, \tag{3.45}$$

$$\nabla_Y V = 0, \;\; h^\ell (Y, V) = -\varepsilon LY, \;\; h^s (Y, V) = -\varepsilon SY, \tag{3.46}$$

$$\nabla_V V = 0, \;\; h^\ell (V, V) = 0, \; h^s (V, V) = 0. \tag{3.47}$$

Proof. The proof follows immediately by using (2.10), (3.1) and (3.4) in (2.6). □

4. Integrability of the distributions

Theorem 2. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, necessary and sufficient conditions for the radical distribution $Rad(TM)$ to be integrable are the following*

(1) $h^\ell (X, \phi Y) = h^\ell (Y, \phi X)$, $h^s (X, \phi Y) = h^s (Y, \phi X)$, $\forall X, Y \in Rad(TM)$.

(2) $\bar{g}\left(h^* (X, \phi Y), \phi\tilde{Z}\right) = \bar{g}\left(h^* (Y, \phi X), \phi\tilde{Z}\right)$, $\forall X, Y \in D_2$, $\phi\tilde{Z} \in \ell tr (TM)$.

(3) $\bar{g}\left(\nabla^{*t}_X \phi Y, \phi\tilde{Z}\right) = \bar{g}\left(\nabla^{*t}_Y \phi X, \phi\tilde{Z}\right)$, $\forall X, Y \in D_1$, $\phi\tilde{Z} \in \ell tr (TM)$.

(4) $\bar{g}\left(h^{*}(X,\phi Y),\phi\tilde{Z}\right)=\bar{g}\left(\nabla^{*t}_{Y}(\phi X),\phi\tilde{Z}\right)$, $\forall X\in D_1, Y\in D_2, \phi\tilde{Z}\in \ell tr(TM)$.
$\bar{g}\left(h^{*}(Y,\phi X),\phi\tilde{Z}\right)=\bar{g}\left(\nabla^{*t}_{X}(\phi Y),\phi\tilde{Z}\right)$, $\forall X\in D_2, Y\in D_1, \phi\tilde{Z}\in \ell tr(TM)$.

(5) $\bar{g}\left(A^{*}_{\phi Y}X,\phi Z_0\right)=\bar{g}\left(A^{*}_{\phi X}Y,\phi Z_0\right)$, $\forall X,Y\in D_1, Z_0\in D_0$.

(6) $\bar{g}\left(\nabla^{*}_{X}\phi Y,\phi Z_0\right)=\bar{g}\left(\nabla^{*}_{Y}(\phi X),\phi Z_0\right)$, $\forall X,Y\in D_2,\ Z_0\in D_0$.

(7) $\bar{g}\left(\nabla^{*}_{X}(\phi Y),\phi Z_0\right)=-\bar{g}\left(A^{*}_{\phi X}Y,\phi Z_0\right)$, $\forall X\in D_1, Y\in D_2,\ Z_0\in D_0$.

Proof. (1) Assume that the radical distribution $RadTM$ is integrable; then, this implies that $[X,Y]\in RadTM$ for any $X,Y\in RadTM$. Using (3.13), we have $h^{\ell}(X,\phi Y)=L\nabla_X Y+Ch^{\ell}(X,Y)$ and $h^{\ell}(Y,\phi X)=L\nabla_Y X+Ch^{\ell}(Y,X)$ this further implies that $h^{\ell}(X,\phi Y)-h^{\ell}(Y,\phi X)=L(\nabla_X Y-\nabla_Y X)=L[X,Y]$. If $[X,Y]\in RadTM$, then $L[X,Y]=0$, therefore we get

$$h^{\ell}(X,\phi Y)=h^{\ell}(Y,\phi X). \tag{4.1}$$

Similarly as a consequence of (3.14), we also have

$$h^{s}(X,\phi Y)=h^{s}(Y,\phi X), \tag{4.2}$$

and $\bar{g}([X,Y],V)=2\varepsilon g(Y,\phi X)=0$. Now using (2.6) for any $X,Y\in RadTM$ and $\tilde{Z}\in\bar{D}$, we get

$\bar{g}\left(\bar{\nabla}_X(\phi Y)+h^{\ell}(X,\phi Y)+h^{s}(X,\phi Y)-\nabla_Y(\phi X)-h^{\ell}(Y,\phi X)-h^{s}(Y,\phi X)\phi\tilde{Z}\right)=0,$

and then by using (4.1) and (4.2), we obtain

$$0=\bar{g}\left(\nabla_X(\phi Y),\phi\tilde{Z}\right)-\bar{g}\left(\nabla_Y(\phi X),\phi\tilde{Z}\right). \tag{4.3}$$

For $\nabla_X(\phi Y)$, $\nabla_Y(\phi X)\in\Gamma(TM)$ and $\phi\tilde{Z}\in S\left(TM^{\perp}\right)$, (4.3) is satisfied.

(2) Let $\phi X,\phi Y\in S(TM)$ and $\phi\tilde{Z}\in\ell tr(TM)$, then applying (2.16) in (4.3), we get

$$\begin{aligned}0 &= \bar{g}\left(\nabla^{*}_{X}(\phi Y),\phi\tilde{Z}\right)-\bar{g}\left(\nabla^{*}_{Y}(\phi X),\phi\tilde{Z}\right)\\ &\quad+\bar{g}\left(h^{*}(X,\phi Y),\phi\tilde{Z}\right)-\bar{g}\left(h^{*}(Y,\phi X),\phi\tilde{Z}\right)\end{aligned} \tag{4.4}$$

where $\nabla^{*}_{X}(\phi Y),\nabla^{*}_{Y}(\phi X)\in S(TM)$ and $h^{*}(X,\phi Y),\ h^{*}(Y,\phi X)\in Rad(TM)$. Hence, (4.4) becomes

$$\bar{g}\left(h^{*}(X,\phi Y),\phi\tilde{Z}\right)=\bar{g}\left(h^{*}(Y,\phi X),\phi\tilde{Z}\right). \tag{4.5}$$

(3) Similarly, for $\phi X,\phi Y\in S(TM)$ and $\phi\tilde{Z}\in\ell tr(TM)$, using (2.17) in (4.3), we get

$$0=\bar{g}\left(A^{*}_{\phi X}Y,\phi\tilde{Z}\right)-\bar{g}\left(A^{*}_{\phi Y}X,\phi\tilde{Z}\right)-\bar{g}\left(A^{*t}_{Y}\phi X,\phi\tilde{Z}\right)+\bar{g}\left(A^{*t}_{X}\phi Y,\phi\tilde{Z}\right), \tag{4.6}$$

where $A^{*}_{\phi X}Y,A^{*}_{\phi Y}X\in S(TM)$ and $\nabla^{*t}_{Y}\phi X,\nabla^{*t}_{X}\phi Y\in Rad(TM)$. Hence, (4.6) becomes

$$\bar{g}\left(\nabla^{*t}_{X}\phi Y,\phi\tilde{Z}\right)=\bar{g}\left(\nabla^{*t}_{Y}\phi X,\phi\tilde{Z}\right). \tag{4.7}$$

(4) Let $\phi X \in Rad\,(TM)$, $\phi Y \in S\,(TM)$ and $\phi \tilde{Z} \in \ell tr\,(TM)$; then using (2.16) and (2.17) in (4.3), we get

$$\bar{g}\left(h^*\,(X, \phi Y)\,, \phi \tilde{Z}\right) = \bar{g}\left(\nabla_Y^{*t}\,(\phi X)\,, \phi \tilde{Z}\right). \tag{4.8}$$

Similarly, by using $\phi X \in S\,(TM)$, $\phi Y \in Rad\,(TM)$ and $\phi \tilde{Z} \in \ell tr\,(TM)$, with the help of (2.16) and (2.17) in (4.3), it follows that

$$\bar{g}\left(h^*\,(Y, \phi X)\,, \phi \tilde{Z}\right) = \bar{g}\left(\nabla_X^{*t}\,(\phi Y)\,, \phi \tilde{Z}\right). \tag{4.9}$$

(5) For $X, Y \in D_1$ and $Z_0 \in D_0$, using (2.6) and (2.17), we have the following relation immediately

$$\bar{g}\left(A^*_{\phi X} Y, \phi Z_0\right) = \bar{g}\left(A^*_{\phi Y} X, \phi Z_0\right). \tag{4.10}$$

(6) For $X, Y \in D_2$ and $Z_0 \in D_0$, by the use of (2.6) and (2.16), we have

$$\bar{g}\left(\nabla^*_X\,(\phi Y)\,, \phi Z_0\right) = \bar{g}\left(\nabla^*_Y\,(\phi X)\,, \phi Z_0\right). \tag{4.11}$$

(7) Finally, let $X \in D_1$, $Y \in D_2$ and $Z_0 \in D_0$; then, using (2.6), (2.16) and (2.17), we obtain

$$\bar{g}\left(\nabla^*_X\,(\phi Y)\,, \phi Z_0\right) = -\bar{g}\left(A^*_{\phi X} Y, \phi Z_0\right). \tag{4.12}$$

If we take $X, Y \in D_2$ and $Z \in \phi D_2$, then on applying(2.17) and (2.16), we get

$$\bar{g}\left(h^*\,(X, \phi Y)\,, \phi Z\right) - \bar{g}\left(h^*\,(Y, \phi X)\,, \phi Z\right) = 0, \tag{4.13}$$

where $\bar{g}\left(h^*\,(X, \phi Y)\,, \phi Z\right) = 0 = \bar{g}\left(h^*\,(Y, \phi X)\,, \phi Z\right)$, for all $\phi Z \in D_2$, this implies (4.13) holds. For $X, Y \in D_1$ and $Z \in \phi D_2$ with the help of (2.6) and (2.17) it is possible to get $g\left(\nabla_X^{*t} \phi Y, \phi Z\right) = g\left(\nabla_Y^{*t} \phi X, \phi Z\right) = 0$. For $X \in D_1$, $Y \in D_2$ and $Z \in \phi D_2$ by using (2.6) and (2.16) and (2.17), we get $\bar{g}\left(h^*\,(X, \phi Y)\,, \phi Z\right) = 0 = \bar{g}\left(h^*\,(Y, \phi X)\,, \phi Z\right)$, for all $\phi Z \in D_2$. □

Theorem 3. *Let $(M, g, S\,(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, the distribution D_0 is never integrable.*

Proof. Assume that the distribution D_0 is integrable, then $g\,([X, Y]\,, V) = 0$ for any $X, Y \in D_0$. By using (2.6), we obtain $\bar{g}\,([X, Y]\,, V) = 2\varepsilon \bar{g}\,(Y, \phi X)$ for $X, Y \in D_0$, then by using the above relation, we have $\bar{g}\,(Y, \phi X) = 0$. Since D_0 is non-degenerate then $\bar{g}\,(Y, \phi X) \neq 0$. This leads to a contradiction and hence the assertion follows. □

Lemma 5. *Let $(M, g, S\,(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, $[X, V] \in D \oplus \{V\}$, for any $X \in D$.*

Proof. Let $X \in D$ and $Y \in \bar{D}$; then using (3.47), we get $\bar{g}\,([X, V]\,, Y) = \varepsilon \bar{g}\,(X, \phi Y) - \bar{g}\,(\nabla_V X, Y)$. Particularly, on taking $\phi Y \in S\left(TM^{\perp}\right)$, we have $\bar{g}\,([X, V]\,, Y) = -\bar{g}\,(\nabla_V X, Y)$ and further on putting $X = \phi X$, then using (2.6), (2.12), (2.13) and (3.47), we obtain $\bar{g}\,([\phi X, V]\,, Y) = \bar{g}\,(h^s\,(V, X)\,, \phi Y) = 0$. In particular, if we take $\phi Y \in \ell tr\,(TM)$ then by using (2.6), (2.15) and (2.11), we get $\bar{g}\,([\phi X, V]\,, Y) = \bar{g}\,(\phi Y, \nabla_V X) = 0$. We know that if $X \in D$ then this implies that $\phi X \in D$ therefore $\bar{g}\,([X, V]\,, Y) = 0$ implies that $[X, V] \in D \oplus \{V\}$ for $X \in D$. □

Theorem 4. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, necessary and sufficient conditions for $D \oplus \{V\}$ to be integrable are $h^{\ell}(X, \phi Y) = h^{\ell}(\phi X, Y)$ and $h^{s}(X, \phi Y) = h^{s}(\phi X, Y)$, for any $X, Y \in D \oplus \{V\}$.*

Proof. Assume that $D \oplus \{V\}$ is integrable then $[X, Y] \in D \oplus \{V\}$, for any $X, Y \in D \oplus \{V\}$. We know that for any $X, Y \in D \oplus \{V\}$, we can write $X = PX + \eta(X) V$ and $Y = PY + \eta(Y) V$, where $PX, PY \in D$. Using these relations, we obtain $[X, Y] - \eta(Y)[PX, V] - \eta(X)[V, PY] = [PX, PY]$. Since $[X, Y]$, $[PX, V]$, $[V, PY] \in D \oplus \{V\}$ then $[PX, PY] \in D \oplus \{V\}$. Thus $Q[PX, PY] = 0$ implies $L[PX, PY] = 0 = S[PX, PY]$. On using these equalities in (3.13), our assertion follows. □

Theorem 5. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, necessary and sufficient condition for the distribution $\bar{D}$ to be integrable is that $A_{\phi X} Y = A_{\phi Y} X$ for $X, Y \in \bar{D}$.*

Proof. Let $\phi X \in S\left(TM^{\perp}\right)$, $Y \in \bar{D}$ and $Z \in S(TM)$, then using (2.6), (2.10) and (2.13), it follows that

$$g(A_{\phi X} Y, Z) = -\bar{g}\left(\bar{\nabla}_{Z}(\phi Y), X\right). \tag{4.14}$$

For $\phi Y \in S\left(TM^{\perp}\right)$, by the use of (2.13) in (4.14), we have

$$g(A_{\phi X} Y, Z) = g(A_{\phi Y} X, Z). \tag{4.15}$$

Also for $\phi Y \in \ell tr(TM)$, by using (2.17) in (4.14), we have

$$g(A_{\phi X} Y, Z) = g(A_{\phi Y} X, Z). \tag{4.16}$$

Hence by the use of (4.15) and (4.16), for any $Z \in S(TM)$, we obtain $A_{\phi X} Y = A_{\phi Y} X$. If $\phi X \in \ell tr(TM)$, $Y \in \bar{D}$ and $Z \in S(TM)$ then from (2.6), (2.16) and (2.17), it follows that $g(A_{\phi X} Y, Z) = \bar{g}(\nabla_Z X, \phi Y)$. Furthermore, particularly on taking $\phi Y \in \ell tr(TM)$ and using (2.16), we obtain $g(A_{\phi X} Y, Z) = \bar{g}(h^{*}(X, Z), \phi Y) = g(A_{\phi Y} X, Z)$. If particularly we take $\phi X \in S\left(TM^{\perp}\right)$, $Y \in \bar{D}$ and $Z \in Rad(TM)$, then using (2.6), (2.10), (2.12) and (2.13), we get

$$\begin{aligned} g(A_{\phi X} Y, Z) &= \bar{g}(h^{s}(Y, Z), \phi X) + \bar{g}\left(Z, D^{\ell}(Y, \phi X)\right) \\ &= \bar{g}(h^{s}(X, Z), \phi Y) + \bar{g}\left(Z, D^{\ell}(Y, \phi X)\right). \end{aligned} \tag{4.17}$$

Now, for any $\phi Y \in S\left(TM^{\perp}\right)$, using (3.13), we get $D^{\ell}(X, \phi Y) = D^{\ell}(Y, \phi X)$. Then, using (4.17) for any $Z \in S(TM)$, we obtain $g(A_{\phi X} Y, Z) = \bar{g}(A_{\phi Y} X, Z)$.

If $\phi X \in \ell tr(TM)$, $Y \in \bar{D}$ and $Z \in S(TM)$; then using (2.6), (2.16) and (2.17), it yields

$$\begin{aligned} g(A_{\phi X} Y, Z) &= \bar{g}(\phi X, \nabla^{*}_{Y} Z) + \bar{g}\left(\nabla^{\ell}_{Y}(\phi X), Z\right) \\ &= -\bar{g}(X, \nabla_{Y}(\phi Z)) + \bar{g}\left(\nabla^{\ell}_{Y}(\phi X), Z\right). \end{aligned} \tag{4.18}$$

On applying (2.16) and (2.17) in (4.18) we get $g(A_{\phi X} Y, Z) = \bar{g}\left(\nabla^{\ell}_{Y}(\phi X), Z\right)$ and further on taking $\phi Y \in \ell tr(TM)$ and using (3.13), we obtain $g(A_{\phi X} Y, Z) =$

$\bar{g}\left(\nabla^{\ell}_{\phi X}(Y), Z\right) = g\left(A_{\phi Y}X, Z\right)$. This implies that on considering $Z \in Rad\,(TM)$, we have $A_{\phi X}Y = A_{\phi Y}X$.

Conversely, let $X, Y \in \bar{D}$; then using $\phi P\nabla_X Y = \phi\left(P_1\nabla_X Y\right) + \phi\left(P_2\nabla_X Y\right) + \phi\left(P_0\nabla_X Y\right) - P_2\nabla_X Y$ and applying (2.6) and (2.10),by the use of the equations from (3.2) to (3.5), it is possible to have

$$\begin{aligned}\bar{\nabla}_X\phi Y &= \bar{g}(X,Y)V + \phi P\nabla_X Y + L\nabla_X Y + S\nabla_X Y \\ &\quad + Th^{\ell}(X,Y) + Ch^{\ell}(X,Y) + Th^{s}(X,Y) + Ch^{s}(X,Y). \quad (4.19)\end{aligned}$$

For $\phi X, \phi Y \in \ell tr\,(TM)$, by using (2.11) in (4.19), we also have

$$\begin{aligned}-A_{\phi Y}X + \nabla^{\ell}_X(\phi Y) + D^{s}(X,\phi Y) &= \bar{g}(X,Y)V + \phi P\nabla_X Y + L\nabla_X Y \\ &\quad + S\nabla_X Y + Th^{\ell}(X,Y) + Ch^{\ell}(X,Y) \\ &\quad + Th^{s}(X,Y) + Ch^{s}(X,Y). \quad (4.20)\end{aligned}$$

On separating the tangential and transversal components of (4.20), we obtain

$$A_{\phi Y}X = -\bar{g}(X,Y)V - \phi P\nabla_X Y - Th^{\ell}(X,Y) - Th^{s}(X,Y), \quad (4.21)$$

$$\nabla^{\ell}_X(\phi Y) + D^{s}(X,\phi Y) = L\nabla_X Y + S\nabla_X Y + Ch^{\ell}(X,Y) + Ch^{s}(X,Y). \quad (4.22)$$

From (4.21), we get $A_{\phi Y}X - A_{\phi X}Y = -\phi P[X,Y]$. Since $A_{\phi Y}X = A_{\phi X}Y$; then we get $P[X,Y] = 0$ and this implies that $[X,Y] \in \bar{D}\oplus\{V\}$. Therefore, $\bar{g}([X,Y],V) = -g(Y,\nabla_X V) + g(X,\nabla_Y V) = 0$, for any $X, Y \in \bar{D}$, $\phi X, \phi Y \in \ell tr\,(TM)$ and $[X,Y] \in \bar{D}$. Similarly, for any $\phi X, \phi Y \in S\left(TM^{\perp}\right)$, by using (2.12) in (4.19), we obtain

$$\begin{aligned}-A_{\phi Y}X + \nabla^{s}_X(\phi Y) + D^{\ell}(X,\phi Y) &= \bar{g}(X,Y)V + \phi P\nabla_X Y + L\nabla_X Y \\ &\quad + S\nabla_X Y + Th^{\ell}(X,Y) + Th^{s}(X,Y) \\ &\quad + Ch^{\ell}(X,Y) + Ch^{s}(X,Y). \quad (4.23)\end{aligned}$$

On separating the tangential and transversal components of (4.23), we get

$$A_{\phi Y}X = -\bar{g}(X,Y)V - \phi P\nabla_X Y - Th^{\ell}(X,Y) - Th^{s}(X,Y), \quad (4.24)$$

$$\nabla^{s}_X(\phi Y) + D^{\ell}(X,\phi Y) = L\nabla_X Y + S\nabla_X Y + Ch^{\ell}(X,Y) + Ch^{s}(X,Y). \quad (4.25)$$

From (4.24), we get $A_{\phi Y}X - A_{\phi X}Y = -\phi P[X,Y]$. Since $A_{\phi Y}X = A_{\phi X}Y$; then $P[X,Y] = 0$. We know that $\bar{g}([X,Y],V) = 0$; therefore, for any $X, Y \in \bar{D}$, $\phi X, \phi Y \in S\left(TM^{\perp}\right)$, it follows that $[X,Y] \in \bar{D}$. Hence, the proof is complete. □

Theorem 6. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, the distribution D defines a totally geodesic foliation in M if $Th(X,Y) = 0$ for any $X, Y \in \Gamma(D)$.*

Proof. From the Definition 1, for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D)$, $W \in \Gamma(S)$, we have $g(\nabla_X Y, \phi Z) = g(\nabla_X Y, \phi W) = 0$. Particularly, from (2.6) and (2.10), for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D_1) \subset Rad\,(TM)$, it follows that

$$g(\nabla_X Y, \phi Z) = -g\left(h^{\ell}(X,\phi Y), Z\right) = 0. \quad (4.26)$$

Similarly, using (2.10) and (2.3), for any $X, Y \in \Gamma(D)$, $W \in \Gamma(S)$, we have

$$g(\nabla_X Y, \phi W) = -g\left(h^{s}(X,\phi Y), W\right) = 0. \quad (4.27)$$

Thus, from (4.26) and (4.27), it is clear that if the distribution D defines a totally geodesic foliation in M then $h^s(X, \phi Y)$ and $h^\ell(X, \phi Y)$ have no components in S and L, respectively. Thus, using these results with (3.2) and (3.3), the proof is complete. □

Theorem 7. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, the distribution $\bar{D}$ defines a totally geodesic foliation in M if and only if $A_{\phi Y}X \in \Gamma(\bar{D})$ for any $X, Y \in \Gamma(\bar{D})$.*

Proof. For the elements $X, Y \in \Gamma(\bar{D})$, by using (3.4), we obtain $\phi(\bar{\nabla}_X Y) = \phi(P\bar{\nabla}_X Y) + \phi(Q\bar{\nabla}_X Y)$. If we set $\phi(P\bar{\nabla}_X Y) = T(\bar{\nabla}_X Y)$ and $\phi(Q\bar{\nabla}_X Y) = C\bar{\nabla}_X Y$; then, by the use of (2.6), (2.8) and (2.9), we further have $-A_{\phi Y}X = T\nabla_X Y - Th(X, Y)$ for any $X, Y \in \Gamma(\bar{D})$. Assume that the distribution $\bar{D}$ is a totally geodesic foliation in M; then, it follows that $A_{\phi Y}X = -Th(X, Y)$. Therefore, $A_{\phi Y}X \in \Gamma(\bar{D})$ for any $X, Y \in \Gamma(\bar{D})$. Conversely, let $A_{\phi Y}X \in \Gamma(\bar{D})$, for any $X, Y \in \Gamma(\bar{D})$ then this implies that $T\nabla_X Y = 0$ and hence $\nabla_X Y \in \Gamma(\bar{D})$. □

Definition 2. *A GCR-lightlike submanifold M is called D-geodesic if $h(X, Y) = 0$, for any $X, Y \in \Gamma(D)$. Using the decomposition of the transversal vector bundle, GCR-lightlike submanifold M is said to be a D-geodesic if $h^\ell(X, Y) = 0$ and $h^s(X, Y) = 0$ for any $X, Y \in \Gamma(D)$. Also, M is said to be a mixed geodesic if $h^\ell(X, Y) = 0$ and $h^s(X, Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(\bar{D})$.*

Theorem 8. *Let $(M, g, S(TM))$ be a GCR-lightlike submanifold of an (ε)-Sasakian manifold $\bar{M}$. Then, the following assertions are equivalent*

(1) *M is mixed totally geodesic.*
(2) *$\nabla^s_D(\phi\bar{D}) \subset \phi\bar{D}$ and $A_{\phi\bar{D}}D \subset D$.*

Proof. Choose $Y \in \bar{D}$ such that $\phi Y \in S(TM^\perp)$; then, there exists a $W \in S(TM^\perp)$ such that $\phi W = TW = Y$. Let $X \in D$ and $W \in S(TM^\perp)$; then, we have $h^s(X, Y) = C(\nabla^s_X W) - S(A_W X)$. Using the hypothesis that M is a mixed totally geodesic; then, for $X \in D$ and $Y \in \bar{D}$, $h^s(X, Y) = 0$ holds and we further obtain $C(\nabla^s_X W) = S(A_W X)$ where $S(A_W X) \in \Gamma(S) \subset S(TM^\perp)$ and $C(\nabla^s_X W) \in \Gamma(S^\perp) \subset S(TM^\perp)$. For any $\nabla^s_X W \in S(TM^\perp)$, on using (3.3), we have

$$\phi(C\nabla^s_X W) = -\nabla^s_X W - \phi(T\nabla^s_X W), \tag{4.28}$$

where $\nabla^s_X W \in S(TM^\perp)$ and $\phi(T\nabla^s_X W) \in \phi\bar{D} \subset S(TM^\perp)$. Using (4.28), it follows that $C\nabla^s_X W \in \{S(TM^\perp) - \phi\bar{D}\}$. Since $S(A_W X) \in \phi\bar{D}$; then, using (4.27), we get $C(\nabla^s_X W) = 0$ and $S(A_W X) = 0$. Thus, from (4.27) and (4.28), we have $\nabla^s_X W = \phi(T\nabla^s_X W), \nabla^s_X W \in \phi\bar{D}$ and $A_W X \in D$, for any $X \in D$ and $W \in \phi\bar{D}$. Consequently, we obtain that $\nabla^s_D \phi\bar{D} \subset \phi\bar{D}$ and $A_{\phi\bar{D}}D \subset D$.

Next, choose $Y \in \bar{D}$ such that there exists a $N \in \ell tr(TM)$ such that $\phi N = TN = Y$, $CN = 0$. Using (3.26), for any $X \in D$ and $N \in \ell tr(TM)$, it follows that $h^\ell(X, Y) = C\nabla^\ell_X N - LA_N X$. Assume that M is the mixed totally geodesic; then, $h^\ell(X, Y) = 0$ where $X \in D$, $Y \in \bar{D}$, therefore we further obtain

$$C\nabla^\ell_X N = LA_N X, \tag{4.29}$$

where $C\left(\nabla^{\ell}_X N\right) \in \Gamma\left(L^{\perp}\right) \subset \ell tr\,(TM)$ and $LA_N X \in \Gamma\,(L) \subset \ell tr\,(TM)$. For $\nabla^{\ell}_X N \in \ell tr\,(TM)$, from (3.2), we also have

$$\phi\left(C\nabla^{\ell}_X N\right) = -\nabla^{\ell}_X N - \phi\left(T\nabla^{\ell}_X N\right), \tag{4.30}$$

where $\nabla^{\ell}_X N \in \ell tr\,(TM)$ and $\phi\left(T\nabla^{\ell}_X N\right) \in \phi\bar{D} \subset \ell tr\,(TM)$. From (4.30), it is obvious that $\phi\left(C\nabla^{\ell}_X N\right) \notin \bar{D}$, this implies that $C\nabla^{\ell}_X N \notin \phi\bar{D}$, that is, $C\nabla^{\ell}_X N \in \left\{\ell tr\,(TM) - \phi\bar{D}\right\}$. Since $L\,(A_N X) \in \phi D$ therefore from (4.29), we get $C\nabla^{\ell}_X N = 0$ and $LA_N X = 0$. As a conclusion, we obtain $\nabla^{\ell}_X N \in \phi\bar{D}$ and $A_N X \in D$, for any $X \in D$ and $N \in \phi\bar{D} \subset \ell tr\,(TM)$. Consequently, we have $\nabla^{\ell}_D \phi\bar{D} \subset \phi\bar{D}$ and $A_{\phi\bar{D}} D \subset D$. Hence the proof is complete. □

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