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Smarandache Curves According to q-Frame in Minkowski 3-Space

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Abstract: In this study, we investigate special Smarandache curves according to q-frame in Minkowski 3-space and we give some differential geometric properties of Smarandache curves.

Keywords: Frenet frame, Natural curvatures, q-frame, Smarandache curves.

1 Introduction

The most well-known adapted frame is the Frenet frame. The Frenet frame plays an important role in classical differential geometry [30], where it is useful to investigate Bertrand curves [27] and tube surfaces [17, 21].

Let $\alpha(t)$ be a regular space curve [6, 8], then the Frenet frame is defined as follows

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{b} = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \mathbf{n} = \mathbf{b} \wedge \mathbf{t}. \tag{1}$$

The curvature κ and the torsion τ are given by

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}. \tag{2}$$

The well-known Frenet formulas are given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \|\alpha'\| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{3}$$

A regular curve in Euclidean 3-space, whose position vector is composed of Frenet frame vectors on another regular curve, is called a Smarandache curve [3]. Recently, Smarandache curves have been stutied in various ambient spaces [1, 10, 16, 26, 28]. Moreover, some special Smarandache curves with reference to Darboux frame in Euclidean 3-space is studied in [4]. A. T. Ali has introduced some special Smarandache curves in the Euclidean space [1]. Taking $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be its moving Serret-Frenet frame, $\mathbf{tn}-$Smarandache curve, $\mathbf{nb}-$Smarandache curve and $\mathbf{tnb}-$Smarandache curve are defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{t} + \mathbf{n}), \beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{n} + \mathbf{b}) \text{ and } \beta(s^*) = \frac{1}{\sqrt{3}}(\mathbf{t} + \mathbf{n} + \mathbf{b}) \tag{4}$$

respectively [1].

However, Frenet frame has several disadvantages in applications. For instance, Frenet frame is undefined wherever the curvature vanishes. Moreover, the main drawback of the Frenet frame is that it has undesirable rotation about tangent vector [6, 18]. Therefore, Bishop [5] introduced a new frame along a space curve which is more suitable for applications. But, it is well known that Bishop frame calculations is not a easy task [29]. In order to construct the 3D curve offset, Coquillart [9] introduced the quasi-normal vector of a space curve. The quasi-normal vector is defined for each point of the curve, and lies in the plane perpendicular to the tangent of the curve at this point [24]. Then using the quasi-normal vector Dede et al. in [11] introduced the q-frame along a space curve. Given a space curve $\alpha(t)$ the q-frame consists of three orthonormal vectors, these being the unit tangent vector $\mathbf{t}$, the quasi-normal $\mathbf{n}_q$ and the quasi-binormal vector $\mathbf{b}_q$. The q-frame $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$ is given by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q \tag{5}$$

where $\mathbf{k}$ is the projection vector [11]. The q-frame has many advantages versus other frames (Frenet, Bishop). For instance the q-frame can be defined even along a line($\kappa = 0$) and the construction of the q-frame doesn't change if the space curve has unit speed or not. Moreover the q-frame can be calculated easily [12].

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For simplicity, we have chosen the projection vector $\mathbf{k} = (0, 0, 1)$ in this paper. However, the q-frame is singular in all cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel. Thus, in those cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (0, 1, 0)$ or $\mathbf{k} = (1, 0, 0)$.

A q-frame along a space curve is shown in Figure 1.

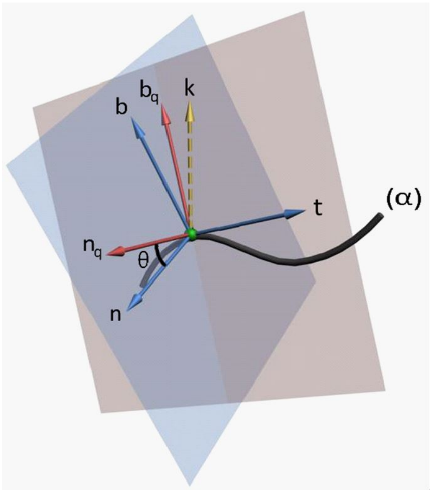

Fig. 1: The q-frame and Frenet frame.

The variation equations of the directional q-frame is given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \|\alpha'\| \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}, \tag{6}$$

where the q-curvatures are expressed as follows

$$k_1 = -\frac{\langle \mathbf{t}, \mathbf{n}_q' \rangle}{\|\alpha'\|}, k_2 = -\frac{\langle \mathbf{t}, \mathbf{b}_q' \rangle}{\|\alpha'\|}, k_3 = \frac{\langle \mathbf{n}_q', \mathbf{b}_q \rangle}{\|\alpha'\|}. \tag{7}$$

In the three dimensional Minkowski space $\mathbb{R}^3_1$, the inner product and the cross product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are defined as

$$< \mathbf{u}, \mathbf{v} >= u_1 v_1 + u_2 v_2 - u_3 v_3$$

and

$$\mathbf{u} \wedge \mathbf{v} = (u_3 v_2 - u_2 v_3, u_1 v_3 - u_3 v_1, u_1 v_2 - u_2 v_1)$$

where $e_1 \wedge e_2 = e_3$, $e_2 \wedge e_3 = -e_1$, $e_3 \wedge e_1 = -e_2$, respectively [2]. If $\mathbf{u}$ and $\mathbf{v}$ are timelike vectors then $\mathbf{u} \wedge \mathbf{v}$ is a spacelike vector [22].

The norm of the vector $\mathbf{u}$ is given by

$$\|\mathbf{u}\| = \sqrt{|\langle u, u \rangle|} \tag{8}$$

We say that a Lorentzian vector $\mathbf{u}$ is spacelike, lightlike or timelike if $\langle \mathbf{u}, \mathbf{u} \rangle > 0$, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ and $\mathbf{u} \neq 0$, $\langle \mathbf{u}, \mathbf{u} \rangle < 0$, respectively. In particular, the vector $\mathbf{u} = 0$ is spacelike.

An arbitrary curve $\alpha(s)$ in $\mathbb{R}^3_1$, can locally be spacelike, timelike or null(lightlike), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null [20]. A null curve α is parameterized by pseudo-arc s if $\langle \alpha''(s), \alpha''(s) \rangle = 1$. On the other hand, a non-null curve α is parameterized by arc-lenght parameter s if $\langle \alpha'(s), \alpha'(s) \rangle = \pm 1$ [7, 23].

Then Frenet formulas of timelike curve may be written as

$$\frac{d}{dt} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = v \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{9}$$

where $v = \|\mathbf{t}\|$. The Minkowski curvature and torsion of timelike curve $\alpha(t)$ are obtained by

$$\kappa = \|\mathbf{t}'\|, \tau =< \mathbf{n}', \mathbf{b} > \tag{10}$$

respectively [2, 22].

As an alternative to the Frenet frame we define a new adapted frame along a timelike space curve, called as the q-frame. Given a regular timelike space curve $\alpha(t)$ the q-frame consists of three orthonormal vectors, these being the unit tangent vector $\mathbf{t}$(timelike), the quasi-normal $\mathbf{n}_q$(spacelike) and the quasi-binormal vector $\mathbf{b}_q$(spacelike). The q-frame $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$ along $\alpha(t)$ is given by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q. \tag{11}$$

For simplicity, we have chosen the projection vector $\mathbf{k} = (0, 1, 0)$(spacelike) in this paper. However, the q-frame is singular in all cases where $\mathbf{t} \wedge \mathbf{k}$ vanishes. Therefore, in those cases where $\mathbf{t} \wedge \mathbf{k}$ vanishes the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (1, 0, 0)$(spacelike). Interestingly if we chose $\mathbf{k} = (0, 0, 1)$(timelike) we get the same results in this paper [14].

The variation equations of the directional q-frame is given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & k_3 \\ k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{12}$$

where the q-curvatures are expressed as follows

$$k_1 = \langle \mathbf{t}', \mathbf{n}_q \rangle, k_2 = \langle \mathbf{t}', \mathbf{b}_q \rangle, k_3 = \langle \mathbf{n}_q', \mathbf{b}_q \rangle. \tag{13}$$

Theorem 1.1. Let $\alpha(s)$ be a timelike curve that is parameterized by arc length s. The relation between the q-curvatures and the Frenet curvatures (the torsion τ and the curvature κ) may be expressed as,

$$\begin{aligned} k_1 &= \kappa \cos\theta, \\ k_2 &= -\kappa \sin\theta, \\ k_3 &= d\theta + \tau, \end{aligned} \tag{14}$$

where θ is the angle between the vectors, the principal normal vector $\mathbf{n}$ and the quasi-normal vector $\mathbf{n}_q$ [13, 14].

We can define the Euclidean angle θ between the principal normal $\mathbf{n}$ and quasi-normal $\mathbf{n}_q$ spacelike vectors. Then, as one can see immediately, the relation matrix may be expressed as

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{15}$$

Thus,

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{16}$$

[11, 15]. Furthermore, from (15) and (11) we have

$$\cos\theta = \langle \mathbf{n}_q, \mathbf{n} \rangle = \frac{\langle \mathbf{t} \wedge \mathbf{k}, \alpha'' \rangle}{\|\mathbf{t} \wedge \mathbf{k}\| \, \|\alpha''\|} = \frac{\det(\alpha'', \alpha', \mathbf{k})}{\|\alpha' \wedge \mathbf{k}\| \, \|\alpha''\|}. \tag{17}$$

In [10], the authors investigated the Smarandache curves with respect to the Bishop frame. Smarandache curves have also been studied in other ambient frames [4, 19, 25]. Taking $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q\}$ be its moving q-frame, $\mathbf{t}\mathbf{n}_q-$Smarandache curve, $\mathbf{t}\mathbf{b}_q-$Smarandache curve and $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve are defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{t} + \mathbf{n}_q), \beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{t} + \mathbf{b}_q) \text{ and } \beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{n}_q + \mathbf{b}_q) \tag{18}$$

respectively [16].

Now, in this paper we investigate special Smarandache curves according to q-frame in Minkowski 3-space and we give some differential geometric properties of Smarandache curves.

2 Smarandache curves of Timelike Space Curve according to q-frame

In this section we will investigate the Smarandache curves of timelike space curve according to q-frame in Minkowski space.

2.1 $\mathbf{t}\mathbf{n}_q$- *Smarandache curves of a timelike curve in* $\mathbb{R}^3_1$

Let $\alpha = \alpha(s)$ be a unit speed regular curve in $\mathbb{R}^3_1$ and $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q\}$ be its moving q-frame. $\mathbf{t}\mathbf{n}_q-$Smarandache curve can be defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{t} + \mathbf{n}_q) \tag{19}$$

We can get q-invariants of $\mathbf{t}\mathbf{n}_q-$Smarandache curves according to $\alpha = \alpha(s)$. From (19) we obtain

$$\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(\mathbf{t}' + \mathbf{n}_q') \tag{20}$$

and

$$\mathbf{t}_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(k_1\mathbf{t} + k_1\mathbf{n}_q + (k_2 + k_3)\mathbf{b}_q) \tag{21}$$

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where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}} |k_2 + k_3| \tag{22}$$

We can write tangent vector of curve β as follow

$$\mathbf{t}_\beta = \frac{1}{|k_2 + k_3|}(k_1\mathbf{t} + k_1\mathbf{n}_q + (k_2 + k_3)\mathbf{b}_q) \tag{23}$$

Let's assume that $k_2 \neq -k_3$. The causal character of $\mathbf{tn}_q-$Smarandache curve is obtained as

$$\left\langle \mathbf{t}_\beta, \mathbf{t}_\beta \right\rangle = \frac{1}{|k_2 + k_3|^2}(k_1^2 \left\langle\, \mathbf{t}, \mathbf{t}\right\rangle + k_1^2 \left\langle \mathbf{n}_q, \mathbf{n}_q \right\rangle + (k_2 + k_3)^2 \left\langle \mathbf{b}_q, \mathbf{b}_q \right\rangle) = 1$$

this shows that $\mathbf{tn}_q-$Smarandache curve is a spacelike curve. Then differentiating (23) with respect to s, we obtain

$$\frac{d\mathbf{t}_\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{|k_2 + k_3|^2}(\zeta_1\mathbf{t} + \ \zeta_2\mathbf{n}_q + \zeta_3\mathbf{b}_q) \tag{24}$$

where

$$\begin{aligned}
\zeta_1 &= -\left(\frac{|k_2 + k_3|}{k_1}\right)' k_1^2 + |k_2 + k_3| \left(k_1^2 + k_2^2 + k_2k_3\right) \\
\zeta_2 &= -\left(\frac{|k_2 + k_3|}{k_1}\right)' k_1^2 + |k_2 + k_3| \left(k_1^2 + k_3^2 - k_2k_3\right) \\
\zeta_3 &= k_1 \left|k_2 + k_3\right| \left(k_2 + k_{3.}\right)
\end{aligned}$$

Substituting (22) into (24), we get

$$\mathbf{t}'_\beta = \frac{\sqrt{2}}{|k_2 + k_3|^3}(\zeta_1\mathbf{t} + \zeta_2\mathbf{n}_q + \zeta_3\mathbf{b}_q) \tag{25}$$

Then, the first curvature and the principal normal vector field of curve β are calculated

$$\kappa_\beta = \sqrt{\left\langle \mathbf{t}'_\beta, \mathbf{t}'_\beta \right\rangle} = \frac{\sqrt{2}}{|k_2 + k_3|^3}\sqrt{\left|-\zeta_1^2 + \zeta_2^2 + \zeta_3^2\right|} \tag{26}$$

and

$$\mathbf{n}_\beta = \frac{1}{\sqrt{\xi}}\left(\zeta_1\mathbf{t} + \zeta_2\mathbf{n}_q + \zeta_3\mathbf{b}_q\right) \tag{27}$$

where $\xi \neq 0$ and $\xi = \left|-\zeta_1^2 + \zeta_2^2 + \zeta_3^2\right|$. Then, the binormal vector of curve β is

$$\mathbf{b}_\beta = \frac{1}{|k_2 + k_3|\sqrt{\xi}}\left(\varsigma_1\mathbf{t} + \varsigma_2\mathbf{n}_q + \varsigma_3\mathbf{b}_q\right) \tag{28}$$

where

$$\varsigma_1 = k_1\zeta_3 - |k_2 + k_3|\,\zeta_2, \quad \varsigma_2 = k_1\zeta_3 - |k_2 + k_3|\,\zeta_1, \quad \varsigma_3 = -k_1\ \zeta_2 + k_1\zeta_1.$$

In order to calculate the torsion, differentiating (20) with respect to s gives

$$\beta'' = \frac{1}{\sqrt{2}}[(k_1' + k_1^2 + k_2(k_2 + k_3))\mathbf{t} + (k_1' + k_1^2 - k_3(k_2 + k_3))\mathbf{n}_q + (k_2' + k_3' + k_1(k_2 + k_3))\mathbf{b}_q] \tag{29}$$

and

$$\beta''' = \frac{1}{\sqrt{2}}\left(\nu_1\mathbf{t} + \nu_2\mathbf{n}_q + \nu_3\mathbf{b}_q\right) \tag{30}$$

where

$$\begin{aligned}
\nu_1 &= k_1'' + 3k_1'k_1 + (k_2(k_2 + k_3))' + k_2(k_2 + k_3)' + k_1(\kappa - k_3^2) \\
\nu_2 &= k_1'' - 3k_1k_1' + k_3'(k_2 + k_3) - 2k_3(k_2 + k_3)' - k_1(\kappa - k_3^2) \\
\nu_3 &= k_2'' + k_3'' + (k_1(k_2 + k_3))' + (k_2 + k_3)(\kappa - k_3^2)
\end{aligned}$$

The torsion of the curve β is

$$\tau_\beta = \left[\frac{\sqrt{2}(\psi(\nu_1 - \nu_2) - (k_2 + k_3)^2(\nu_1k_3 + \nu_2k_2 - \nu_3k_1))}{(\psi - k_3(k_2 + k_3)^2)^2 + (\psi + k_2(k_2 + k_3)^2)^2 - (k_1(k_2 + k_3)^2)^2}\right] \tag{31}$$

where $\psi = -\left(\frac{|k_2+k_3|}{k_1}\right)' k_1^2$. Using (15) and (16) to compute the quasi-normal and quasi-binormal vectors of the curve β, we obtain

$$\mathbf{n}_q^\beta = \frac{1}{|k_2+k_3|\sqrt{\xi}} \begin{bmatrix} (|k_2+k_3|\,\zeta_1\cos\theta_\beta + \varsigma_1\sin\theta_\beta)\mathbf{t} \\ +(|k_2+k_3|\,\zeta_2\cos\theta_\beta + \varsigma_2\sin\theta_\beta)\mathbf{n}_q \\ +(|k_2+k_3|\,\zeta_3\cos\theta_\beta + \varsigma_3\sin\theta_\beta)\mathbf{b}_q \end{bmatrix} \tag{32}$$

and

$$\mathbf{b}_q^\beta = \frac{-1}{|k_2+k_3|\sqrt{\xi}} \begin{bmatrix} (|k_2+k_3|\,\zeta_1\sin\theta_\beta - \varsigma_1\cos\theta_\beta)\mathbf{t} \\ +(|k_2+k_3|\,\zeta_2\sin\theta_\beta - \varsigma_2\cos\theta_\beta)\mathbf{n}_q \\ +(|k_2+k_3|\,\zeta_3\sin\theta_\beta - \varsigma_3\cos\theta_\beta)\mathbf{b}_q \end{bmatrix} \tag{33}$$

Using (13) to calculate q-curvatures of the curve β, we get

$$k_1^\beta = \frac{\sqrt{2}}{|k_2+k_3|\sqrt{\xi}}\left(|k_2+k_3|\,\xi\cos\theta_\beta + \sin\theta_\beta(-\varsigma_1\zeta_1 + \varsigma_2\,\zeta_2 + \varsigma_3\zeta_3)\right) \tag{34}$$

$$k_2^\beta = \frac{-\sqrt{2}}{|k_2+k_3|\sqrt{\xi}}\left(|k_2+k_3|\,\xi\sin\theta_\beta - \cos\theta_\beta(-\varsigma_1\zeta_1 + \varsigma_2\zeta_2 + \varsigma_3\zeta_3)\right) \tag{35}$$

and

$$k_3^\beta = \frac{1}{|k_2+k_3|\sqrt{\xi}} \begin{bmatrix} \cos\theta_\beta\sin\theta_\beta\begin{pmatrix} \sqrt{|k_2+k_3|}(\zeta_1'\varsigma_1 + \zeta_2'\varsigma_2 + \zeta_3'\varsigma_3) \\ +\theta_\beta'(\varsigma_1^2 + \varsigma_2^2 + \varsigma_3^2 + |k_2+k_3|\,\xi) \end{pmatrix} \\ +\sqrt{|k_2+k_3|}\theta_\beta'(\zeta_1\varsigma_1 + \zeta_2\varsigma_2 + \ \zeta_3\varsigma_3) \\ +|k_2+k_3|\cos^2\theta_\beta(\zeta_1\zeta_1' + \zeta_2\zeta_2' + \zeta_3\zeta_3') \end{bmatrix}. \tag{36}$$

EXAMPLE: In this example, we derived the Smarandache curve of a timelike curve parametrized by

$$\alpha(s) = (2\cosh s, \sqrt{3}s, 2\sinh s)$$

for $\mathbf{k} = (0,1,0)$(spacelike), the q-frame of the curve is obtained by

$$\mathbf{t} = (2\sinh s, \sqrt{3}, 2\cosh s) \tag{37}$$

$$\mathbf{n}_q = (\cosh s, 0, \sinh s) \tag{38}$$

and

$$\mathbf{b}_q = (\sqrt{3}\sinh s, 2, \sqrt{3}\cosh s). \tag{39}$$

Thus, $\mathbf{tn}_q-$Smarandache curve is obtained by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(2\sinh s + \cosh s, \sqrt{3}, 2\cosh s + \sinh s) \tag{40}$$

and its Frenet curvatures as follows

$$\kappa_\beta = \frac{\sqrt{6}}{3} \text{ and } \tau_\beta = 0$$

The q-frame and q-curvatures of the $\mathbf{tn}_q-$Smarandache curve are calculated by

$$\begin{aligned} \mathbf{n}_q^\beta &= \frac{-1}{3}\left[\sqrt{2}\cos\theta_\beta(2\sinh s + \cosh s), \sin\theta_\beta, \sqrt{2}\cos\theta_\beta(2\cosh s + \sinh s)\right] \\ \mathbf{b}_q^\beta &= \frac{1}{3}\left[\sqrt{2}\sin\theta_\beta(2\sinh s + \cosh s), -\cos\theta_\beta, \sqrt{2}\sin\theta_\beta(2\cosh s + \sinh s)\right] \end{aligned}$$

and

$$\begin{aligned} k_1^\beta &= \cos\theta_\beta = -\frac{\sqrt{6}}{3} \\ k_2^\beta &= -\sin\theta_\beta = 0 \\ k_3^\beta &= \frac{5}{3}\sin^2\theta_\beta\theta_\beta' = 0 \end{aligned}$$

respectively. Finally the curve (black) and the $\mathbf{tn}_q-$Smarandache curve (red) are shown in Figure 2.

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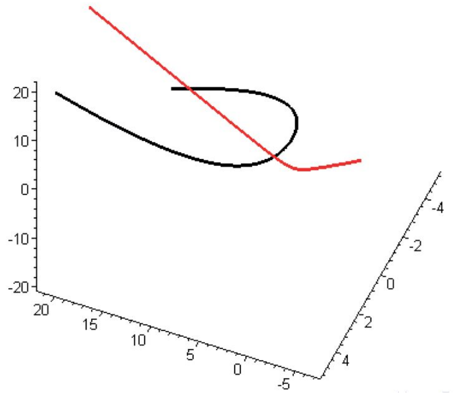

Fig. 2: The curve and $\mathbf{t}\mathbf{n}_q$-Smarandache curve.

2.2 $\mathbf{n}_q\mathbf{b}_q$- *Smarandache curves of a timelike curve in* $\mathbb{R}_1^3$

Let $\alpha = \alpha(s)$ be a unit speed regular curve in $\mathbb{R}_1^3$ and $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q\}$ be its moving q-frame. $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve can be defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{n}_q + \mathbf{b}_q) \tag{41}$$

We can get q-invariants of $\mathbf{n}_q\mathbf{b}_q-$Smarandache curves according to $\alpha = \alpha(s)$. From (41) we have

$$\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(\mathbf{n}_q' + \mathbf{b}_q') \tag{42}$$

It follows that

$$\mathbf{t}_\beta \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}((k_1 + k_2)\mathbf{t} - k_3\mathbf{n}_q + k_3\mathbf{b}_q) \tag{43}$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\sqrt{\left|2k_3^2 - (k_1 + k_2)^2\right|} \tag{44}$$

We can write tangent vector of curve β as follow

$$\mathbf{t}_\beta = \frac{1}{\sqrt{\left|2k_3^2 - (k_1 + k_2)^2\right|}}((k_1 + k_2)\mathbf{t} - k_3\mathbf{n}_q + k_3\mathbf{b}_q) \tag{45}$$

Let's assume that $2k_3^2 \neq (k_1 + k_2)^2$ and investigate the causal character of $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve, we get

$$\langle \mathbf{t}_\beta, \mathbf{t}_\beta \rangle = \frac{2k_3^2 - (k_1 + k_2)^2}{\left|2k_3^2 - (k_1 + k_2)^2\right|}$$

Therefore, there are two possibilities for the causal character of $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve; the β curve is spacelike if

$$2k_3^2 > (k_1 + k_2)^2 \Rightarrow \langle \mathbf{t}_\beta, \mathbf{t}_\beta \rangle > 0$$

and the β curve is timelike if

$$2k_3^2 < (k_1 + k_2)^2 \Rightarrow \langle \mathbf{t}_\beta, \mathbf{t}_\beta \rangle < 0$$

Let's assume that $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve is a spacelike curve. Then differentiating (45) with respect to s, we obtain

$$\frac{d\mathbf{t}_\beta}{ds^*}\frac{ds^*}{ds} = \frac{1}{|\upsilon|^{\frac{3}{2}}}\left(\widetilde{\zeta}_1\mathbf{t} + \widetilde{\zeta}_2\mathbf{n}_q + \widetilde{\zeta}_3\mathbf{b}_q\right) \tag{46}$$

where $\upsilon = 2k_3^2 - (k_1 + k_2)^2$ and

$$\begin{aligned}
\widetilde{\zeta}_1 &= 2k_3^3\left(\frac{k_1 + k_2}{k_3}\right)' - \upsilon k_3(k_1 - k_2) \\
\widetilde{\zeta}_2 &= -(k_1 + k_2)k_3^2\left(\frac{k_1 + k_2}{k_3}\right)' + \upsilon(k_1^2 - k_3^2 + k_1k_2) \\
\widetilde{\zeta}_3 &= (k_1 + k_2)k_3^2\left(\frac{k_1 + k_2}{k_3}\right)' + \upsilon(k_2^2 - k_3^2 + k_1k_2)
\end{aligned}$$

Substituting (44) into (46), we get

$$\mathbf{t}_\beta' = \frac{\sqrt{2}}{\upsilon^2}\left(\widetilde{\zeta}_1\mathbf{t} + \widetilde{\zeta}_2\mathbf{n}_q + \widetilde{\zeta}_3\mathbf{b}_q\right) \tag{47}$$

Then, the first curvature and the principal normal vector field of curve β are respectively

$$\kappa_\beta = \left\|\mathbf{t}_\beta'\right\| = \sqrt{\left\langle \mathbf{t}_\beta', \mathbf{t}_\beta' \right\rangle} = \frac{1}{\upsilon^2}\sqrt{2\left|-\widetilde{\zeta}_1^2 + \widetilde{\zeta}_2^2 + \widetilde{\zeta}_3^2\right|} \tag{48}$$

and

$$\mathbf{n}_\beta = \frac{1}{\sqrt{\widetilde{\xi}}}\left(\widetilde{\zeta}_1\mathbf{t} + \widetilde{\zeta}_2\mathbf{n}_q + \widetilde{\zeta}_3\mathbf{b}_q\right) \tag{49}$$

where $\xi \neq 0$ and $\xi = \left| -\widetilde{\zeta}_1^2 + \widetilde{\zeta}_2^2 + \widetilde{\zeta}_3^2 \right|$. Then, the binormal vector of curve β is

$$\mathbf{b}_\beta = \frac{1}{\sqrt{\widetilde{\xi}}\,|\upsilon|}\left(\widetilde{\varsigma}_1\mathbf{t} + \widetilde{\varsigma}_2\mathbf{n}_q + \widetilde{\varsigma}_3\mathbf{b}_q\right) \tag{50}$$

where

$$\widetilde{\varsigma}_1 = -k_3\widetilde{\zeta}_3 - k_3\widetilde{\zeta}_2, \quad \widetilde{\varsigma}_2 = (k_1+k_2)\widetilde{\zeta}_3 - k_3\widetilde{\zeta}_1, \quad \widetilde{\varsigma}_3 = -(k_1+k_2)\widetilde{\zeta}_2 - k_3\widetilde{\zeta}_1.$$

In order to calculate the torsion, differentiating (20) with respect to s gives

$$\beta'' = \frac{1}{\sqrt{2}}[(k_1' + k_2' + k_3(k_2 - k_1))\mathbf{t} + (-k_3' - k_3^2 + k_1(k_1+k_2))\mathbf{n}_q + (k_3' - k_3^2 + k_2(k_1+k_2))\mathbf{b}_q] \tag{51}$$

and

$$\beta''' = \frac{1}{\sqrt{2}}\begin{bmatrix} (k_1'' + k_2'' + (k_3(k_2-k_1))' + k_3'(k_2-k_1) + (k_1+k_2)(\kappa - k_3^2))\mathbf{t} \\ +(-k_3'' - 3k_3k_3' + (k_1(k_1+k_2))' + k_1(k_1+k_2)' - k_3(\kappa - k_3^2))\mathbf{n}_q \\ +(k_3'' - 3k_3k_3' + (k_2(k_1+k_2))' + k_2(k_1+k_2)' + k_3(\kappa - k_3^2))\mathbf{b}_q \end{bmatrix} \tag{52}$$

where

$$\begin{aligned}
\widetilde{\nu}_1 &= k_1'' + k_2'' + (k_3(k_2-k_1))' + k_3'(k_2-k_1) + (k_1+k_2)(\kappa - k_3^2) \\
\widetilde{\nu}_2 &= -k_3'' - 3k_3k_3' + (k_1(k_1+k_2))' + k_1(k_1+k_2)' - k_3(\kappa - k_3^2) \\
\widetilde{\nu}_3 &= k_3'' - 3k_3k_3' + (k_2(k_1+k_2))' + k_2(k_1+k_2)' + k_3(\kappa - k_3^2)
\end{aligned}$$

The torsion of the curve β is

$$\tau_\beta = \left[\frac{-\sqrt{2}(\delta(\widetilde{\nu}_2 + \widetilde{\nu}_3) + \upsilon(k_3\widetilde{\nu}_1 + k_2\widetilde{\nu}_2 + k_1\widetilde{\nu}_3))}{(k_3\upsilon)^2 + (\delta + k_2\upsilon)^2 + (\delta + k_1\upsilon)^2}\right] \tag{53}$$

where $\delta = \left(\frac{k_1+k_2}{k_3}\right)' k_3^2$. The quasi-normal and quasi-binormal vectors of curve β are as follow.

$$\mathbf{n}_q^\beta = \frac{1}{\sqrt{\widetilde{\xi}}\,|\upsilon|}\begin{bmatrix} (|\upsilon|\,\widetilde{\zeta}_1\cos\theta_\beta + \widetilde{\varsigma}_1\sin\theta_\beta)\mathbf{t} \\ +(|\upsilon|\,\widetilde{\zeta}_2\cos\theta_\beta + \widetilde{\varsigma}_2\sin\theta_\beta)\mathbf{n}_q \\ +(|\upsilon|\,\widetilde{\zeta}_3\cos\theta_\beta + \widetilde{\varsigma}_3\sin\theta_\beta)\mathbf{b}_q \end{bmatrix} \tag{54}$$

and

$$\mathbf{b}_q^\beta = \frac{-1}{\sqrt{\widetilde{\xi}}\,|\upsilon|}\begin{bmatrix} (|\upsilon|\,\widetilde{\zeta}_1\sin\theta_\beta - \widetilde{\varsigma}_1\cos\theta_\beta)\mathbf{t} \\ +(|\upsilon|_2\sin\theta_\beta - \widetilde{\varsigma}_2\cos\theta_\beta)\mathbf{n}_q \\ +(|\upsilon|\,\widetilde{\zeta}_3\sin\theta_\beta - \widetilde{\varsigma}_3\cos\theta_\beta)\mathbf{b}_q \end{bmatrix} \tag{55}$$

We can calculate q-curvatures of curve β, so from (13) we get

$$k_1^\beta = \frac{\sqrt{2}}{\sqrt{\widetilde{\xi}}\,|\upsilon|}\left(|\upsilon|\,\widetilde{\xi}\cos\theta_\beta + \sin\theta_\beta(-\widetilde{\varsigma}_1\widetilde{\zeta}_1 + \widetilde{\varsigma}_2\widetilde{\zeta}_2 + \widetilde{\varsigma}_3\widetilde{\zeta}_3)\right) \tag{56}$$

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and

$$k_2^\beta = \frac{-\sqrt{2}}{\sqrt{\widetilde{\xi}}\,|\upsilon|}\left(|\upsilon|\,\widetilde{\xi}\sin\theta_\beta - \cos\theta_\beta(-\widetilde{\varsigma}_1\widetilde{\zeta}_1 + \widetilde{\varsigma}_2\widetilde{\zeta}_2 + \widetilde{\varsigma}_3\widetilde{\zeta}_3)\right) \tag{57}$$

and

$$k_3^\beta = \frac{1}{|\upsilon|\,\sqrt{\widetilde{\xi}}}\begin{bmatrix} \cos\theta_\beta\sin\theta_\beta\begin{pmatrix} \sqrt{|\upsilon|}(\widetilde{\zeta}_1'\widetilde{\varsigma}_1 + \widetilde{\zeta}_2'\widetilde{\varsigma}_2 + \widetilde{\zeta}_3'\widetilde{\varsigma}_3) \\ +\theta_\beta'(\widetilde{\varsigma}_1^2 + \widetilde{\varsigma}_2^2 + \widetilde{\varsigma}_3^2 + |\upsilon|\,\widetilde{\xi}) \end{pmatrix} \\ +\sqrt{|\upsilon|}\theta_\beta'(\widetilde{\zeta}_1\widetilde{\varsigma}_1 + \widetilde{\zeta}_2\widetilde{\varsigma}_2 + \widetilde{\zeta}_3\widetilde{\varsigma}_3) \\ +|\upsilon|\cos^2\theta_\beta(\widetilde{\zeta}_1\widetilde{\zeta}_1' + \widetilde{\zeta}_2\widetilde{\zeta}_2' + \widetilde{\zeta}_3\widetilde{\zeta}_3') \end{bmatrix}. \tag{58}$$

EXAMPLE: In this example, we derived the Smarandache curve of a timelike curve parametrized by

$$\alpha(s) = (\cosh s, 1, \sinh s)$$

for $\mathbf{k} = (0, 1, 0)$(spacelike), the q-frame of the curve is obtained by

$$\mathbf{t} = (\sinh s, 0, \cosh s) \tag{59}$$

$$\mathbf{n}_q = (\cosh s, 0, \sinh s) \tag{60}$$

and

$$\mathbf{b}_q = (0, 1, 0). \tag{61}$$

Thus, $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve is obtained by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\cosh s, 1, \sinh s) \tag{62}$$

and its Frenet curvatures as follows

$$\kappa_\beta = \sqrt{2} \text{ and } \tau_\beta = 0$$

The q-frame and q-curvatures of the $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve is obtained by

$$\begin{aligned} \mathbf{n}_q^\beta &= (\cosh s, 0, \sinh s) \\ \mathbf{b}_q^\beta &= (0, 1, 0) \end{aligned}$$

$$\begin{aligned} k_1^\beta &= \cos\theta_\beta = \sqrt{2} \\ k_2^\beta &= -\sin\theta_\beta = 0 \\ k_3^\beta &= \theta_\beta' = 0 \end{aligned}$$

respectively. Finally the curve (blue) and the $\mathbf{n}_q\mathbf{b}_q-$Smarandache curve (red) are shown in Figure 3.

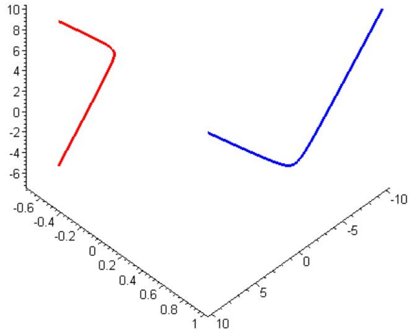

Fig. 3: The curve and $\mathbf{n}_q\mathbf{b}_q$- Smarandache curve.

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