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SOME RESULTS ON PRIME RINGS AND (σ, τ) - LIE IDEALS

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Abstract

Let R be a prime ring with characteristic not 2, $\sigma, \tau, \alpha, \beta, \lambda$ and μ automorphisms of R and $d : R \longrightarrow R$ a nonzero (σ, τ)-derivation. Suppose that $a \in R$. In this paper, we give some results on (σ, τ)-Lie ideals and prove that: **(1)** If $[a, d(R)]_{\alpha,\beta} = 0$ and $d\sigma = \sigma d$, $d\tau = \tau d$, then $a \in C_{\alpha,\beta}$. **(2)** Let d_1 be a nonzero (σ, τ)-derivation and d_2 an (α, β)-derivation of R such that $d_2\alpha = \alpha d_2$, $d_2\beta = \beta d_2$. If $[d_1(R), d_2(R)]_{\lambda,\mu} = 0$ then R is commutative. **(3)** If I is a nonzero ideal of R and $d(x, y) = 0$ for all $x, y \in I$, then R is commutative. **(4)** If $d(R, a) = 0$ then $(d(R), a)_{\sigma,\tau} = 0$.

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1. Introduction

Let $\sigma, \tau, \alpha, \beta, \lambda, \mu$ be automorphisms of a ring R and U an additive subgroup of R. The definition of (σ, τ)-Lie ideal is given in [6] as follows.

(i) U is a right (σ, τ)-Lie ideal of R, if $[U, R]_{\sigma,\tau} \subset U$.
(ii) U is a left (σ, τ)-Lie ideal of R, if $[R, U]_{\sigma,\tau} \subset U$.
(iii) U is a (σ, τ)-Lie ideal of R if U is both a left (σ, τ)-Lie ideal of R and a right (σ, τ)-Lie ideal of R.

It is clear that every Lie ideal of R is a (1,1)-Lie ideal of R.

An additive mapping $d : R \longrightarrow R$ is called a (σ, τ)-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in$R. We write $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$, $[x, y] = xy - yx$, $C_{\sigma,\tau} = \{c \in R \mid c\sigma(r) = \tau(r)c$ for all $r \in R\}$ and use the following commutator identities extensively.

(A): $[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$
(B): $[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z)$
(C): $(xy, z)_{\sigma,\tau} = x(y, z)_{\sigma,\tau} - [x, \tau(z)]y = x[y, \sigma(z)] + (x, z)_{\sigma,\tau}y$
(D): $(x, yz)_{\sigma,\tau} = \tau(y)(x, z)_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z)$

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Suppose that a is an element of R such that $ad(x) = d(x)a$ for all $x \in R$. Then, a must be central due to Herstein's theorem [4]. In [2], J. C.Chang extended this result by assuming that $[a, \delta(x)] = 0$ for all $x \in R$, where δ is an (α, β)-derivation of R such that $\delta\alpha = \alpha\delta$, $\delta\beta = \beta\delta$. One of the goals of this paper is to generalize the preceding results in the form expressed in abstract (1). In [3,Theorem 2] Herstein proved that if $[d(x), d(y)] = 0$ for all $x \in R$ then R is commutative. J. C.Chang extended this result in [2,Theorem-2(i)] by assuming that $[\delta(x), \delta(y) = 0$ for all $x, y \in R$, where δ is an (α, β)-derivation of R such that $\delta\alpha = \alpha\delta$, $\delta\beta = \beta\delta$. In this paper, we generalize this result in the form expressed in abstract (2). Furthermore, we give some results on (σ, τ)-Lie ideals in prime rings.

2. Results

2.1. Lemma. [7, Lemma 3] *Let R be a prime ring. If $b, ab \in C_{\sigma,\tau}$ then $a \in Z$ or $b = 0$.*

2.2. Lemma. [5, Lemma 2] *Let U be a nonzero left (σ, τ)-Lie ideal of R and $d : R \longrightarrow R$ a nonzero derivation. If $d(U) = 0$ then $[U, \sigma(U)] = 0$ and $[\sigma(U), \tau(U)] = 0$.*

2.3. Lemma. [8, Lemma 1] *Let U be a nonzero ideal of R and $d : R \longrightarrow R$, a nonzero (σ, τ)-derivation such that $d\sigma = \sigma d$, $d\tau = \tau d$. If $d^2(U) = 0$ then $d = 0$.*

2.4. Lemma. *Let U be a nonzero left (σ, τ)-Lie ideal of R. If $U \subset C_{\alpha,\beta}$ then $U \subset Z$.*

Proof. For any $r, x \in R$, $v \in U$, we have

$$\begin{aligned} 0 &= [[r\sigma(v), v]_{\sigma,\tau}, x]_{\alpha,\beta} \\ &= [r[\sigma(v), \sigma(v)] + [r, v]_{\sigma,\tau}\sigma(v), x]_{\alpha,\beta} \\ &= [r, v]_{\sigma,\tau}[\sigma(v), \alpha(x)] + [[r, v]_{\sigma,\tau}, x]_{\alpha,\beta}\sigma(v) \\ &= [r, v]_{\sigma,\tau}[\sigma(v), \alpha(x)]. \end{aligned}$$

That is:

(2.1) $\quad [r, v]_{\sigma,\tau}[\sigma(v), \alpha(x)] = 0,$ for all $r, x \in R, v \in U.$

Replacing x by $xz, z \in R$ in (2.1) and using the primeness of R we get

(2.2) $\quad [r, v]_{\sigma,\tau} = 0,$ for all $r \in R$ or $[\sigma(v), R] = 0.$

If $[r, v]_{\sigma,\tau} = 0$ for all $r \in R$, then $0 = [rt, v]_{\sigma,\tau} = r[t, v]_{\sigma,\tau} + [r, \tau(v)]t = [r, \tau(v)]t$, for all $r, t \in R$. Since R is prime we obtain $v \in Z$ from the last relation. That is, $U \subset Z$ is obtained from (2.2). □

The following lemma is a generalization of [3,Lemma 5.1].

2.5. Lemma. *Let d be a nonzero (σ, τ)-derivation on R. If $d(R) \subset C_{\lambda,\mu}$, then R is commutative.*

Proof. For any $x, y, r \in R$ we have

$$\begin{aligned} 0 &= [d(xy), r]_{\lambda,\mu} \\ &= [d(x)\sigma(y) + \tau(x)d(y), r]_{\lambda,\mu} \\ &= d(x)[\sigma(y), \lambda(r)] + [d(x), r]_{\lambda,\mu}\sigma(y) + \tau(x)[d(y), r]_{\lambda,\mu} + [\tau(x), \mu(r)]d(y) \\ &= d(x)[\sigma(y), \lambda(r)] + [\tau(x), \mu(r)]d(y). \end{aligned}$$

Replacing r by $\mu^{-1}\tau(x)$ in the last relation we have,

(2.3) $\quad 0 = d(x)[\sigma(y), \lambda\mu^{-1}\tau(x)],$ for all $x, y \in R.$

If we take yz instead of y in (2.3), and use the primeness of R we have $d(x) = 0$ or $x \in Z$. Let us consider Brauer's Trick. Note that $K = \{x \in R \mid x \in Z\}$ and

$L = \{x \in R \mid d(x) = 0\}$ are subgroups of R, furthermore $R = K \cup L$. This gives $R = K$ or $R = L$ by Brauer's Trick. Since d is nonzero, we obtain that $R = K$, and so R is commutative. □

2.6. Theorem. *If d is a nonzero (σ,τ)-derivation of R such that $d\sigma = \sigma d$, $d\tau = \tau d$ and $[a, d(R)]_{\alpha,\beta} = 0$, then $a \in C_{\alpha,\beta}$.*

Proof. Let $[a, d(R)]_{\alpha,\beta} = 0$. For any $x, y \in R$ we have

$$\begin{aligned} 0 &= [a, d(xy)]_{\alpha,\beta} \\ &= [a, d(x)\sigma(y) + \tau(x)d(y)]_{\alpha,\beta} \\ &= \beta d(x)[a, \sigma(y)]_{\alpha,\beta} + [a, \tau(x)]_{\alpha,\beta}\alpha d(y), \end{aligned}$$

for all $x, y \in R$. Replacing x by $\tau^{-1}d(x)$ in the last relation and using the hypothesis, we get

$$(2.4) \qquad \beta d\tau^{-1}d(x)[a, \sigma(y)]_{\alpha,\beta} = 0, \text{ for all } x, y \in R.$$

If we take $yz,\ z \in R$ instead of y in (2.4) we obtain $\beta d\tau^{-1}d(x)\beta\sigma(y)[a, \sigma(z)]_{\alpha,\beta} = 0$, for all $x, y, z \in R$. Since R is prime and σ, β are onto we have:

$$(2.5) \qquad d\tau^{-1}d(R) = 0 \text{ or } [a, R]_{\alpha,\beta} = 0.$$

Now $d\tau = \tau d$ and $d\tau^{-1}d(R) = 0$ imply that $d^2(R) = 0$. Thus $d = 0$ by Lemma 2.3. Hence $a \in C_{\alpha,\beta}$ follows from (2.5) and the hypothesis. □

2.7. Corollary. *Let U be a nonzero right (σ,τ)-Lie ideal of R and d a nonzero derivation on R such that $d\sigma = \sigma d$, $d\tau = \tau d$. If $d(U) = 0$ then $U \subset C_{\sigma,\tau}$.*

Proof. We have

$$\begin{aligned} 0 &= d[v, r]_{\sigma,\tau} \\ &= d(v\sigma(r) - \tau(r)v) \\ &= vd\sigma(r) - d\tau(r)v, \end{aligned}$$

for all $r \in R$, $v \in U$. So we obtain $[v, d(r)]_{\sigma,\tau} = 0$ for all $r \in R$, $v \in U$. This implies that $U \subset C_{\sigma,\tau}$ by Theorem 2.6. □

2.8. Theorem. (1) *Let U be a nonzero left (σ,τ)-Lie ideal of R and d a nonzero (α,β)-derivation on R such that $d\alpha = \alpha d$, $d\beta = \beta d$. If $[U, d(R)]_{\lambda,\mu} = 0$ then $U \subset Z$.*

(2) *Let d_1 be a nonzero (σ,τ)-derivation, d_2 a nonzero (α,β)-derivation on R such that $d_2\alpha = \alpha d_2$ and $d_2\beta = \beta d_2$. If $[d_1(R), d_2(R)]_{\lambda,\mu} = 0$ then R is commutative.*

Proof. **(1)** If $[U, d(R)]_{\lambda,\mu} = 0$ then we have $U \subset C_{\lambda,\mu}$ by Theorem 2.6. This implies that $U \subset Z$ by Lemma 2.4.

(2) If $[d_1(R), d_2(R)]_{\lambda,\mu} = 0$ then $d_1(R) \subset C_{\lambda,\mu}$ by Theorem 2.6. This implies that R is commutative by Lemma 2.5. □

2.9. Theorem. *Let d be a nonzero (σ,τ)-derivation and $a \in R$. If $d(R, a) = 0$ then $(d(R), a)_{\sigma,\tau} = 0$.*

Proof. For any $r \in R$, using the hypothesis, we have:

$$\begin{aligned} 0 &= d(ar, a) = d(a(r, a) - [a, a]r) \\ &= d(a(r, a)) \\ &= d(a)\sigma(r, a) + \tau(a)d(r, a) \\ &= d(a)\sigma(r, a). \end{aligned}$$

That is,

$$d(a)\sigma(r, a) = 0, \text{ for all } r \in R. \tag{2.6}$$

Replacing r by $rx,\ x \in R$ in (2.6) we get, $0 = d(a)\sigma(r)\sigma[x, a] + d(a)\sigma(r, a)\sigma(x)$. Thus we obtain

$$d(a)\sigma(r)\sigma[x, a] \ = 0, \text{ for all } x, r \in R. \tag{2.7}$$

Since R is prime we have $d(a) = 0$ or $a \in Z$ by (2.7). If $a \in Z$ then we can deduce that $d(a) = 0$ as follows. Firstly,

$$\begin{aligned} 0 &= d(r, a) \\ &= 2d(ra) \\ &= 2d(r)\sigma(a) + 2\tau(r)d(a) \end{aligned}$$

for all $r \in R$. Replacing r by (r, a) in the preceding relation and using that char$R \neq 2$, we have

$$\tau(r, a)d(a) = 0, \text{ for all } r \in R. \tag{2.8}$$

Since $a \in Z$ and char$R \neq 2$ we have $aR\tau^{-1}d(a) = 0$ by (2.8) and so $d(a) = 0$ is obtained. Thus, we have, $0 = d(r, a) = (d(r), a)_{\sigma,\tau} + (d(a), r)_{\sigma,\tau} = (d(r), a)_{\sigma,\tau}$, for all $r \in R$. □

2.10. Lemma. *Let U be a nonzero left (σ, τ)-Lie ideal of R and d a nonzero derivation of R such that $d\sigma = \sigma d$ and $d\tau = \tau d$. If $d(U) = 0$ then U is commutative.*

Proof. For any $r \in R,\ v \in U$ we have

$$\begin{aligned} 0 &= d[r, v]_{\sigma,\tau} \\ &= d(r\sigma(v) - \tau(v)r) \\ &= d(r)\sigma(v) + rd\sigma(v) - d\tau(v)r - \tau(v)d(r) \\ &= d(r)\sigma(v) - \tau(v)d(r). \end{aligned}$$

That is,

$$d(r)\sigma(v) = \tau(v)d(r \text{ for all } r \in R,\ v \in U. \tag{2.9}$$

Replacing r by $rx,\ x \in R$ in (2.9) and using (2.9) again we get:

$$\begin{aligned} 0 &= d(rx)\sigma(v) \ - \tau(v)d(rx) \\ &= d(r)x\sigma(v) + rd(x)\sigma(v) - \tau(v)d(r)x - \tau(v)rd(x) \\ &= d(r)x\sigma(v) + r\tau(v)d(x) - d(r)\sigma(v)x - \tau(v)rd(x), \end{aligned}$$

for all $x, r \in R,\ v \in U$. That is,

$$d(r)[x, \sigma(v)] + [r, \tau(v)]d(x) = 0, \text{ for all } x, r \in R,\ v \in U. \tag{2.10}$$

If we take $\sigma(w),\ w \in U$ instead of x in (2.10) we obtain, $d(R)[\sigma(w), \sigma(v)] = 0$,for all $v, w \in U$. Since R is prime we have $d = 0$ or $\sigma[U, U] = 0$. Since $d \neq 0$ we get $[U, U] = 0$. □

2.11. Lemma. *Let U be a nonzero left (σ, τ)-Lie ideal of R and d a nonzero derivation of R such that $d\sigma = \sigma d$, $d\tau = \tau d$. If $d^2(U) = 0$ and $d(U) \subset Z$ then U is commutative.*

Proof. For all $x\in R$ and $u\in U$ we have

$$U\ni[\tau(u)x,u]_{\sigma,\tau}=\tau(u)[x,u]_{\sigma,\tau}[\tau(u),\tau(u)]x=\tau(u)[x,u]_{\sigma,\tau}.$$

That is, $\tau(u)[x,u]_{\sigma,\tau}\in U$, for all $x\in R$, $u\in U$. Thus,

$$\begin{aligned}0&=d^2(\tau(u)[x,u]_{\sigma,\tau})\\&=d(d\tau(u)[x,u]_{\sigma,\tau}+\tau(u)d[x,u]_{\sigma,\tau})\\&=d^2\tau(u)[x,u]_{\sigma,\tau}+d\tau(u)d[x,u]_{\sigma,\tau}+d\tau(u)d[x,u]_{\sigma,\tau}+\tau(u)d^2[x,u]_{\sigma,\tau},\end{aligned}$$

gives

$$(2.11)\qquad d\tau(u)d[x,u]_{\sigma,\tau}=0,\ \text{for all } x\in R,\ u\in U.$$

Replacing u by $u+v,\ v\in U$ in (2.11) we obtain,

$$(2.12)\qquad d\tau(u)d[x,v]_{\sigma,\tau}+d\tau(v)d[x,u]_{\sigma,\tau}=0,\ \text{for all } x\in R,\ u,v\in U.$$

If we multiply (2.12) on the by left by $d\tau(u)$ and use that $d(U)\subset Z$ and $d\tau=\tau d$, we have that

$$(2.13)\qquad (d\tau(u))^2d[R,U]_{\sigma,\tau}=0,\ \text{for all } u\in U.$$

On the other hand, for any $x\in R$ and $v\in U$ we obtain:

$$\begin{aligned}[x\sigma(v),v]_{\sigma,\tau}&=x[\sigma(v),\sigma(v)]+[x,v]_{\sigma,\tau}\sigma(v)\\&=[x,v]_{\sigma,\tau}\sigma(v)\in[R,U]_{\sigma,\tau}.\end{aligned}$$

That is, $d([x,v]_{\sigma,\tau}\sigma(v))\in d[R,U]_{\sigma,\tau}$. If we consider this relation in (2.13) we have,

$$\begin{aligned}0&=(d\tau(u))^2d([x,v]_{\sigma,\tau}\sigma(v))\\&=(d\tau(u))^2d[x,v]_{\sigma,\tau}\sigma(v)+(d\tau(u))^2[x,v]_{\sigma,\tau}d\sigma(v).\end{aligned}$$

That is,

$$(2.14)\qquad (d\tau(u))^2[x,v]_{\sigma,\tau}d\sigma(v)=0,\ \text{for all } x\in R,\ u,v\in U.$$

Taking $v+w,\ w\in U$ instead of v in (2.14) we get

$$\begin{aligned}0&=(d\tau(u))^2[x,v+w]_{\sigma,\tau}d\sigma(v+w)\\&=(d\tau(u))^2[x,v]_{\sigma,\tau}d\sigma(v)+(d\tau(u))^2[x,w]_{\sigma,\tau}d\sigma(v)+(d\tau(u))^2[x,v]_{\sigma,\tau}d\sigma(w)\\&\qquad+(d\tau(u))^2[x,w]_{\sigma,\tau}d\sigma(w).\end{aligned}$$

If we use (2.14), we obtain:

$$(2.15)\qquad (d\tau(u))^2[x,v]_{\sigma,\tau}d\sigma(w)+(d\tau(u))^2[x,w]_{\sigma,\tau}d\sigma(v)=0,\ \text{for all } x\in R,\ u,v,w\in U.$$

Let us multiply (2.15) by $d\sigma(v)$ on the right hand side, and use that $d(U)\subset Z$ and (2.14). Then we have,

$$(2.16)\qquad (d\tau(u))^2[x,w]_{\sigma,\tau}(d\sigma(v))^2=0,\ \text{for all } x\in R,\ u,v,w\in U.$$

Since $d(U)\subset Z$ and R is prime we obtain:

$$(2.17)\qquad (d\tau(u))^2[x,w]_{\sigma,\tau}=0,\ \text{for all } x\in R,\ u,w\in U \text{ or } (d\sigma(v))^2=0,\ \text{for all } v\in U.$$

If we recall that $d(U)\subset Z$ and $d\sigma=\sigma d,\ d\tau=\tau d$,we obtain $d(U)=0$ or $[R,U]_{\sigma,\tau}=0$.

Case 1. If $[R,U]_{\sigma,\tau}=0$ then for all $x,y\in R,\ v\in U$ we have,

$$\begin{aligned}0&=[xy,v]_{\sigma,\tau}\\&=x[y,\sigma(v]+[x,v]_{\sigma,\tau}y\\&=x[y,\sigma(v].\end{aligned}$$

That is, $R[R,\sigma(U)]=0$. Since R is prime we obtain $U\subset Z$.

Case 2. If $d(U) = 0$ then U is commutative by Lemma 2.10. □

2.12. Theorem. *Let U be a nonzero left (σ,τ)-Lie ideal of R and d a nonzero derivation of R such that $d\sigma = \sigma d$ and $d\tau = \tau d$. If $d(U) \subset Z$ then U is commutative.*

Proof. Let $x, y \in R$ and $u, v \in U$. Then we have,

$$\begin{aligned} Z \ni d[d(v)x, u]_{\sigma,\tau} &= d(d(v)[x, u]_{\sigma,\tau} + [d(v), \tau(u)]x) \\ &= d(d(v)[x, u]_{\sigma,\tau}) \\ &= d^2(v)[x, u]_{\sigma,\tau} + d(v)d[x, u]_{\sigma,\tau} \end{aligned}$$

for all $x \in R$, $u, v \in U$. Since $d(v)d[x, u]_{\sigma,\tau} \in Z$ we have:

$$d^2(v)[x, u]_{\sigma,\tau} \in Z, \text{ for all } x \in R,\ u, v \in U. \tag{2.18}$$

If we recall that $d(U) \subset Z$, then Lemma 2.1 and (2.18) give $d^2(v) = 0$, for all $v \in U$, or $[x, u]_{\sigma,\tau} \in Z$, for all $x \in R$, $u \in U$.

Case 1. If $d^2(U) = 0$, then U is commutative by Lemma 7.

Case 2. If $[x, u]_{\sigma,\tau} \in Z$, for all $x \in R,\ u \in U$, then

$$Z \ni [x\sigma(u), u]_{\sigma,\tau} = x[\sigma(u), \sigma(u)] + [x, u]_{\sigma,\tau}\sigma(u) = [x, u]_{\sigma,\tau}\sigma(u)$$

for all $x \in R,\ u \in U$. Again applying Lemma 2.1 in the last relation we obtain,

$$[x, u]_{\sigma,\tau} = 0, \text{ for all } x \in R, \text{ or } u \in Z. \tag{2.19}$$

If $[x, u]_{\sigma,\tau} = 0$, for all $x \in R$ then,

$$\begin{aligned} 0 &= [xr, u]_{\sigma,\tau} \\ &= x[r, \sigma(u)] + [x, u]_{\sigma,\tau}r \\ &= x[r, \sigma(u)] \end{aligned}$$

for all $x, r \in R,\ u \in U$. That is , $R[R, \sigma(u)] = 0$. Since R is prime, the last equation gives us $u \in Z$. So, we have $u \in Z$ for the two cases in (2.19). Hence we obtain $U \subset Z$, so again U is commutative. □

2.13. Theorem. *Let U be a nonzero left (σ,τ)-Lie ideal of R and d a nonzero derivation of R such that $d\sigma = \sigma d$ and $d\tau = \tau d$. If $d(U) = 0$ and $u^2 \in Z$ for all $u \in U$ then $U \subset Z$.*

Proof. If $d(U) = 0$ then $[U, \sigma(U)] = 0$ by Lemma 2.2, and U is commutative by Lemma 2.10. For any $u, v \in U$ we have $(u+v)^2 = u^2 + v^2 + 2uv \in Z$. Since char $R \neq 2$ we have $uv \in Z$ for all $u, v \in U$. Now let us take the arbitrary elements r, s of R and u, v of U. Then we get

$$[r, u]_{\sigma,\tau}[s, v]_{\sigma,\tau} \in Z, \text{ for all } r, s \in R,\ u, v \in U. \tag{2.20}$$

Replacing s by $sx,\ x \in R$, in (2.20), we have

$$Z \ni [r, u]_{\sigma,\tau}[sx, v]_{\sigma,\tau} = [r, u]_{\sigma,\tau}s[x, \sigma(v)] + [r, u]_{\sigma,\tau}[s, v]_{\sigma,\tau}x.$$

Taking $w \in U$ instead of x, and using that $[U, \sigma(U)] = 0$ in the preceding relation, we get:

$$[r, u]_{\sigma,\tau}[s, v]_{\sigma,\tau}w \in Z, \text{ for all } r, s \in R,\ u, v, w \in U. \tag{2.21}$$

From the (2.20), (2.21) and Lemma 2.1 we have,

$$[r, u]_{\sigma,\tau}[s, v]_{\sigma,\tau} = 0, \text{ for all } r, s \in R,\ u, v \in U, \text{ or } w \in Z, \text{ for all } w \in U. \tag{2.22}$$

If $[r,u]_{\sigma,\tau}[s,v]_{\sigma,\tau}=0$ for all $r,s\in R,\ u,v\in U$, then

$$\begin{aligned}0&=[rt,u]_{\sigma,\tau}[s,v]_{\sigma,\tau}\\&=r[t,u]_{\sigma,\tau}[s,v]_{\sigma,\tau}+[r,\tau(u)]t[s,v]_{\sigma,\tau}\\&=[r,\tau(u)]t[s,v]_{\sigma,\tau},\end{aligned}$$

for all $r,t,s\in R,\ u,v\in U$. This gives that $[R,\tau(U)]R[R,U]_{\sigma,\tau}=0$. On the other hand, $[R,U]_{\sigma,\tau}=0$ implies that $U\subset Z$ as we saw in the proof of Lemma 2.11. □

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