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Some New Integral Inequalities for n-Times Differentiable Trigonometrically Convex Functions

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Abstract

In this manuscript, by using an integral identity together with both the Hölder, Hölder-İşcan and the Power-mean integral inequalities we obtain several new inequalities for n-time differentiable trigonometrically convex functions.

1. Preliminaries

$\Omega : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $r, s \in I$ with $r < s$. The inequality

$$\Omega\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \Omega(u)\,du \leq \frac{\Omega(r)+\Omega(s)}{2}$$

is well known in the literature as Hermite-Hadamard's (H-H) integral inequality for convex functions [13]. The classical H-H inequality provides estimates of the mean value of a continuous convex or concave function. In recent years, significant improvements and generalizations have been found on convexity theory and H-H inequality; see for example [1–6, 8, 13].

Definition 1.1. *A function $\Omega : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$\Omega(\varepsilon r + (1-\varepsilon)s) \leq \varepsilon\Omega(r) + (1-\varepsilon)\Omega(s)$$

is valid for all $r, s \in I$ and $\varepsilon \in [0,1]$. If this inequality reverses, then Ω is said to be concave on interval $I \neq \varnothing$.

For some inequalities, generalizations and applications concerning convexity see [2–4, 6, 11–15]. Recently, in the literature there are so many papers about n-times differentiable functions on several kinds of convexities. In references [2, 4, 8, 14], readers can find some results about this issue. Many papers have been written by a number of mathematicians concerning inequalities for different classes of convex functions see for instance the recent papers [1, 3, 5, 6] and the references within these papers.
In [9], Kadakal gave the concept of the trigonometrically convex functions and related Hermite-Hadamard type inequalities.

Definition 1.2 ([9]). *A non-negative function $\Omega : I \to \mathbb{R}$ is called trigonometrically convex function on interval $[r,s]$, if for each $r, s \in I$ and $\varepsilon \in [0,1]$,*

$$\Omega(\varepsilon r + (1-\varepsilon)s) \leq \left(\sin\frac{\pi\varepsilon}{2}\right)\Omega(r) + \left(\cos\frac{\pi\varepsilon}{2}\right)\Omega(s).$$

If this inequality reveresed, then the function is called trigonometrically concave.

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Theorem 1.3 ([9]). *Let the function $\Omega : [r,s] \to \mathbb{R},\ s>0$, be a trigonometrically convex function. If $0 \le r < s$ and $\Omega \in L[r,s]$, then the following inequality holds:*

$$\frac{1}{s-r}\int_r^s \Omega(x)dx \le \frac{2}{\pi}\left[\Omega(r)+\Omega(s)\right].$$

Remark 1.4. *It is easily seen that, if the function $\Omega : [r,s] \to \mathbb{R},\ s>0$, be a trigonometrically concave function, then for $0 \le r < s$ and $\Omega \in L[r,s]$, then the following inequality holds:*

$$\frac{1}{s-r}\int_r^s \Omega(x)dx \ge \frac{2}{\pi}\left[\Omega(r)+\Omega(s)\right].$$

Theorem 1.5 ([9]). *Let the function $\Omega : [r,s] \to \mathbb{R},\ s>0$,be a trigonometrically convex function. If $0 \le r < s$ and $\Omega \in L[r,s]$, then the following inequalities holds:*

$$\Omega\left(\frac{s+r}{2}\right) \le \frac{\sqrt{2}}{s-r}\int_r^s \Omega(x)dx.$$

Remark 1.6. *It is easily seen that, if the function $\Omega : [r,s] \to \mathbb{R},\ s>0$, be a trigonometrically concave function, then for $0 \le r < s$ and $\Omega \in L[r,s]$, then the following inequality holds:*

$$\Omega\left(\frac{a+b}{2}\right) \ge \frac{\sqrt{2}}{s-r}\int_r^s \Omega(x)dx.$$

A refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows:

Theorem 1.7 (Hölder-İşcan Integral Inequality [7]). *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on interval $[r,s]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[r,s]$ then*

$$\int_r^s |f(x)g(x)|\,dx \le \frac{1}{s-r}\left\{\left(\int_r^s (s-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_r^s (s-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}} + \left(\int_r^s (x-r)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_r^s (x-r)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}$$

Let $0<r<s$, throughout this paper we will use

$$A(r,s) = \frac{r+s}{2}$$

$$L_p(r,s) = \left(\frac{s^{p+1}-r^{p+1}}{(p+1)(s-r)}\right)^{\frac{1}{p}},\ \ r \ne s,\ p \in \mathbb{R},\ \ p \ne -1,0$$

for the arithmetic and generalized logarithmic mean, respectively.

2. Main Results

We will use the following Lemma for obtain our main results.

Lemma 2.1 ([10]). *Let $\Omega : I \subseteq \mathbb{R} \to \mathbb{R}$ be n-times differentiable mapping on I° for $n \in \mathbb{N}$ and $\Omega^{(n)} \in L[r,s]$, where $r,s \in I^\circ$ with $r<s$, we have the identity*

$$\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right) - \int_r^s \Omega(x)dx = \frac{(-1)^{n+1}}{n!}\int_r^s x^n\Omega^{(n)}(x)dx \tag{2.1}$$

where an empty sum is understood to be nil.

Theorem 2.2. *For $n \in \mathbb{N}$; let $\Omega : I \subseteq (0,\infty) \to \mathbb{R}$ be n-times differentiable function on I° and $r,s \in I^\circ$ with $r<s$. If $\Omega^{(n)} \in L[r,s]$ and $\left|\Omega^{(n)}\right|^q$ for $q>1$ is trigonometrically convex function on the interval $[r,s]$, then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right) - \int_r^s \Omega(x)dx\right| \le \frac{s-r}{n!}\left(\frac{4}{\pi}\right)^{\frac{1}{q}} L_{np}^n(s,r)A^{\frac{1}{q}}\left(\left|\Omega^{(n)}(r)\right|^q,\left|\Omega^{(n)}(s)\right|^q\right). \tag{2.2}$$

Proof. If the function $\left|\Omega^{(n)}\right|^q$ for $q>1$ is trigonometrically convex on the interval $[r,s]$, using Lemma 2.1, the Hölder integral inequality and

$$\left|\Omega^{(n)}(x)\right|^q = \left|\Omega^{(n)}\left(\frac{s-x}{s-r}r+\frac{x-r}{s-r}s\right)\right|^q \le \sin\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(r)\right|^q + \cos\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(s)\right|^q,$$

we get

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right| \\
&\le \frac{1}{n!}\int_r^s x^n\left|\Omega^{(n)}(x)\right|dx \\
&\le \frac{1}{n!}\left(\int_r^s x^{np}dx\right)^{\frac{1}{p}}\left(\int_r^s\left|\Omega^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}} \\
&\le \frac{1}{n!}\left(\int_r^s x^{np}dx\right)^{\frac{1}{p}}\left(\int_r^s\left[\sin\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(a)\right|^q+\cos\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(b)\right|^q\right]dx\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}\left(\int_r^s x^{np}dx\right)^{\frac{1}{p}}\left(\left|\Omega^{(n)}(r)\right|^q\int_r^s\sin\frac{\pi(s-x)}{2(s-r)}dx+\left|\Omega^{(n)}(s)\right|^q\int_r^s\cos\frac{\pi(s-x)}{2(s-r)}dx\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}\left(\frac{s^{np+1}-r^{np+1}}{np+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}(s-r)\left|\Omega^{(n)}(r)\right|^q+\frac{2}{\pi}(s-r)\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}} \\
&= \frac{1}{n!}(s-r)^{\frac{1}{p}}(s-r)^{\frac{1}{q}}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}\left(\frac{s^{np+1}-r^{np+1}}{(np+1)(s-r)}\right)^{\frac{1}{p}}\left[\frac{\left|\Omega^{(n)}(r)\right|^q+\left|\Omega^{(n)}(s)\right|^q}{2}\right]^{\frac{1}{q}} \\
&= \frac{s-r}{n!}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}\left[\frac{s^{np+1}-r^{np+1}}{(np+1)(s-r)}\right]^{\frac{1}{p}}\left[\frac{\left|\Omega^{(n)}(r)\right|^q+\left|\Omega^{(n)}(s)\right|^q}{2}\right]^{\frac{1}{q}} \\
&= \frac{s-r}{n!}\left(\frac{4}{\pi}\right)^{\frac{1}{q}}L_{np}^n(r,s)A^{\frac{1}{q}}\left(\left|\Omega^{(n)}(r)\right|^q,\left|\Omega^{(n)}(s)\right|^q\right).
\end{aligned}$$

□

Corollary 2.3. *Under the conditions Theorem 2.2 for $n=1$ we have the following inequality:*

$$\left|\frac{\Omega(s)s-\Omega(s)s}{s-r}-\frac{1}{s-r}\int_r^s\Omega(x)dx\right|\le\left(\frac{4}{\pi}\right)^{\frac{1}{q}}L_p(r,s)\left[\frac{|\Omega'(r)|^q+|\Omega'(s)|^q}{2}\right]^{\frac{1}{q}}.$$

Proposition 2.4. *Let $r,s\in(0,\infty)$ with $r<s$, $q>1$ and $m\in(-\infty,0]\cup[1,\infty)\setminus\{-2q,-q\}$, we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(r,s)\le\left(\frac{4}{\pi}\right)^{\frac{1}{q}}L_p(r,s)A^{\frac{1}{q}}(r^m,s^m)$$

Proof. Under the assumption of the Proposition, let $\Omega(x)=\frac{q}{m+q}x^{\frac{m}{q}+1}$, $x\in(0,\infty)$. Then

$$\left|\Omega'(x)\right|^q=x^m$$

is trigonometrically convex on $(0,\infty)$ and the result follows directly from Corollary 2.3. □

Theorem 2.5. *For $n\in\mathbb{N}$; let $\Omega: I\subseteq(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° and $r,s\in I^\circ$ with $r<s$. If $\Omega^{(n)}\in L[r,s]$ and $\left|\Omega^{(n)}\right|^q$ for $q>1$ is trigonometrically convex function on the interval $[r,s]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right| \\
&\le \frac{(s-r)^{\frac{1}{q}}}{n!}\left(\left[sL_{np}^{np}(r,s)-L_{np+1}^{np+1}(r,s)\right]\right)^{\frac{1}{p}}\left(\frac{4}{\pi^2}\left|\Omega^{(n)}(r)\right|^q+\frac{2(\pi-2)}{\pi^2}\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}} \\
&+\frac{(s-r)^{\frac{1}{q}}}{n!}\left(\left[L_{np+1}^{np+1}(r,s)-aL_{np}^{np}(r,s)\right]\right)^{\frac{1}{p}}\left(\frac{2(\pi-2)}{\pi^2}\left|\Omega^{(n)}(r)\right|^q+\frac{4}{\pi^2}\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.3}$$

Proof. If the function $\left|\Omega^{(n)}\right|^q$ for $q>1$ is trigonometrically convex on the interval $[r,s]$, using Lemma 2.1, the Hölder-İşcan integral inequality and

$$\left|\Omega^{(n)}(x)\right|^q=\left|\Omega^{(n)}\left(\frac{s-x}{s-r}r+\frac{x-r}{s-r}s\right)\right|^q\le\sin\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(r)\right|^q+\cos\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(s)\right|^q,$$

we get

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right|\\
&\leq\frac{1}{n!}\int_r^s x^n\left|\Omega^{(n)}(x)\right|dx\\
&\leq\frac{1}{n!}\left(\int_r^s x^{np}dx\right)^{\frac{1}{p}}\left(\int_r^s\left|\Omega^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\\
&\leq\frac{1}{n!\,(s-r)}\left(\int_r^s(s-x)x^{np}dx\right)^{\frac{1}{p}}\left(\int_r^s(s-x)\left[\sin\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(a)\right|^q+\cos\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(b)\right|^q\right]dx\right)^{\frac{1}{q}}\\
&+\frac{1}{n!\,(s-r)}\left(\int_r^s(x-r)x^{np}dx\right)^{\frac{1}{p}}\left(\int_r^s(x-r)\left[\sin\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(r)\right|^q+\cos\frac{\pi(s-x)}{2(s-r)}\left|\Omega^{(n)}(s)\right|^q\right]dx\right)^{\frac{1}{q}}\\
&=\frac{1}{n!\,(s-r)}\left(\int_r^s(s-x)x^{np}dx\right)^{\frac{1}{p}}\left(\left|\Omega^{(n)}(r)\right|^q\int_r^s(s-x)\sin\frac{\pi(s-x)}{2(s-r)}dx+\left|\Omega^{(n)}(s)\right|^q\int_r^s(b-x)\cos\frac{\pi(s-x)}{2(s-r)}dx\right)^{\frac{1}{q}}\\
&+\frac{1}{n!\,(s-r)}\left(\int_r^s(x-r)x^{np}dx\right)^{\frac{1}{p}}\left(\left|\Omega^{(n)}(r)\right|^q\int_r^s(x-r)\sin\frac{\pi(s-x)}{2(s-r)}dx+\left|\Omega^{(n)}(s)\right|^q\int_r^s(x-r)\cos\frac{\pi(s-x)}{2(s-r)}dx\right)^{\frac{1}{q}}\\
&=\frac{1}{n!\,(s-r)}\left((s-r)\left[sL_{np}^{np}(r,s)-L_{np+1}^{np+1}(r,s)\right]\right)^{\frac{1}{p}}\left(\frac{4(s-r)^2}{\pi^2}\left|\Omega^{(n)}(r)\right|^q+\frac{2(\pi-2)(s-r)^2}{\pi^2}\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}}\\
&+\frac{1}{n!\,(s-r)}\left((s-r)\left[L_{np+1}^{np+1}(r,s)-rL_{np}^{np}(r,s)\right]\right)^{\frac{1}{p}}\left(\frac{2(\pi-2)(s-r)^2}{\pi^2}\left|\Omega^{(n)}(r)\right|^q+\frac{4(s-r)^2}{\pi^2}\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}}\\
&=\frac{(s-r)^{\frac{1}{q}}}{n!}\left(\left[sL_{np}^{np}(r,s)-L_{np+1}^{np+1}(r,s)\right]\right)^{\frac{1}{p}}\left(\frac{4}{\pi^2}\left|\Omega^{(n)}(r)\right|^q+\frac{2(\pi-2)}{\pi^2}\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}}\\
&+\frac{(s-r)^{\frac{1}{q}}}{n!}\left(\left[L_{np+1}^{np+1}(r,s)-rL_{np}^{np}(r,s)\right]\right)^{\frac{1}{p}}\left(\frac{2(\pi-2)}{\pi^2}\left|\Omega^{(n)}(r)\right|^q+\frac{4}{\pi^2}\left|\Omega^{(n)}(s)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

□

Theorem 2.6. *For $n\in\mathbb{N}$; let $\Omega: I\subseteq(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° and $r,s\in I^\circ$ with $r<s$. If $\Omega^{(n)}\in L[r,s]$ and $\left|\Omega^{(n)}\right|^q$ for $q\geq 1$ is trigonometrically convex on the interval $[r,s]$, then the following inequality holds:*

$$
\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right|\leq\frac{1}{n!}(s-r)^{1-\frac{1}{q}}L_n^{n\left(1-\frac{1}{q}\right)}\left\{\left|\Omega^{(n)}(r)\right|^q S_1(r,s)+\left|\Omega^{(n)}(s)\right|^q S_2(r,s)\right\}^{\frac{1}{q}},
$$

where

$$
S_1(r,s)=\int_r^s x^n\sin\frac{\pi(s-x)}{2(s-r)}dx,\qquad S_2(r,s)=\int_r^s x^n\cos\frac{\pi(s-x)}{2(s-r)}dx.
$$

Proof. From Lemma 2.1 and Power-mean integral inequality, we have

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right|\\
&\leq\frac{1}{n!}\int_r^s x^n\left|\Omega^{(n)}(x)\right|dx\\
&\leq\frac{1}{n!}\left(\int_r^s x^n dx\right)^{1-\frac{1}{q}}\left(\int_r^s x^n\left|\Omega^{(n)}(x)\right|^q dx\right)^{\frac{1}{q}}\\
&\leq\frac{1}{n!}\left(\int_r^s x^n dx\right)^{1-\frac{1}{q}}\left(\int_r^s x^n\left[\sin\frac{\pi(s-x)}{2(s-r))}\left|\Omega^{(n)}(r)\right|^q+\cos\frac{\pi(s-x)}{2(s-r))}\left|\Omega^{(n)}(s)\right|^q\right]dx\right)^{\frac{1}{q}}\\
&=\frac{1}{n!}\left(\int_r^s x^n dx\right)^{1-\frac{1}{q}}\left(\left|\Omega^{(n)}(r)\right|^q\int_r^s x^n\sin\frac{\pi(s-x)}{2(s-r))}dx+\left|\Omega^{(n)}(s)\right|^q\int_r^s x^n\cos\frac{\pi(s-x)}{2(s-r))}dx\right)^{\frac{1}{q}}\\
&=\frac{1}{n!}(s-r))^{1-\frac{1}{q}}\left[\frac{s^{n+1}-r^{n+1}}{(n+1)(s-r)}\right]^{1-\frac{1}{q}}\left\{\left|\Omega^{(n)}(s)\right|^q S_1(r,s)+\left|\Omega^{(n)}(r)\right|^q S_2(r,s)\right\}^{\frac{1}{q}}\\
&=\frac{1}{n!}(s-r)^{1-\frac{1}{q}}L_n^{n\left(1-\frac{1}{q}\right)}\left\{\left|\Omega^{(n)}(r)\right|^q S_1(r,s)+\left|\Omega^{(n)}(s)\right|^q S_2(r,s)\right\}^{\frac{1}{q}}.
\end{aligned}
$$

□

Corollary 2.7. *Under the conditions Theorem 2.6 for $n=1$ we have the following inequality:*

$$\left|\frac{\Omega(s)s-\Omega(s)s}{s-r}-\frac{1}{s-r}\int_r^s\Omega(x)dx\right|\le\left(\frac{r+s}{2}\right)^{1-\frac{1}{q}}\left[\frac{2\pi s-4(s-r)}{\pi^2}\left|\Omega'(r)\right|^q+\frac{4(s-r)-2\pi r}{\pi^2}\left|\Omega'(s)\right|^q\right]^{\frac{1}{q}}.$$

Proposition 2.8. *Let $r,s\in(0,\infty)$ with $r<s$, $q>1$ and $m\in(-\infty,0]\cup[1,\infty)\setminus\{-2q,-q\}$, we have*

$$L_{\frac{m}{q}+1}^{\frac{m}{q}+1}(r,s)\le A^{1-\frac{1}{q}}(r,s)\left[\frac{2\pi s-4(s-r)}{\pi^2}r^m+\frac{4(s-r)-2\pi r}{\pi^2}s^m\right]^{\frac{1}{q}}.$$

Proof. The result follows directly from Corollary 2.7 for the function

$$\Omega(x)=\frac{q}{m+q}x^{\frac{m}{q}+1}, x\in(0,\infty).$$

This completes the proof of Proposition. □

Corollary 2.9. *Using Proposition 2.8. for $m=1$, we have following inequality:*

$$L_{\frac{1}{q}+1}^{\frac{1}{q}+1}(r,s)\le A^{1-\frac{1}{q}}(r,s)\left[\frac{4(s-r)^2}{\pi^2}\right]^{\frac{1}{q}}.$$

Corollary 2.10. *Using Proposition 2.8 for $q=1$, we have following inequality:*

$$L_{m+1}^{m+1}(r,s)\le\frac{2\pi s-4(s-r)}{\pi^2}r^m+\frac{4(s-r)-2\pi r}{\pi^2}s^m.$$

Corollary 2.11. *Using Corollary 2.10 for $m=1$, we have following inequality:*

$$L_2^2(r,s)\le\frac{4(s-r)^2}{\pi^2}.$$

Corollary 2.12. *With the conditions of the Theorem 2.6 for $q=1$ we have the following inequality:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right|\le\frac{1}{n!}\left\{\left|\Omega^{(n)}(r)\right|S_1(r,s)+\left|\Omega^{(n)}(s)\right|S_2(r,s)\right\}$$

Theorem 2.13. *For $n\in\mathbb{N}$; let $\Omega:I\subset(0,\infty)\to\mathbb{R}$ be n-times differentiable function on I° and $r,s\in I^\circ$ with $r<s$. If $\Omega^{(n)}\in L[r,s]$ and $\left|\Omega^{(n)}\right|^q$ for $q>1$ is trigonometrically concave on the interval $[a,b]$, then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right|\le\frac{s-r}{n!}\left(\frac{1}{2}\right)^{\frac{1}{2q}}L_{np}^n(r,s)\left|\Omega^{(n)}\left(\frac{r+s}{2}\right)\right|.$$

Proof. Since $\left|\Omega^{(n)}\right|^q$ for $q>1$ is trigonometrically concave on the interval $[r,s]$, with respect to Hermite-Hadamard inequality we can write

$$\int_r^s\left|\Omega^{(n)}(x)\right|^qdx\le\frac{s-r}{\sqrt{2}}\left|\Omega^{(n)}\left(\frac{r+s}{2}\right)\right|^q.$$

Using Lemma 2.1 and the Hölder integral inequality we have

$$\begin{aligned}\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{\Omega^{(k)}(s)s^{k+1}-\Omega^{(k)}(r)r^{k+1}}{(k+1)!}\right)-\int_r^s\Omega(x)dx\right|&\le\frac{1}{n!}\int_r^sx^n\left|\Omega^{(n)}(x)\right|dx\\&\le\frac{1}{n!}\left(\int_r^sx^{np}dx\right)^{\frac{1}{p}}\left(\int_r^s\left|\Omega^{(n)}(x)\right|^qdx\right)^{\frac{1}{q}}\\&\le\frac{1}{n!}\left(\int_r^sx^{np}dx\right)^{\frac{1}{p}}\left(\frac{s-r}{\sqrt{2}}\left|f^{(n)}\left(\frac{r+s}{2}\right)\right|^q\right)^{\frac{1}{q}}\\&=\frac{s-r}{n!}\left(\frac{1}{2}\right)^{\frac{1}{2q}}\left[\frac{s^{np+1}-r^{np+1}}{(np+1)(s-r)}\right]^{\frac{1}{p}}\left|\Omega^{(n)}\left(\frac{r+s}{2}\right)\right|\\&=\frac{s-r}{n!}\left(\frac{1}{2}\right)^{\frac{1}{2q}}L_{np}^n(r,s)\left|\Omega^{(n)}\left(\frac{r+s}{2}\right)\right|.\end{aligned}$$

□

Corollary 2.14. *With the conditions of the Theorem 2.13 for $n=1$ we have the following inequality:*

$$\left|\frac{\Omega(s)s-\Omega(s)s}{s-r}-\frac{1}{s-r}\int_r^s\Omega(x)dx\right|\le\left(\frac{1}{2}\right)^{\frac{1}{2q}}L_p(r,s)\left|\Omega'\left(\frac{r+s}{2}\right)\right|.$$

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