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Spherical Curves in Finsler 3-Space

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Abstract: In this work, we investigate the general characteristics of the Finslerian spherical curves in Finsler 3-space. We obtain some characterizations for these curves. Moreover, we give various examples and visualized their images on Randers sphere.

Keywords: Finsler space, Special curves and surfaces.

1 Introduction

Finsler geometry is introduced with the doctoral thesis of Finsler. This geometry has more general metric and includes the Riemannian metric. Therefore, it has numerous applications in thermodynamics, optics, ecology, evolution, biology, geometry, physics, engineering, and computer sciences, etc. [1, 3–5, 7, 14]. Remizov [13] researched the singularities of geodesics flows in two-dimensional Finsler space. In [15, 16], Yıldırım, et al. investigated the helices in Finsler space. Ergüt, et al. gave the characterizations of AW(k)-type curves in three-dimensional Finsler manifold [9]. Furthermore, there exist various studies in literature examining methodology to use spherical curves to construct some specialized curves (see [2, 8, 10, 11, 17]). Deshmukh et al. study rectifying curves via the dilation of unit speed curves on the unit sphere $\mathbb{S}^2$ [8] in the Euclidean space $\mathbb{E}^3$. In [10], Izuyama and Takeuchi, created Bertrand curves using the spherical curve. In [11], the authors examined the spherical images of the tangent vector and binormal vector of a slant helix and gave various characterizations of these curve. The authors, in [6] gave the spherical curve characterization in the Sasakian 3-space. The spherical spirals are studied by the author, [12].

The purpose of this study is to introduce the Finslerian spherical curves and give some characterizations of these curves on the Finslerian 3 sphere.

2 Preliminaries

Let $c : I \to M;\ s \to x^i(s)$ be a smooth curve in M and $c' : I \to TM;\ s \to (x^i(s), \dot{x}^i(s))$ be the tangent bundle of the curve c in M' then Finslerian 3-manifold can be denoted by $\mathbb{F}^3 = \big(M, M', F\big)$. Also, the notion of the one dimensional Finsler submanifold of $\mathbb{F}^3$has the notion $\mathbb{F}^1 = (c, c', F_1)$. c has unit Finslerian speed on the condition that

$$F(x, \dot{x}) = 1.$$

The unit speed curve c has the following representation

$$x^i = x^i(s) :\ i \in \{1, 2, ..., m+1\}, \quad s \in (a, b) \tag{1}$$

Let (s, v) denoted by the coordinates on c' and s be an arc length parameter of the curve c. Then, we have the following equalities

$$\frac{\partial}{\partial s} = \frac{dx^i}{ds}\frac{\partial}{\partial x^i} + v\frac{d^2x^i}{ds^2}\frac{\partial}{\partial y^i} \text{ and } \frac{\partial}{\partial v} = \frac{dx^i}{ds}\frac{\partial}{\partial y^i}. \tag{2}$$

Here, $\dfrac{\partial}{\partial v}$ is a unit Finslerian vector field. These imply following equation

$$y^i(s, v) = v\frac{dx^i}{ds}, \ \ i \in \{1, 2, ..., m+1\} \tag{3}$$

where $\left\{\dfrac{\partial}{\partial s}, \dfrac{\partial}{\partial v}\right\}$ is the natural frame on c'.

A Finsler vector field X on $\mathbb{F}^{m+1}$ along c' is projectable on c. Then, it can be expressed as follows:

$$X(x(s), vx'(s)) = X^i(s)\frac{\partial}{\partial y^i}\left(x\left(s\right), vx'(s)\right) \tag{4}$$

at any point $\left(x(s), vx'(s)\right) \in c'$. From here, a vector field X^* on c denoted by the following formula

$$X^*(x(s)) = X^i(s)\frac{\partial}{\partial x^i}\left(x\left(s\right)\right). \tag{5}$$

Thus, the vector field $X^*(x(s))$ considered as the projection of the Finsler vector $X(x(s), vx'(s))$ on the tangent space TM of M at $x(s) \in c$ *(see for details* [4]).

The covariant derivatives according to Cartan connection of any projectable Finsler vector field X in the direction of $\frac{\partial}{\partial v}$ vanish identically on c'.

$$(\nabla^*_{\frac{\partial}{\partial v}}X)\left(x\left(s\right), vx'(s)\right) = 0,\ s \in (-\varepsilon, \varepsilon) \tag{6}$$

and

$$\nabla^*_{\frac{\partial}{\partial v}}\frac{\partial}{\partial v} = 0 \tag{7}$$

which enable us to express that the vertical covariant derivatives along c with respect to Cartan connection do not give any Frenet frame for c [4].

Let $c = c(s)$ be a smooth curve in $\mathbb{F}^3$ and s be an arc length parameter of the curve c. Suppose that the moving Frenet frame along the curve c denoted by $\{T := \frac{\partial}{\partial v}, N, B\}$ in the Finsler space $\mathbb{F}^3$. Then, the Frenet formulas of the curve c are given by

$$\begin{aligned} \nabla^*_{\frac{\partial}{\partial s}}T(s) &= \varkappa(s)N(s), \\ &= -\varkappa(s)T(s) + \tau(s)B(s), \\ \nabla^*_{\frac{\partial}{\partial s}}B(s) &= -\tau(s)N(s), \end{aligned} \tag{8}$$

where the vector fields $N(s)$ and $B(s)$ are Finslerian principal normal and binormal vector fields of c, respectively. The Finslerian curvature and torsion of the curve c are defined by

$$\varkappa(s) = \left\{g_{ij}(s)(c''^i(s) + 2G^i(s))(c''^j(s) + 2G^j(s))\right\}^{\frac{1}{2}}, \tag{9}$$

$$\tau(s) = -g\left(\nabla^*_{\frac{\partial}{\partial s}}N, B\right)(s) = -g_{ij}(s)B^i(s)\left\{\frac{\partial N^j}{\partial s} + N^k(s)S^j_k(s)\right\}$$

[4].

3 Spherical curves in Finslerian 3-space $\mathbb{F}^3$

In this section, we investigate the Finslerian spherical curves. The similar characterizations of those curves are found in Euclidean and Lorentzian spaces. However, since the Finsler metric is derived from the Minkowski norm, the results are quite different in practice. To explain this difference, we give various examples with the help of the Randers metric that is described in the Remark 1.

Theorem 1. *Let c be a smooth curve in* $3-$*dimensional Finsler space* $\mathbb{F}^3$ *with an arc length parameter* s. *Then the center of the osculating sphere of the curve c is written as*

$$A(s) = c(s) + A_2N + A_3B \tag{10}$$

where $A_2 = \frac{1}{\varkappa}$, $A_3 = \frac{1}{\tau(s)}\nabla^*_{\frac{\partial}{\partial s}}A_2$.

Proof: Suppose that c is a smooth curve in $3-$dimensional Finsler space $\mathbb{F}^3$ and $A(s)$ is the center of a Finslerian sphere having four common neighbor points with c. Let us define the following function

$$f : I \to R;\ s \to f(s) = g_F(A(s) - c(s), A(s) - c(s)) - r^2. \tag{11}$$

If the Finslerian sphere has four common neighbor points with c then we have

$$f(s) = \nabla^*_{\frac{\partial}{\partial s}}f(s) = \nabla^{2*}_{\frac{\partial}{\partial s}}f(s) = \nabla^{3*}_{\frac{\partial}{\partial s}}f(s) = 0. \tag{12}$$

If $f(s) = 0$ then we obtain

$$g_F(A(s) - c(s), A(s) - c(s)) = r^2. \tag{13}$$

Using the equation $\nabla^*_{\frac{\partial}{\partial s}} f(s) = 0$ we reach

$$g_F(A(s) - c(s), T(s)) = 0. \tag{14}$$

If $\nabla^{2*}_{\frac{\partial}{\partial s}} f(s) = 0$ then we get

$$g_F(A(s) - c(s), N) = \frac{1}{\varkappa}. \tag{15}$$

Then the equation $\nabla^{3*}_{\frac{\partial}{\partial s}} f(s) = 0$ gives us

$$g_F(A(s) - c(s), B) = \frac{1}{\tau(s)} \nabla^*_{\frac{\partial}{\partial s}} \frac{1}{\varkappa(s)}. \tag{16}$$

Since $A(s) - c(s) \in sp\{T, N, B\}$, we can write

$$A(s) - c(s) = A_1 T + A_2 N + A_3 B. \tag{17}$$

This gives $A_1 = g_F(A(s) - c(s), T(s)) = 0$, $A_2 = g_F(A(s) - c(s), N) = \frac{1}{\varkappa}$, and $A_3 = g_F(A(s) - c(s), B) = \frac{1}{\tau(s)} \nabla^*_{\frac{\partial}{\partial s}} \frac{1}{\varkappa(s)}$. This completes the proof. □

Corollary 1. *Let $\mathbb{FS}^2$ be a contact Finslerian sphere whose center is at the origin. Then the contact osculating Finslerian sphere of the curve γ is $\mathbb{FS}^2$, for all $s \in (-\varepsilon, \varepsilon)$.*

Proof: From the Theorem 1, we have

$$A(s) = c(s) + A_2 N + A_3 B, \tag{18}$$

here $A_2 = -g_F(c(s), N)$ and $A_3 = -g_F(c(s), B)$. On the other hand, we can write

$$c(s) = \zeta_1 T + \zeta_2 N + \zeta_3 B. \tag{19}$$

Then, we calculate that $\zeta_1 = 0$, $\zeta_2 = g_F(c(s), N)$,and $\zeta_3 = g_F(c(s), B)$. Therefore, using the eq.(18) and eq.(19), we obtain

$$A(s) = c(s) - c(s) = 0 \tag{20}$$

which completes the proof. □

Theorem 2. *Let $\mathbb{FS}^2$ be a contact Finslerian sphere whose center is denoted by $A(s)$. The radius of contact osculating sphere is constant, for all $s \in (-\varepsilon, \varepsilon)$ if and only if the centers of contact osculating spheres are the same constants.*

Proof: From the Theorem 1, we have

$$g_F(A(s) - c(s), A(s) - c(s)) = r^2, \tag{21}$$

and

$$A(s) - c(s) = A_2 N + A_3 B. \tag{22}$$

Therefore, we reach the following equation

$$A_2^2 + A_3^2 = r^2. \tag{23}$$

If we differentiate the eq.(23), we deduce

$$A_2 \nabla^*_{\frac{\partial}{\partial s}} A_2 + A_3 \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0. \tag{24}$$

Besides, we have

$$\nabla^*_{\frac{\partial}{\partial s}} A_2 = \tau(s) A_3. \tag{25}$$

If we consider with the eq.(24) and eq.(25) we compute

$$A_2 \tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0. \tag{26}$$

On the other hand, differentiating eq.(22) we obtain

$$\nabla^*_{\frac{\partial}{\partial s}} A(s) = (A_2 \tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3) B = 0. \tag{27}$$

Thus, $A(s)$ is a constant for all $s \in (-\varepsilon, \varepsilon)$.

Conversely, if $A(s)$ is a constant then we have $A_2 \tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0$. If we differentiate the eq.(23) we get

$$r \nabla^*_{\frac{\partial}{\partial s}} r = A_2 \nabla^*_{\frac{\partial}{\partial s}} A_2 + A_3 \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0. \tag{28}$$

Therefore r is a constant. □

Theorem 3. *Let c be a smooth curve in the 3−dimensional Finsler space $\mathbb{F}^3$ with an arc length parameter s. Then c is a spherical curve if and only if the following equation holds*

$$A_2\tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0 \tag{29}$$

for all $s \in I$.

Proof: From the Theorem 1 we have

$$A(s) = c(s) + A_2 N + A_3 B, \tag{30}$$

then using the Theorem 2 we get

$$\nabla^*_{\frac{\partial}{\partial s}} A(s) = (A_2\tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3)B = 0. \tag{31}$$

These imply

$$A_2\tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0. \tag{32}$$

Conversely, if we have $A_2\tau(s) + \nabla^*_{\frac{\partial}{\partial s}} A_3 = 0$ then it gives $\nabla^*_{\frac{\partial}{\partial s}} A(s) = 0$. Therefore we can say that c is a spherical curve. □

Remark 1. *The curves in the examples throughout the article are the unit speed curves according to the Randers metric. The Randers metric defines as*

$$F(x, y) := \alpha(x, y) + \beta(x, y) \tag{33}$$

here, $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ be Riemannian metric and $\beta(x, y) = b_i(x)y^i$ be a 1-form on a manifold M. The authors, in [7], obtained the coefficients of the Randers metric as follows:

$$g_{ij} = \frac{F}{\alpha}\{a_{ij} - \frac{y^i}{\alpha}\frac{y^j}{\alpha} + \frac{\alpha}{F}(b_i + \frac{y_i}{\alpha})(b_j + \frac{y_j}{\alpha})\}$$

where $y_i := a_{ij}y^j$. Since the bilinear form $\left(g_{ij}\right)$ is positive definite, then the length of β is less then 1, i.e., $\|\beta\|_\gamma := \sqrt{a^{ij}b_ib_j} < 1$ where $\left(a^{ij}\right) := \left(a_{ij}\right)^{-1}$. A Minkowski norm in the form eq.(33) is called as the Randers norm [7].

4 Examples of the Finslerian spherical curves

The Finslerian sphere with center origin O and radius r according to the Randers metric can be parameterized as follows

$$\mathbb{FS}^2(u, v) = (\frac{r\cos u \sin v - b}{1 - b^2}, \frac{r\cos u\cos v}{\sqrt{1 - b^2}}, \frac{r\sin u}{\sqrt{1 - b^2}}).$$

In this subsection we obtain various spherical curves and visulated their images on the Finslerian sphere $\mathbb{FS}^2$.

Examle 1. *We consider an arc length parameterized curve γ in Finsler space $\mathbb{F}^3$ is defined by*

$$\gamma(s) = \begin{pmatrix} \frac{R}{2}\frac{(1+k)\cos(1-k)s-(1-k)\cos(1+k)s-2b}{1-b^2}, \\ \frac{R}{2}\frac{(1+k)\sin(1-k)s-(1-k)\sin(1+k)s}{\sqrt{1-b^2}}, \\ \frac{R\sqrt{1-k^2}\cos(ks)}{\sqrt{1-b^2}} \end{pmatrix}$$

where $b \in (0, 1)$ is a constant real number. From the Theorem 3, the curves γ are spherical curve illustrated in Figure 1.

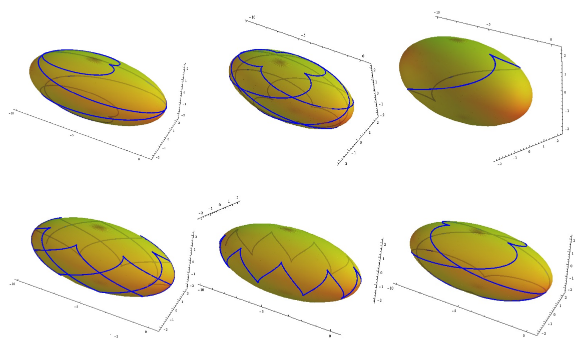

Figure 1. Curves on the Finslerian sphere $\mathbb{FS}^2$ for $b = 0.9$.

4.1 Finslerian spherical spirals

The spiral curve γ defined as a curve for which the tangent vector of γ makes a constant angle with fixed line. First we obtain Finslerian spherical spirals that obtained the intersection of the Finslerian sphere and the spherical helicoid.

Let f be a positive continuous function then the surface of revolution generated by rotating the curve $y = f(z)$, $z \in [a, b]$ around the $oz-$axis has the parametric equation

$$\begin{cases} x = \frac{f(t)\cos s - b}{1-b^2}, \\ y = \frac{f(t)\sin s}{\sqrt{1-b^2}}, \\ z = \frac{t}{\sqrt{1-b^2}} \end{cases}, \quad (t, s) \in [a, b] \times [0, 2\pi).$$

From here, the helicoid is obtained as

$$\begin{cases} x = \frac{t\cos s - b}{1-b^2}, \\ y = t\frac{\sin s}{\sqrt{1-b^2}}, \\ z = \frac{ct}{\sqrt{1-b^2}} \end{cases}, \quad (t, s) \in [a, b] \times [0, 2\pi),$$

the sphere is obtained as

$$\begin{cases} x = \frac{\sqrt{r^2-t^2}\cos s - b}{1-b^2}, \\ y = \frac{\sqrt{r^2-t^2}\sin s}{\sqrt{1-b^2}}, \\ z = \frac{t}{\sqrt{1-b^2}} \end{cases}, \quad (t, s) \in [a, b] \times [0, 2\pi).$$

As the intersection of the circular helicoid and Finslerian sphere gives rice to a three-dimensional spiral obtained as follows:

$$\begin{cases} x = \frac{\sqrt{r^2-t^2}\cos \frac{t}{c} - b}{1-b^2}, \\ y = \frac{\sqrt{r^2-t^2}\sin \frac{t}{c}}{\sqrt{1-b^2}}, \\ z = \frac{t}{\sqrt{1-b^2}} \end{cases}, \quad t \in [a, b].$$

From the above calculation we can give the following parametrization of the Finslerian spherical helicoid.

Finslerian Spherical helicoid: The sphere of radius r is generated by the function $f : [-r, r] \to [0, r]$, $f(z) = \sqrt{r^2 - z^2}$. Then the Finslerian spherical helicoid are given in the following parametric representation:

$$\begin{cases} x = \frac{u\sqrt{r^2-v^2}\cos \frac{v}{c} - b}{1-b^2}, \\ y = \frac{u\sqrt{r^2-v^2}\sin \frac{v}{c}}{\sqrt{1-b^2}}, \\ z = \frac{v}{\sqrt{1-b^2}} \end{cases}, \quad (u, v) \in [0, 1] \times [-r, r].$$

Now we give various examples of the Finslerian spherical helical curves and the Finslerian spherical helicoid. The images of the spirals and Finslerian spherical helicoid are plotted in Figure 2.

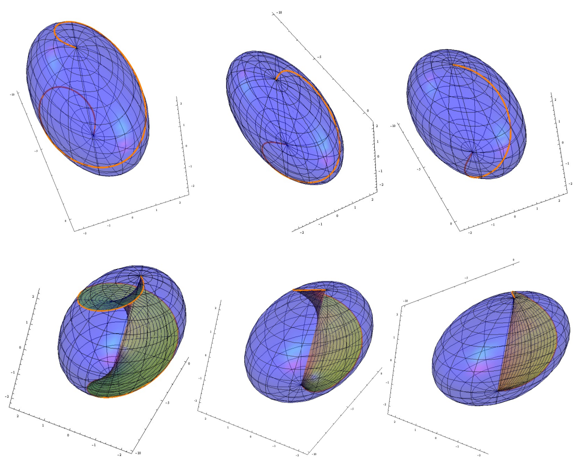

Figure 2. Spirals and helicoid on the $\mathbb{FS}^2$ for $b = 0.9$.

Let us observe that if we take different values of c, then we obtain different shapes of the spherical helical curves. Moreover, the $x - y$ projections of these curves are different. The family of these curves, which reminds us the Limacon of Pascal with a different view, is given by the following parametric representation:

$$\begin{cases} x = \frac{\sqrt{r^2-t^2}\cos\frac{t}{c}-b}{1-b^2}, \\ y = \frac{\sqrt{r^2-t^2}\sin\frac{t}{c}}{\sqrt{1-b^2}}, \end{cases}, \quad t \in [-r, r].$$

The images of the curve are illustrated in Figure 3. The first image demonstrate the projections of the values $c = 1$ $(green)$, $c = \frac{1}{2}$ $(orange)$ and $c = \frac{1}{4}$ $(blue)$. The second images show projection of the value$c = \frac{1}{6}$ and $c = \frac{1}{8}$, respectively, in Figure 3.

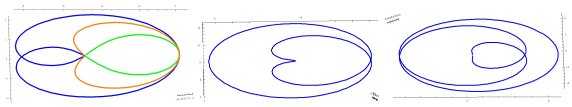

Figure 3. projections of the spirals.

Now we obtain the Viviani's curve obtained the intersection of a circular cylinder and a sphere in Euclidean 3 space Gray (1997, p. 201). This curve can be given the intersection of the Finslerian cylinder of radius a and center $(a, 0)$

$$F_S(u, v) = (\frac{\cos u + a - b}{1 - b^2}, \frac{\sin u}{\sqrt{1 - b^2}} v)$$

and the Finslerian sphere with center $(0, 0, 0)$ and radius $2a$.

$$\mathbb{FS}^2(u, v) = (\frac{2a\cos u \sin v - b}{1 - b^2}, \frac{2a\cos u\cos v}{\sqrt{1 - b^2}}, \frac{2a\sin u}{\sqrt{1 - b^2}})$$

Then the Viviani's curve on the Finslerian sphere has the following parametric representation

$$\gamma(t) = (\frac{a(1 + \cos t) - b}{1 - b^2}, \frac{a\sin t}{\sqrt{1 - b^2}}, \frac{2a\sin\frac{t}{2}}{\sqrt{1 - b^2}}).$$

The images of this curve is shown in Figure 4.

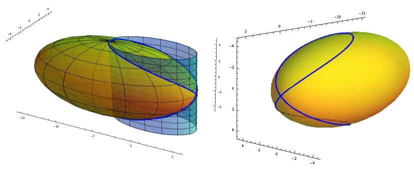

Figure 4. Viviani's curve on the $\mathbb{FS}^2$ for $b = 0.9$.

5 References

[1] H. Akbar-Zadeh, *Initiation to global Finslerian geometry*, North-Holland Math Library, 2006.
[2] I. Arslan Güven and S. Kayan Nurkan, *The relation among Bishop spherical indicatrix curves*, International Mathematical Forum **6**(25) (2011), 1209-1215.
[3] A. Asanjarani, B. Bidabad, *Classification of complete Finsler manifolds through a second order differential equation*, Differential Geom Appl. **26** (2008), 434-444.
[4] A. Bejancu, H. R. Farran, *Geometry of pseudo-Finsler submanifolds*, Kluwer Academic Publishers, 2000.
[5] B. Bidabad, Z. Shen, *Circle-preserving transformations on Finsler spaces*, Publication Mathematicae, in press.
[6] Ç. Camcı, Yaylı, Y., Hacısalihoğlu, H.H., *On the characterization of spherical curves in 3-dimensional Sasakian spaces*, J. Math. Anal. Appl. **342** (2008) 1151–1159.
[7] S.S. Chern, Z. Shen, *Riemann Finsler Geometry*, World Scientific, 2005.
[8] S. Deshmukh, B.Y. Chen, S.H. Alshammari, *On rectifying curves in Euclidean* $3-$*space*, Turk J Math. **42** (2018), 609-620.
[9] M. Ergüt, M. Külahcı *Special curves in three dimensional Finsler manifold* $\mathbb{F}^3$, TWMS J Pure Appl Math. **5**(2) (2014), 147-151.
[10] S. Izumiya, N. Takeuchi, *Generic properties of helices and Bertrand curves*, J. Geom. **74** (2002), 97-109.
[11] L. Kula and Y. Yayli, *On the slant helix and its spherical indicatrix*, Applied Mathematics and Computation **169** (2005), 600-607.
[12] C., Lăzureanu, Spirals on surfaces of revolution, Unpublished manuscript.
[13] A.O. Remizov, *Geodesics in generalized Finsler spaces: singularities in dimension two*, Journal of Singularities. **14** (2016), 172-193
[14] Z. Shen, *Lecture on Finsler geometry*, World Scientific Publishing Co, 2001.
[15] M.Y. Yildirim, M. Bektas, *Helices of the 3-dimensional Finsler manifold*, J Advanced Math Stud, **2**(1) (2009), 107-113.
[16] M.Y. Yildirim, *Biharmonic general helices in* $3-$*dimensional Finsler manifold*, Karaelmas Fen ve Muh Derg. **7**(1) (2017), 1-4.
[17] S. Yilmaz, E. Özyilmaz and M. Turgut, *New spherical indicatrices and their characterizations*, An. St. Univ. Ovidius Constanta **18**(2) (2010), 337-354.