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Special Helices on the Ellipsoid

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Abstract: In this study, we investigate three types of special helices whose axis is a fixed constant Killing vector field on the Ellipsoid $\mathbb{S}^2_{a_1,a_2,a_3}$ in $\mathbb{R}^3_{a_1,a_2,a_3}$. Then, we obtain the curvatures of all special helices on the ellipsoid $\mathbb{S}^2_{a_1,a_2,a_3}$ and give some characterizations of these curves. Moreover, we present various examples and visualize their images using the Mathematica program.

Keywords: Frame fields, Killing vector field, Special curves and surfaces.

1 Introduction

The spherical curves are the special space curves that lie on the sphere. If the sphere is constructed by using the elliptical inner product, then the elliptical 2-sphere is obtained. This sphere is an ellipsoid according to the Euclidean sense. We summarize some studies about spherical curves: Firstly, Wong proved the condition for a curve to be on a sphere and gave some characterizations for this curve [10, 11]. In [3], Breuer et al. gave an explicit characterization of the spherical curve. In [6], the author investigated the characterization of the dual spherical curve. Then, in [2], the author obtained a differential equation for characterizing of the dual spherical curves. Besides, in [4], İlarslan presented the spherical curve characterization for non-null regular curves in Lorentzian 3-space. Ayyıldız introduced the dual Lorentzian spherical curves [1]. Moreover, Izumiya and Takeuchi defined the slant helices and conical geodesic curve and gave a classification of special developable surfaces under the condition of the existence of such a special helix as a geodesic [5]. Scofield derived a curve of constant precession and proved that this curve is tangent indicatrix of a spherical helix [9].

In the present work, we give some characterizations for the special helices whose axis is the fixed constant Killing vector field on the elliptical 2-sphere. Furthermore, we give various examples and draw their images by using the Mathematica program.

2 Preliminaries

Let we take $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $a_1, a_2, a_3 \in \mathbb{R}^+$ then the elliptical inner product defined as

$$B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R};\ B(u, v) = a_1x_1y_1 + a_2x_2y_2 + a_3x_3y_3. \tag{1}$$

The 3-dimensional real vector space $\mathbb{R}^3$ equipped with the elliptical inner product will be represented by $\mathbb{R}^n_{a_1,a_2,a_3}$. The norm of a vector associated with the scalar product B is defined as

$$\|u\|_B = \sqrt{B(u, u)}. \tag{2}$$

Two vectors u and v are called elliptically orthogonal vectors if $B(u, v) = 0$. In addition, if u is an elliptically orthonormal vector then $B(u, u) = 1$. The cosine of the angle between two vectors u and v is defined as

$$\cos\theta = \frac{B(u, v)}{\|u\|_B\,\|v\|_B}, \tag{3}$$

where θ is compatible with the parameters of the angular parametric equations of ellipse or ellipsoid. The cross product of two vector fields $X, Y \in \mathbb{R}^3_{a_1,a_2,a_3}$ is given by

$$X \times_E Y = \Delta \begin{vmatrix} \frac{e_1}{a_1} & \frac{e_2}{a_2} & \frac{e_3}{a_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}, \tag{4}$$

where $\Delta = \sqrt{a_1a_2a_3}$, $a_1, a_2, a_3 \in \mathbb{R}^+$ [7].

Let us take the ellipsoid denoted by $\mathbb{S}^2_{a_1,a_2,a_3}$ in $\mathbb{R}^3_{a_1,a_2,a_3}$. Then, the sectional curvature of the ellipsoid generated by the non-degenerated plane $\{u, v\}$ is defined as

$$K(u,v) = \frac{B(R(u,v)u,v)}{B(u,u)B(v,v) - B(u,v)^2}, \tag{5}$$

where R is the Riemannian curvature tensor given by

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z. \tag{6}$$

The ellipsoid has the constant sectional curvature. Therefore, the curvature tensor R is written as follows

$$R(X,Y)Z = C\{B(Z,X)Y - B(Z,Y)X\}, \tag{7}$$

where C is the constant sectional curvature.

A curve γ on the ellipsoid $\mathbb{S}^2_{a_1,a_2,a_3}$ defined by $\gamma(s) = \varphi(\alpha(s))$ and a unit normal vector field Z along the surface $\mathbb{S}^2_{a_1,a_2,a_3}$ defined

$$Z = \frac{\varphi_u \times_E \varphi_v}{\|\varphi_u \times \varphi_v\|}. \tag{8}$$

Since $\mathbb{S}^2_{a_1,a_2,a_3}$ is sphere according to the elliptical inner product, the unit normal vector field Z along the surface $\mathbb{S}^2_{a_1,a_2,a_3}$ equal to the position vector of the curve γ. Then, we found an orthonormal frame $\{t = \gamma', y = \gamma \times_E \gamma', \gamma\}$ which is called the elliptical Darboux frame along the curve γ. The corresponding Darboux formulae of γ is written as

$$\begin{aligned} t' &= -\gamma + k_{g_E} y, \\ \gamma' &= t, \\ y' &= -k_{g_E} t, \end{aligned} \tag{9}$$

where $k_{n_E} = -1$, $k_{g_E} = B(\gamma'', y)$ and $\tau_r = 0$ are *geodesic curvature*, *asymptotic curvature*, and *principal curvature* of γ on the surface $\mathbb{S}^2_{a_1,a_2,a_3}$, respectively. Moreover, it is found as the following relation

$$y \times_E t = \gamma,\ z \times_E y = t,\ z \times_E t = -y, \tag{10}$$

[8].

Lemma 1. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathbb{S}^2_{a_1,a_2,a_3}$. Provided that V be a vector field along the curve γ then the variation of γ defined by $\Gamma : I \times (-\varepsilon, \varepsilon) \to \mathbb{S}^2_{a_1,a_2,a_3}(C)$ with $\gamma(s,0)$ the initial curve satisfy $\Gamma(s,0) = \gamma(s)$. The variations of the geodesic curvature function $k_{g_E}(s,w)$ and the speed function $v(s,w)$ at $w = 0$ are calculated as follows:*

$$\begin{aligned} V(v) &= \left(\frac{\partial v}{\partial w}(s,w)\right)\Big|_{w=0} = -v\rho, \\ V(k_{g_E}) &= \left(\frac{\partial k_{g_E}}{\partial w}(s,w)\right)\Big|_{w=0} = B(-R(V,t)t + \nabla_t^2 V, y) - \frac{1}{k_{g_E}} B(-R(V,t)t + \nabla_t^2 V, \gamma), \end{aligned} \tag{11}$$

where $\rho = B(\nabla_t V, t)$ and R stands for the curvature tensor of $\mathbb{S}^2_{a_1,a_2,a_3}$ [8].

Proposition 1. *If $V(s)$ is the restriction to $\gamma(s)$ of a Killing vector field V of $\mathbb{S}^2_{a_1,a_2,a_3}$ then the variations of the elliptical Darboux curvature functions and speed function of γ satisfy:*

$$V(v) = V(k_{g_E}) = 0, \tag{12}$$

[8].

3 Special helices on the ellipsoid $\mathbb{S}^2_{a_1,a_2,a_3}$

Definition 1. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathbb{S}^2_{a_1,a_2,a_3}$. Then we say that γ is a type-1 special helix, type-2 special helix, or type-3 special helix if $B(V,t) = const.$, $B(V,\gamma) = const.$, and $B(V,y) = const.$, respectively.*

Theorem 1. *Let $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}^3$, $\varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on $\mathbb{S}^2_{a_1,a_2,a_3}$ and V be a Killing vector field along the curve γ. Then γ is a type-1 special helix with the axis V if and only if the geodesic curvature of the curve γ satisfy the following equation:*

$$k_{g_E} = \cot\theta,$$

where θ satisfy

$$\theta'' \sin^2\theta - \omega\theta' \cos\theta = 0,$$

[8].

Now, we can give the following corollary without proof. The proof of the corollary similar to Scofield's work [9].

Corollary 1. *Let $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}^3$, $\varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \to U$ be a type-1 special helix with the Killing axis V on $\mathbb{S}^2_{a_1,a_2,a_3}$. Then, the integral curve of γ is an elliptical constant procession curve on the elliptical hyperboloid.*

Theorem 2. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathbb{S}^2_{a_1,a_2,a_3}$. Then γ is a type-2 special helix with the axis V if and only if the geodesic curvature of the curve γ satisfy the following equation:*

$$k_{g_E} = \frac{c_1}{\sin\theta} - \theta', \tag{13}$$

here θ satisfies

$$\theta = const.\ or\ (C+1)\sin^4\theta - c_1\theta'\sin\theta + c_1 = 0,$$

where c is a constant.

Proof: If γ is a type-2 special helix with the Killing axis V then V is written as

$$V = \cos\theta t + c_1\gamma + \sin\theta y, c_1 = const. \tag{14}$$

Differentiating eq.(14) with respect to s, we found the following equation

$$\begin{aligned} \nabla_T V &= ((-\theta' - k_{g_E})\sin\theta + c_1)t + (-\cos\theta)\gamma \\ &\quad + (\cos\theta k_{g_E} + \theta'\cos\theta)y. \end{aligned} \tag{15}$$

Using the equation $V(v) = 0$ in Lemma 1, we found

$$k_{g_E} = \frac{c_1}{\sin\theta} - \theta' \tag{16}$$

The differentiation of eq.(15) is obtained as

$$\nabla^2_T V = (-1 - k^2_{g_E} + k_{g_E}\theta')\cos\theta t + \theta'\sin\theta\gamma + ((k_{g_E} + \theta')\cos\theta)'y. \tag{17}$$

Moreover, we have the following equation

$$R(V,t)t = C(B(t,V)t - B(t,t)V). \tag{18}$$

Using the Darboux frame equations and eq.(14), we deduce

$$R(V,t)t = -C(c_1\gamma + \sin\theta y). \tag{19}$$

Considering the eq.(17) and eq.(19) with the second equation in Lemma 1 and the Proposition 1, we reach the following equations

$$\theta = const.\ \text{or}\ (C+1)\sin^4\theta - c_1\theta'\sin\theta + c_1 = 0.$$

□

Corollary 2. *Let γ be a type-2 special helix on the ellipsoid with the axis*

$$V = \cos\theta t + c_1\gamma + \sin\theta y, \quad \theta = const.,$$

then γ has the following parametric representation

$$\gamma(s) = A_1 + \frac{A_2}{\sqrt{1+\frac{c_1^2}{\sin^2\theta}}}\cos(\sqrt{1+\frac{c_1^2}{\sin^2\theta}}) + \frac{A_3}{\sqrt{1+\frac{c_1^2}{\sin^2\theta}}}\sin(\sqrt{1+\frac{c_1^2}{\sin^2\theta}}s),$$

where $A_1, A_2, A_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_1 \in \mathbb{R}$.

Proof: Let γ be a type-2 special helix on the ellipsoid with the axis

$$V = \cos\theta t + c_1\gamma + \sin\theta y, \quad \theta = const., \tag{20}$$

then the elliptical curvature of γ calculated as

$$k_{g_E} = \frac{c_1}{\sin\theta}. \tag{21}$$

On the other hand, from the Darboux frame equations γ satisfy the following third order differential equation

$$k_{g_E}\gamma''' - k'_{g_E}\gamma'' + (k^3_{g_E} + k_{g_E})\gamma' - k'_{g_E}\gamma = 0. \tag{22}$$

If k_{g_E} is written in the eq.(22) and the differential equation is solved then it is obtained that γ has the following parametric representation

$$\gamma(s) = A_1 + \frac{A_2}{\sqrt{1+\frac{c_1^2}{\sin^2\theta}}}\cos(\sqrt{1+\frac{c_1^2}{\sin^2\theta}}) + \frac{A_3}{\sqrt{1+\frac{c_1^2}{\sin^2\theta}}}\sin(\sqrt{1+\frac{c_1^2}{\sin^2\theta}}s), \tag{23}$$

where $A_1, A_2, A_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_1 \in \mathbb{R}$. □

Theorem 3. *Let $\varphi : U \subset \mathbb{R}^2 \to \mathbb{R}^3_{a_1,a_2,a_3}$, $\varphi(U) = \mathbb{S}^2_{a_1,a_2,a_3}$ be an ellipsoid and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on the $\mathbb{S}^2_{a_1,a_2,a_3}$. Then γ is type-3 special helix with the axis V if and only if the geodesic curvature of the curve γ satisfy the following equation:*

$$k_{g_E} = \frac{(1-\theta')\sin\theta}{c_2}, \tag{24}$$

here θ satisfies

$$(1-\theta')\sin\theta(-c_2^2\theta' - \theta''\sin\theta\cos\theta + (1-\theta')\theta'\cos 2\theta) - \theta'\sin\theta - c_2^2\theta''\cos\theta = 0,$$

where c_2 is a constant.

Proof: If γ is a type-3 special helix with the Killing axis V then V is written as

$$V = \cos\theta t + \sin\theta\gamma + c_2 y. \tag{25}$$

By differentiating eq.(25), we get

$$\nabla_T V = ((1-\theta')\sin\theta - c_2 k_{g_E})t + (1-\theta')\cos\theta\gamma + k_{g_E}\cos\theta y. \tag{26}$$

By using the equation $V(v) = 0$ in Lemma 1, we reach

$$k_{g_E} = \frac{(1-\theta')\sin\theta}{c_2}. \tag{27}$$

If we take the differentiation of eq.(26), we obtain

$$\begin{aligned}\nabla_T^2 V &= ((1-\theta')\cos\theta - k_{g_E}^2\cos\theta)t + (-\theta''\cos - (1-\theta')\theta'\sin\theta)\gamma \\ &+ (k_{g_E}^2\cos\theta - k_{g_E}\theta'\sin\theta)y.\end{aligned} \tag{28}$$

Furthermore, we have the following equation

$$R(V,t)t = C(B(t,V)t - B(t,t)V). \tag{29}$$

By using the Darboux frame equations and eq.(25), we obtain

$$R(V,T)T = C(-\sin\theta\gamma - c_2 y). \tag{30}$$

If we consider the eq.(28) and eq.(30) with the second equation in Lemma 1 and the Proposition 1, we deduce

$$\theta = const. \tag{31}$$

or satisfy the following equation

$$(1-\theta')\sin\theta(-c_2^2\theta' - \theta''\sin\theta\cos\theta + (1-\theta')\theta'\cos 2\theta) - \theta'\sin\theta - c_2^2\theta''\cos\theta = 0. \tag{32}$$

□

Corollary 3. *Let γ be a type-3 special helix on the ellipsoid with the axis*

$$V = \cos\theta t + \sin\theta\gamma + c_2 y, \quad \theta = const., \tag{33}$$

then γ has the following parametric representation

$$\gamma(s) = B_1 + \frac{B_2}{\sqrt{1+\frac{\sin^2\theta}{c_2^2}}}\cos(\sqrt{1+\frac{\sin^2\theta}{c_2^2}}) + \frac{B_3}{\sqrt{1+\frac{\sin^2\theta}{c_2^2}}}\sin(\sqrt{1+\frac{\sin^2\theta}{c_2^2}}s),$$

where $B_1, B_2, B_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_2 \in \mathbb{R}$.

Proof: Let γ be a type-3 special helix on the ellipsoid with the axis

$$V = \cos\theta t + \sin\theta\gamma + c_2 y, \quad \theta = const., \tag{34}$$

then the elliptical curvature of γ calculated as

$$k_{g_E} = \frac{\sin\theta}{c_2}. \tag{35}$$

On the other hand, from the Darboux frame equations, γ satisfy the following third order differential equation

$$k_{g_E}\gamma''' - k_{g_E}'\gamma'' + (k_{g_E}^3 + k_{g_E})\gamma' - k_{g_E}'\gamma = 0. \tag{36}$$

If k_{g_E} is written in the eq.(34) and the differential equation is solved then it is obtained that γ has the following parametric representation

$$\gamma(s) = B_1 + \frac{B_2}{\sqrt{1+\frac{\sin^2\theta}{c_2^2}}}\cos(\sqrt{1+\frac{\sin^2\theta}{c_2^2}}) + \frac{B_3}{\sqrt{1+\frac{\sin^2\theta}{c_2^2}}}\sin(\sqrt{1+\frac{\sin^2\theta}{c_2^2}}s), \tag{37}$$

where $B_1, B_2, B_3 \in \mathbb{R}^3_{a_1,a_2,a_3}$ and $c_2 \in \mathbb{R}$. □

In the following examples we give various special helices on the ellipsoid.

Examle 1. *Let us take the curve parameterized as*

$$\gamma(s) = \frac{1}{2}\frac{(1+k)\cos(1-k)t-(1-k)\cos(1+k)t}{2}\frac{1}{2}\frac{(1+k)\sin(1-k)t-(1-k)\sin(1+k)t}{4}\frac{\sqrt{1-k^2}\cos kt}{9}. \tag{38}$$

The elliptical curvature of the helix calculated as

$$k_{g_E}(s) = \cot(ks). \tag{39}$$

Thus, we can easily see that γ is a type-1 special helix. It is illustrated in Figure 1.

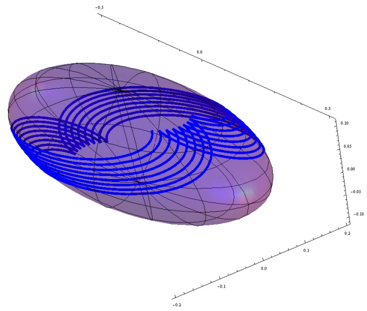

Figure1. Type-1 special Helices on the Ellipsoid $\mathbb{S}^2_{2,4,9}, k = 0.505$.

Examle 2. *Type-2 (type-3) special helices corresponding to different values of the A_i, B_i, $i = 1, 2, 3$. are illustrated in Figure 2.*

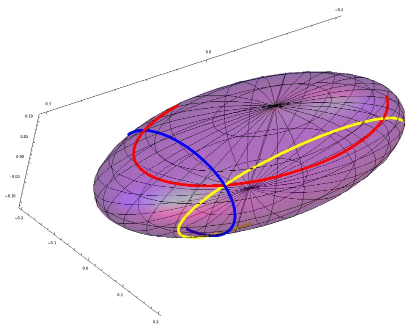

Figure2. Type-2 (type-3) Special Helices on the Ellipsoid $\mathbb{S}^2_{2,4,9}$.

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