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Research Article

On square Tribonacci Lucas numbers

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Abstract

The Tribonacci-Lucas sequence $\{S_n\}$ is defined by the recurrence relation $S_{n+3} = S_{n+2} + S_{n+1} + S_n$ with $S_0 = 3,\ S_1 = 1,\ S_2 = 3$. In this note, we show that 1 is the only perfect square in Tribonacci-Lucas sequence for $n \not\equiv 1 \pmod{32}$ and $n \not\equiv 17 \pmod{96}$.

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1. Introduction

For $n \geq 1$, the Fibonacci sequence $\{F_n\}_{n\geq 0}$ is given by $F_{n+1} = F_n + F_{n-1}$ with $F_0 = 0,\ F_1 = 1$. The Lucas sequence $\{L_n\}_{n\geq 0}$ satisfies the same recursive relation with the initials $L_0 = 2,\ L_1 = 1$.

Tribonacci sequence $\{T_n\}_{n\geq 0}$ is defined by $T_n = T_{n-1}+T_{n-2}+T_{n-3}$ with $T_0 = 0,\ T_1 = 0$ and $T_2 = 1$. The associated sequence of Tribonacci numbers is known Tribonacci-Lucas sequence $\{S_n\}_{n\geq 0}$ which satisfies the same relation with $S_0 = 3,\ S_1 = 1$ and $S_3 = 3$. The Binet formulas of Tribonacci and Tribonacci-Lucas sequences are

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\overline{\beta})} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\overline{\beta})} + \frac{\overline{\beta}^{n+1}}{(\overline{\beta}-\alpha)(\overline{\beta}-\beta)}$$

and

$$S_n = \alpha^n + \beta^n + \overline{\beta}^n$$

where α, β and $\overline{\beta}$ are the roots of the equation $x^3 - x^2 - x - 1 = 0$. A few terms of these sequences are given by the following table.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	...
$\mathbf{F_n}$	0	1	1	2	3	5	8	13	21	34	...
$\mathbf{L_n}$	2	1	3	4	7	11	18	29	47	76	...
$\mathbf{T_n}$	0	0	1	1	2	4	7	13	23	36	...
$\mathbf{S_n}$	3	1	3	7	11	21	39	71	131	241	...

To find perfect powers in recursive sequences is very popular and historical topic in number theory. Firstly, the well-known result was given by Cohn [2] and Wylie [8], indepently. The authors proved that 0, 1 and 144 are only perfect Fibonacci squares. Alfred [1] showed that 1 and 4 are two squares in Lucas sequences. Other known results for second order linear recursive sequences can be found in the papers [4, 7].

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In 1996, Pethő [5] proposed the following problem at 7^{th} International Research Conference on Fibonacci numbers and Their Applications.

Problem 1.1. Are the only squares $T_0 = T_1 = 0,\ T_2 = T_3 = 1,\ T_5 = 4,\ T_{10} = 81,\ T_{16} = 3136 = 56^2$ and $T_{18} = 10609 = 103^2$ among the number T_n?

This problem is still unsolved. By the motivation of this problem and the paper of Alfred [1], it is natural to ask that what are the squares in Tribonacci-Lucas sequences if they exist? In this paper, we answer this question under some weak conditions. Our result is following:

Theorem 1.2. *Let n be nonnegative integer with $n \not\equiv 1 \pmod{32}$ and $n \not\equiv 17 \pmod{96}$. Then $S_1 = 1$ is only square in Tribonacci-Lucas sequence.*

This theorem gives a motivation to proposed the following conjecture

Conjecture 1.3. *The solution of the equation $S_n = x^2$ is only $(n, x) = (1, 1)$.*

Our proof depends on $2-$adic order of the terms $S_n \mp 1$ and congruence identities. Before going further, we present several lemmas for the proof of theorem.

2. Auxiliary results

The p-adic order of r, $\nu_p(r)$, is the exponent of the highest power of a prime p which divides r.

Lemma 2.1. *Let t be integer with $t \not\equiv 0 \pmod 8$. Then*

$$\nu_2(4t + 32) = \nu_2(4t)$$

follows.

Proof. We will follow the method of Theorem 1 in [3]. Assume that t is an odd integer. Since $t + 8$ and t are odd integers, then we have

$$\nu_2(4(t + 8)) = \nu_2(4t) = 2.$$

If t is even integer, then it has the form $t = 2^a s$ where s is odd and $a \in \{1, 2\}$. Then

$$\nu_2(4(t + 8)) = 2 + a = \nu_2(4t)$$

follows as claimed. □

The following lemma gives the recursive relation with arithmetic progressions for Tribonacci-Lucas numbers.

Lemma 2.2. *Let $n,\ r,\ s$ nonnegative integers with $0 \le s \le r - 1$. We get*

$$\begin{aligned} S_{r(n+3)+s} &= \left(\alpha^r + \beta^r + \left(\overline{\beta}\right)^r\right) S_{r(n+2)+s} \\ &\quad - \left((\alpha\beta)^r + \left(\beta\overline{\beta}\right)^r + \left(\alpha\overline{\beta}\right)^r\right) S_{r(n+1)+s} \\ &\quad + S_{rn+s} \end{aligned}$$

where $\alpha, \beta, \overline{\beta}$ are the roots of the equation $x^3 - x^2 - x - 1 = 0$.

Proof. By using the Binet formula for the Tribonacci-Lucas sequence with the fact $\alpha\beta\overline{\beta} = 1$,

$$\begin{aligned}
& \left(\alpha^r + \beta^r + \left(\overline{\beta}\right)^r\right) S_{r(n+2)+s} - \left((\alpha\beta)^r + \left(\beta\overline{\beta}\right)^r + \left(\alpha\overline{\beta}\right)^r\right) S_{r(n+1)+s} + S_{rn+s} \\
= & \left(\alpha^r + \beta^r + \left(\overline{\beta}\right)^r\right) \left(\alpha^{r(n+2)+s} + \beta^{r(n+2)+s} + \overline{\beta}^{r(n+2)+s}\right) \\
- & \left((\alpha\beta)^r + \left(\beta\overline{\beta}\right)^r + \left(\alpha\overline{\beta}\right)^r\right) \left(\alpha^{r(n+1)+s} + \beta^{r(n+1)+s} + \overline{\beta}^{r(n+1)+s}\right) \\
+ & \left(\alpha^{r(n)+s} + \beta^{r(n)+s} + \overline{\beta}^{r(n)+s}\right) \\
= & \left(\alpha^{r(n+3)+s} + \beta^{r(n+3)+s} + \overline{\beta}^{r(n+3)+s}\right) = S_{r(n+3)+s}
\end{aligned}$$

follows as claimed. □

Now, we present the characterization of the term $\nu_2 (S_n \pm 1)$.

Lemma 2.3. *Let* $n \not\equiv 1 \pmod{32}$. *We have that*

$$\nu_2 (S_n - 1) = \begin{cases} 1 & \text{if } n \equiv 0, 2, 3 \pmod 4 \\ \nu_2 ((n+31)(n-1)) - 2 & \text{if } n \equiv 1 \pmod 4 \end{cases}.$$

Proof. Assume that $n \equiv 2 \pmod 4$. We use the induction method on n. It is obvious that $\nu_2 (S_2 - 1) = \nu_2 (2) = 1$, $\nu_2 (S_6 - 1) = \nu_2 (38) = 1$ and $\nu_2 (S_{10} - 1) = \nu_2 (442) = 1$. Assume that $\nu_2 (S_{4n+2} - 1) = \nu_2 \left(S_{4(n+1)+2} - 1\right) = \nu_2 \left(S_{4(n+2)+2} - 1\right) = 1$. So, there exist the odd integers k_1, k_2 and k_3 such that $S_{4n+2} - 1 = 2k_1$, $S_{4(n+1)+2} - 1 = 2k_2$ and $S_{4(n+2)+2} - 1 = 2k_3$ follow. Our aim to show that $\nu_2 \left(S_{4(n+3)+2} - 1\right) = 1$. By Lemma 2.2,

$$\begin{aligned}
S_{4(n+3)+2} - 1 &= 11 S_{4(n+2)+2} + 5 S_{4(n+1)+2} + S_{4n+2} - 1 \\
&= 11 (2k_1 + 1) + 5 (2k_2 + 1) + (2k_3 + 1) - 1 \\
&= 2 (11k_1 + 5k_2 + k_3 + 8)
\end{aligned}$$

Since k_1, k_2 and k_3 are odd integers, then $11k_1 + 5k_2 + k_3 + 8$ is odd integer. This yields $\nu_2 \left(S_{4(n+3)+2} - 1\right) = 1$ as claimed. The other cases $n \equiv j \pmod 4$, $j \in \{0, 3\}$ can be proven similarly. Therefore, we omit these cases.

Now, assume that $n \equiv 1 \pmod 4$. Since $n \not\equiv 1 \pmod{32}$, then we have that $n \equiv 5, 9, 13, 17, 21, 25, 29 \pmod{32}$. Then

$$\nu_2 ((n+31)(n-1)) - 2 = \begin{cases} 2, & \text{if } n \equiv 5, 13, 21, 29 \pmod{32} \\ 4, & \text{if } n \equiv 25, 9 \pmod{32} \\ 6, & \text{if } n \equiv 17 \pmod{32} \end{cases}. \tag{2.1}$$

Let $n \equiv 5 \pmod{32}$. One can see that $\nu_2 (S_5 - 1) = \nu_2 (S_{37} - 1) = \nu_2 (S_{69} - 1) = 2$. By the induction hypothesis, assume that $S_{32n+5} - 1 = 2^2 l_1$, $S_{32(n+1)+5} - 1 = 2^2 l_2$ and $S_{32(n+2)+5} - 1 = 2^2 l_3$ where l_1, l_2, l_3 are odd integers. Together with Lemma 2.2,

$$\begin{aligned}
S_{32(n+3)+5} - 1 &= 294294531 S_{32(n+2)+5} - 29699 S_{32(n+1)+5} + S_{32n+5} - 1 \\
&= 294294531 \left(2^2 l_1 + 1\right) - 29699 \left(2^2 l_2 + 1\right) \\
&\quad + \left(2^2 l_3 - 1\right) + 1 \\
&= 2^2 (294294531 l_1 - 29699 l_2 + l_3)
\end{aligned}$$

holds. Since $294294531 l_1 - 29699 l_2 + l_3$ is odd integer, then this gives our aim, namely $\nu_2 \left(S_{32(n+3)+5} - 1\right) = 2$ follows. The other cases can be proven similarly. To cut unnecessary repetition, we do not give the proof of other cases. □

Lemma 2.4. *If* $n \equiv 1 \pmod 4$, *then* $\nu_2(S_n + 1) = 1$ *holds. Otherwise,* $\nu_2(S_n + 1) \geq 2$ *follows.*

Proof. Assume that $n \equiv 1 \pmod 4$. It is clear that $\nu_2(S_1+1) = \nu_2(S_5+1) = \nu_2(S_9+1) = 1$. By Lemma 2.2 and assuming $S_{4n+1} = 2w_1$, $S_{4(n+1)+1} = 2w_2$, and $S_{4(n+2)+1} = 2w_3$ (w_1, w_2, w_3 are odd integers), we obtain the claimed result. Assume that $n \equiv 2 \pmod 4$. It is obvious that $\nu_2(S_2 + 1) \geq 2$, $\nu_2(S_6 + 1) \geq 2$ and $\nu_2(S_{10} + 1) \geq 2$. By induction method, we suppose that the congruences holds for $n \equiv 2 \pmod 4$. Namely, assume that $S_{4n+2} = 2^{a_1}q_1$, $S_{(n+1)+2} = 2^{a_2}q_2$ and $S_{4(n+2)+2} = 2^{a_3}q_3$ where q_1, q_2, q_3 are odd integers and $\min\{a_1, a_2, a_3\} \geq 2$. Let $\min\{a_1, a_2, a_3\} = a_3$. By the using Lemma 2.2, we get that

$$\begin{aligned} S_{4(n+3)+2} &= 11S_{4(n+2)+2} + 5S_{4(n+1)+2} + S_{4n+2} \\ &= 11 \cdot 2^{a_1}q_1 + 5 \cdot 2^{a_2}q_2 + 2^{a_3}q_3 \\ &= 2^{a_3}\left(11 \cdot 2^{a_1-a_3}q_1 + 5 \cdot 2^{a_2-a_3}q_2 + q_3\right). \end{aligned}$$

This gives that $\nu_2\left(S_{4(n+3)+2}\right) \geq a_3 \geq 2$ as desired. The cases $n \equiv 0, 3 \pmod 4$ can be proven similarly. Therefore, we omit them. □

Lemma 2.5. *For* $n \in \mathbb{Z}^+ \cup \{0\}$, *the followings hold:*

(i) $S_{8n+1} \equiv 3, 5, 6 \pmod 7$, *if* n *odd integer*
(ii) $S_{32(3n+1)+17} \equiv 10 \pmod{17}$
(iii) $S_{32(3n+2)+17} \equiv 14 \pmod{17}$
(iv) $(8n \mp 1)^2 \equiv 0, 1, 2, 4 \pmod 7$, *if* n *odd integer*
(v) $(32n \mp 1)^2 \equiv 1, 8, 13, 16, 0, 13, 8, 1, 9, 15, 2, 4, 4, 2, 15, 9 \pmod{17}$ *if* n *odd integer*

Proof. The items (i), (ii) and (iii) can be proven by using the Lemma 2.2. The period of (iv) and (v) can be seen easily. □

3. Proof of the theorem

Proof. Assume that $n \not\equiv 1 \pmod{32}$ and $n \not\equiv 17 \pmod{96}$. The terms of the Tribonacci-Lucas sequence are odd integers by the recurrence relation of the sequence. So, we are looking for the solution of the equation $S_n = (2k+1)^2$. From now on, assume that k is even integer. If $k = 0$, then it gives the solution $S_1 = 1^2$. Now, assume that $k \geq 1$. This yields that $n \geq 4$. Assume that the pair (n, k) is the solution of $S_n = (2k+1)^2$. Then we obtain that

$$S_n + 1 = (2k+1)^2 + 1.$$

After taking 2−adic order of both sides,

$$\begin{aligned} \nu_2(S_n + 1) &= \nu_2\left((2k+1)^2 + 1\right) \\ &= \nu_2\left(4k^2 + 4k + 2\right) = 1 \end{aligned}$$

follows. This gives that $n \equiv 1 \pmod 4$ by using Lemma 2.4. Now, subtract 1 from both side of the equation $S_n = (2k+1)^2$. Then we have the followings

$$\begin{aligned} \nu_2(S_n - 1) &= \nu_2\left((2k+1)^2 - 1\right) \\ &= \nu_2\left(4k^2 + 4k\right) \\ &= 2 + \nu_2(k). \end{aligned} \tag{3.1}$$

Together with (3.1) and Lemma 2.3, we deduce that

$$\nu_2((n+31)(n-1)) = 4 + \nu_2(k). \tag{3.2}$$

The equation (3.2) gives that

$$2^{4+\nu_2(k)} \mid \ (n+31)(n-1).$$

The Lemma 2.1 yields that

$$2^{2+\frac{\nu_2(k)}{2}} \mid (n+31) \quad \text{and} \quad 2^{2+\frac{\nu_2(k)}{2}} \mid (n-1). \tag{3.3}$$

By (3.3), we have

$$2^{2+\frac{\nu_2(k)}{2}} \mid \gcd(n+31, n-1). \tag{3.4}$$

By (2.1), it is obvious that $\nu_2(k)$ is even nonnegative integer. So, $\frac{\nu_2(k)}{2} \in \mathbb{Z}^+ \cup \{0\}$. Since $n \equiv 1 \pmod 4$ and $n \not\equiv 1 \pmod{32}$, there exists an integer t such that $n = 4t+1$ with $t \not\equiv 0 \pmod 8$.

If $t \equiv 1, 3, 5, 7 \pmod 8$, then we obtain that $\gcd(n+31, n-1) = \gcd(4t+32, 4t) = 4$. By (3.4),

$$2^{2+\frac{\nu_2(k)}{2}} \mid 4$$

follows. It gives that $\nu_2(k) = 0$. Since we assume k even integer with $k \geq 1$, we arrive at a contradiction.

If $t \equiv 2, 6 \pmod 8$, then $\gcd(n+31, n-1) = 8$. It gives that

$$2^{2+\frac{\nu_2(k)}{2}} \mid 8$$

yielding $\nu_2(k) = 0, 2$. If $\nu_2(k) = 2$, then $k = 4b_1$ where w_1 is odd integer. The equation (1) gives that $\nu_2(n-1) = \nu_2(n+31) = 3$. So, we have $n = 8b_2 + 1$ where w_2 is odd integer. Then we obtain the following equation

$$S_{8b_2+1} = (8b_1+1)^2.$$

This is impossible together with Lemma 2.5 (i) and (iv).

If $t \equiv 4 \pmod 8$, then we have

$$2^{2+\frac{\nu_2(k)}{2}} \mid 16$$

since $\gcd(n+31, n-1) = 16$. So, we have $\nu_2(k) = 0, 2, 4$. If $\nu_2(k) = 4$, then there exist the integers c_1, c_2 such that $(2k+1) = (32c_1+1)$ (c_1 odd) and $n = 4t+1 = 4(8c_2+4) = 32c_2+17$. So the equation turns to

$$S_{32c_2+17} = (32c_1+1)^2$$

Since we assume $n \not\equiv 17 \pmod{96}$, then we arrive at a contradiction by using Lemma 2.5 (ii), (iii) and (v).

If k is odd, we get the similar calculations. Therefore, the proof is completed. □

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