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ON THE CAUCHY PROBLEM FOR MATRIX FACTORIZATIONS OF THE HELMHOLTZ EQUATION

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Abstract. In the paper it is considered the regularization of the Cauchy problem for systems of elliptic type equations of the first order with constant coefficients factorisable Helmholtz operator in two-dimensional unbounded domain. Using the results of the works [20], [21], [22], [23], [24], [25] and [26], we construct in explicit form Carleman matrix and based on the regularized solution of the Cauchy problem.

1. Introduction

This problem concerns ill-posed problems, i.e. it is unstable. It is known that the Cauchy problem for elliptic equations is unstable relatively small change in the data, i.e. incorrect (example Hadamard, see, for instance [10], p. 39). There is a sizable literature on the subject (see, e.g. [4]-[9], [13]). Tarkhanov [2] has published a criterion for the solvability of a larger class of boundary value problems for elliptic systems. In unstable problems, the image of the operator is not is closed, therefore, the solvability condition can not be is written in terms of continuous linear functionals. So, in the Cauchy problem for elliptic equations with data on part of the boundary of the domain the solution is usually unique, the problem is solvable for everywhere dense a set of data, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than theory of solvability of Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg, A.M. Kytmanov, N.N. Tarkhanov (see, for instance [3]).

The uniqueness of the solution follows from Holmgren's general theorem (see [14]). The conditional stability of the problem follows from the work of A.N. Tikhonov (see [13]), if we restrict the class of possible solutions to a compactum.

In this paper we construct a family of vector-functions $U_{\sigma(\delta)}(x) = U(x, f_\delta)$ depending on a parameter σ, and prove that under certain conditions and a special choice of the parameter $\sigma = \sigma(\delta)$, at $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ converges in the usual sense to a solution $U(x)$ at a point $x \in G$.

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Following A.N. Tikhonov (see [13]), a family of vector-valued functions $U_{\sigma(\delta)}(x)$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem. For special domains, the problem of extending bounded analytic functions in the case when the data are given only on a part of the boundary was considered by Carleman (see [4]). The researches of T. Carleman were continued by G.M. Goluzin and V.I. Krylov (see [12]). A multidimensional analogue of Carleman's formula for analytic functions of several variables was constructed in (see [11]). The use of the classical Green's formula for constructing a regularized solution of the Cauchy problem for the Laplace equation was proposed by Academician M.M. Lavrent'ev (see, for instance [5], [6]). Using the ideas of M.M. Lavrent'ev and Sh. Yarmukhamedov, a regularized solution of the Cauchy problem for the Laplace and Helmholtz equations was constructed in explicit form (see, for instance [7], [8], [9]). In [1] an integral formula is proved for systems of equations of elliptic type of the first order, with constant coefficients in a bounded domain. In [16], the Cauchy problem for the Helmholtz equation in an arbitrary bounded plane domain with Cauchy data, known only on the region boundary, is considered. The solvability criterion for the Cauchy problem for the Laplace equation in the space $\mathbb{R}^m$ it was considered by Shlapunov in work [17].

For systems of equations of elliptic type of the first order with constant coefficients, the factorizing operator of Helmholtz, in [22] the validity of the integral formula in a three-dimensional unbounded domain was proved.

The construction of the Carleman matrix for elliptic systems was carried out by: Sh. Yarmukhamedov, N.N. Tarkhanov, A.A. Shlapunov, I.E. Niyozov, D.A. Juraev and others. The system considered in this paper was introduced by N.N. Tarkhanov. For this system, he studied correct boundary value problems and found an analogue of the Cauchy integral formula in a bounded domain.

The system considered in this paper was introduced by N.N. Tarkhanov. For this system, he studied correct boundary value problems and found an analogue of the Cauchy integral formula in a bounded domain (see, for instance [3]).

In many well-posed problems for systems of equations of elliptic type of the first order with constant coefficients that factorize the Helmholtz operator, it is not possible to calculate the values of the vector function on the entire boundary. Therefore, the problem of reconstructing the solution of systems of equations of first order elliptic type with constant coefficients, factorizing the Helmholtz operator (see, for instance [20], [21], [22], [23], [24], [25] and [26]), is one of the topical problems in the theory of differential equations.

For the last decades, interest in classical ill-posed problems of mathematical physics has remained. This direction in the study of the properties of solutions of the Cauchy problem for the Laplace equation was started in [5]-[8], [10] and subsequently developed in [1]-[3], [17]-[26].

In this paper, we present an explicit formula for the approximate solution of the Cauchy problem for the matrix factorizations of the Helmholtz equation in a bounded region on the plane. The two-dimensional case requires special consideration, in contrast to three or more dimensions in many mathematical problems. Our formula for an approximate solution also includes the construction of a family of fundamental solutions for the Helmholtz operator on the plane. This family is parametrized by some entire function $K(w)$, the choice of which depends on the dimension of the space. This motivates a separate study of regularization formulas in flat domains and leads to improved estimates compared to the three-dimensional case.

2. The integral formula in an unbounded domain

Let $\mathbb{R}^2$ be the two-dimensional real Euclidean space,

$$x = (x_1, x_2) \in \mathbb{R}^2, \; y = (y_1, y_2) \in \mathbb{R}^2.$$

$G \subset \mathbb{R}^2$ be an unbounded simply-connected domain with piecewise smooth boundary consisting of the plane T: $y_2 = 0$ and some smooth curve S lying in the half-space $y_2 > 0$, i.e., $\partial G = S \bigcup T$.

We introduce the following notation:

$$r = |y - x|\,, \; \alpha = |y_1 - x_1|, \; w = i\sqrt{u^2 + \alpha^2} + y_2, \; u \geq 0,$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)^T, \; \frac{\partial}{\partial x} \to \xi^T, \, \xi^T = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \text{ be a transposed vector } \xi,$$

$$U(x) = (U_1(x), ..., U_n(x))^T, \, u^0 = (1, ..., 1) \in \mathbb{R}^n, \, n = 2^m, \, m = 2,$$

$$E(z) = \left\| \begin{matrix} z_1 \ldots 0 \\ \ldots\ldots. \\ 0 \ldots z_n \end{matrix} \right\| \text{ be a diagonal matrix, } z = (z_1, ..., z_n) \in \mathbb{R}^n.$$

Let $D(\xi^T)$, $(n{\times}n)-$ dimensional matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the following condition is satisfied:

$$D^*(\xi^T)D(\xi^T) = E((|\xi|^2 + \lambda^2)u^0),$$

where $D^*(\xi^T)$ is the Hermitian conjugate matrix $D(x^T)$, λ is a real number.

We consider in the region G a system of differential equations

$$D\left(\frac{\partial}{\partial x}\right)U(x) = 0, \tag{2.1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of first-order differential operators.

We denote by $A(G)$ the class of vector functions in a domain G continuous on $\overline{G} = G \bigcup \partial G$ and satisfying system (2.1).

If G is a bounded and $U(y) \in A(G)$, then the following integral formula of Cauchy type is valid (see [22])

$$U(x) = \int\limits_{\partial G} M(y, x)U(y)ds_y, \;\; x \in G, \tag{2.2}$$

where

$$M(y, x) = \left(E\left(-\frac{i}{4}H_0^{(1)}(\lambda r)u^0\right)D^*\left(\frac{\partial}{\partial y}\right)\right)D(t^T).$$

Here $t = (t_1, t_2)$ is the unit external normal, drawn at a point y, the curve ∂G, $-\frac{i}{4}H_0^{(1)}(\lambda r)$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^2$. [15].

We denote by $K(w)$ is an entire function taking real values for real w, ($w = u + iv$, $u, v-$real numbers) and satisfying the following conditions:

$$K(u) \neq 0, \; \sup_{v \geq 1}\left|v^p K^{(p)}(w)\right| = \mathrm{M}(u, p) < \infty, \;\; -\infty < u < \infty, \; p = 0, 1, 2. \tag{2.3}$$

We define a function $\Phi(y, x)$ at $y \neq x$ for the following equation:

$$\Phi(y, x) = -\frac{1}{2\pi K(x_2)} \int_0^\infty \mathrm{Im} \frac{K(w)}{w - x_2} \frac{u I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du. \tag{2.4}$$

Here $I_0(\lambda u) = J_0(i\lambda u)$ is the Bessel function of the first kind is of zero order [14].

Formula (2.2) is true if instead $-\frac{i}{4}H_0^{(1)}(\lambda r)$ of substituting the function

$$\Phi(y, x) = -\frac{i}{4}H_0^{(1)}(\lambda r) + g(y, x), \tag{2.5}$$

where $g(y, x)$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then formula (2.2) has the following form

$$U(x) = \int_{\partial G} M(y, x)U(y)ds_y, \quad x \in G, \tag{2.6}$$

where

$$M(y, x) = \left(E \left(\Phi(y, x)u^0 \right) D^* \left(\frac{\partial}{\partial y} \right) \right) D(t^T).$$

Formula (2.6) is generalized for the case when G is the unbounded domain.

Let $G \subset \mathbb{R}^2$ be an unbounded domain, with a piecewise smooth boundary ∂G ($\partial G-$extends to infinity).

We denote by G_R the part G lying inside the circle of radius R with center at zero:

$$G_R = \{y : y \in G,\ |y| < R\}, G_R^\infty = G \backslash G_R, R > 0.$$

Theorem 2.1. *Let $U(y) \in A(G)$, G be a finitely connected unbounded domain in $\mathbb{R}^2$, with piecewise-smooth boundary ∂G. If for each fixed $x \in G$ we have the equality*

$$\lim_{R\to\infty} \int_{G_R^\infty} M(y, x)U(y)ds_y = 0, \tag{2.7}$$

then the formula (2.6) is true.

Proof. Indeed, for a fixed $x \in G$ ($|x| < R$) and taking (2.6) into account, we have

$$\int_{\partial G} M(y, x)U(y)ds_y = \int_{\partial G_R} M(y, x)U(y)ds_y +$$

$$+ \int_{\partial G_R^\infty} M(y, x)U(y)ds_y = U(x) + \int_{\partial G_R^\infty} M(y, x)U(y)ds_y,\ x \in G_R.$$

Taking into account condition (2.7), for $R \to \infty$, we obtain (2.6).

Suppose G that an unbounded domain lies inside a strip of the smallest width defined by inequality

$$0 < y_2 < h,\ h = \frac{\pi}{\rho},\ \rho > 0,$$

and ∂G extends to infinity.

Suppose that for some $b_0 > 0$ the length ∂G satisfies the growth condition

$$\int_{\partial G} \exp\left[-b_0\rho_0 |y_1|\right] ds_y < \infty, \;\; 0 < \rho_0 < \rho. \tag{2.8}$$

Suppose $U(y) \in A(G)$ that it satisfies the boundary growth condition

$$|U(y)| \leq \exp\left[\exp\rho_2 |y_1|\right], \;\; \rho_2 < \rho, \; y \in G. \tag{2.9}$$

In (2.4) we put

$$\begin{gathered} K(w) = \exp\left[-bi\rho_1\left(w - \frac{h}{2}\right) - b_1 i\rho_0\left(w - \frac{h}{2}\right)\right], \\ K(x_2) = \exp\left[b\cos\rho_1\left(x_2 - \frac{h}{2}\right) + b_1\cos i\rho_0\left(x_2 - \frac{h}{2}\right)\right], \\ 0 < \rho_1 < \rho, \; 0 < x_2 < h, \end{gathered} \tag{2.10}$$

where

$$b = 2a\exp\left(\rho_1 |x_1|\right), \;\; b_1 > \frac{b_0}{\cos\left(\rho_0\frac{h}{2}\right)}, \; a \geq 0, \; b > 0.$$

Then the integral representation (2.6) is true.

For a fixed $x \in G$ and $y \to \infty$, we estimate the function $\Phi(y, x)$ and its derivatives $\dfrac{\partial\Phi(y,x)}{\partial y_j}$, $j = 1, 2$. For the estimation $\dfrac{\partial\Phi(y,x)}{\partial y_j}$ we use equalities

$$\begin{gathered} -2\pi K(x_2)\frac{\partial\Phi(y,x)}{\partial y_1} = \frac{(y_1 - x_1)\operatorname{Re}K(w_0) - sign(y_1 - x_1)(y_2 - x_2)\operatorname{Im}K(w_0)}{r^2} - \\ -(y_1 - x_1)\lambda\int_0^\infty \frac{\sqrt{u^2+\alpha^2}\operatorname{Re}K(w) - (y_2 - x_2)\operatorname{Im}K(w)}{u^2 + r^2}\cdot\frac{I_1(\lambda u)du}{\sqrt{u^2+\alpha^2}}, \\ y \neq x, \; w_0 = i|y_1 - x_1| + y_2, \;\; I_1(\lambda u) = I_0'(\lambda u) \end{gathered} \tag{2.11}$$

and

$$\begin{gathered} -2\pi K(x_2)\frac{\partial\Phi(y,x)}{\partial y_2} = \frac{(y_2 - x_2)\operatorname{Re}K(w_0) - (y_1 - x_1)\operatorname{Im}K(w_0)}{r^2} - \\ -\lambda\int_0^\infty \frac{(y_2 - x_2)\operatorname{Re}K(w) - \sqrt{u^2+\alpha^2}\operatorname{Im}K(w)}{u^2 + r^2} I_1(\lambda u)du, \;\; y_1 \neq x_1, \end{gathered} \tag{2.12}$$

which are obtained from (2.4).

Really,

$$\left|\exp\left[-bi\rho_1\left(w-\frac{h}{2}\right)-b_1 i\rho_0\left(w-\frac{h}{2}\right)\right]\right|=$$

$$=\exp\operatorname{Re}\left[-bi\rho_1\left(w-\frac{h}{2}\right)-b_1 i\rho_0\left(w-\frac{h}{2}\right)\right]=$$

$$=\exp\left[-b\rho_1\sqrt{u^2+\alpha^2}\cos\rho_1\left(y_2-\frac{h}{2}\right)-b_1\rho_0\sqrt{u^2+\alpha^2}\cos\rho_0\left(y_2-\frac{h}{2}\right)\right].$$

As

$$-\frac{\pi}{2}\le-\frac{\rho_1}{\rho}\cdot\frac{\pi}{2}\le\frac{\rho_1}{\rho}\cdot\frac{\pi}{2}<\frac{\pi}{2},$$

$$-\frac{\pi}{2}\le-\frac{\rho_1}{\rho}\cdot\frac{\pi}{2}\le\rho_0\left(y_2-\frac{h}{2}\right)\le\frac{\rho_1}{\rho}\cdot\frac{\pi}{2}<\frac{\pi}{2}.$$

Consequently,

$$\cos\rho\left(y_2-\frac{h}{2}\right)>0,\cos\rho_0\left(y_2-\frac{h}{2}\right)\ge\cos\frac{h\rho_0}{2}>\delta_0>0,$$

It does not vanish in the region G and

$$|\Phi(y,x)|=\mathrm{O}\left[\exp\left(-\varepsilon\rho_1\,|y_1|\right)\right],\ \varepsilon>0,\ y\to\infty,\ y\in G\bigcup\partial G,$$

$$\left|\frac{\partial\Phi(y,x)}{\partial y_1}\right|=\mathrm{O}\left[\exp\left(-\varepsilon\rho_1\,|y_1|\right)\right],\ \varepsilon>0,\ y\to\infty,\ y\in G\bigcup\partial G,$$

$$\left|\frac{\partial\Phi(y,x)}{\partial y_2}\right|=\mathrm{O}\left[\exp\left(-\varepsilon\rho_1\,|y_1|\right)\right],\ \varepsilon>0,\ y\to\infty,\ y\in G\bigcup\partial G.$$

We now choose ρ_1 with the condition $\rho_2<\rho_1<\rho$. Then condition (2.8) is fulfilled and the integral formula (2.6) is true. Theorem 2.1 is proved. □

Condition (2.10) can be weakened.

We denote by $A_\rho(G)$ is the class of vector-valued functions from $A(G)$, satisfying the following growth condition:

$$A_\rho(G)=\left\{U(y):\ U(y)\in A(G),\ \ |U(y)|\le\exp\left[\mathrm{o}\left(\exp\rho\,|y_1|\right)\right],\ y\to\infty,\ y\in G\right\}. \tag{2.13}$$

The following is valid

Theorem 2.2. *Suppose $U(y)\in A_\rho(G)$ that it satisfies the growth condition*

$$|U(y)|\le C\exp\left[a\cos\rho_1\left(y_2-\frac{h}{2}\right)\exp\left(\rho_1\,|y_1|\right)\right], \tag{2.14}$$

$$a\ge 0,\ 0<\rho_1<\rho,\ y\in\partial G,$$

where $C-$is some constant. Then formula (2.6) is valid.

Proof. Divide the area G by a line $y_2=\dfrac{h}{2}$ into two areas

$$G_1=\left\{y:\ 0<y_2<\frac{h}{2}\right\}\text{ and }G_2=\left\{y:\ \frac{h}{2}<y_2<h\right\}.$$

Consider the domain G_1. In the formula (2.4) together $K(w)$ we put $K_1(w)$

$$K_1(w) = K(w)\exp\left[-\delta i\tau\left(w-\frac{h}{2}\right)-\delta_1 i\rho\left(w-\frac{h}{2}\right)\right], \tag{2.15}$$

$$\rho<\tau<2\rho,\ \delta>0,\ \delta_1>o,$$

Here $K(w)$ it is determined from (2.10). With this notation, (2.8) is true.

Really,

$$\left|\exp\left[-i\tau\left(w-\frac{h}{4}\right)-\delta_1 i\rho\left(w-\frac{h}{4}\right)\right]\right| =$$

$$=\exp\left[-\delta\tau\sqrt{u^2+\alpha^2}\cos\tau\left(y_2-\frac{h}{4}\right)\right]=$$

$$=\exp\left[-\delta\tau\sqrt{u^2+\alpha^2}\right]\le\exp\left[-\delta\exp\tau\left|y_1\right|\right],$$

as

$-\frac{\pi}{2}\le-\tau\frac{\pi}{4}\le\tau\left(y_2-\frac{h}{4}\right)\le\tau\frac{\pi}{2}<\frac{h}{2}$ and $\cos\tau\left(y_2-\frac{h}{4}\right)\ge\cos\tau\frac{h}{4}\ge\delta_0>0.$

We denote the corresponding $\Phi(y,x)$ by $\Phi^+(y,x)$.

As

$$\cos\tau\left(y_2-\frac{h}{4}\right)\ge\delta_0,\ y\in G_1\bigcup\partial G_1,$$

then for fixed $x\in G_1$, $y\in G_1\bigcup\partial G_1$, for $\Phi^+(y,x)$ and its derivatives are true asymptotic estimates

$$\left|\Phi^+(y,x)\right| = \mathrm{O}\left[\exp(-\delta_0\exp\left(\tau\left|y_1\right|\right)\right],\ y\to\infty,\ \rho<\tau<2\rho,$$

$$\left|\frac{\partial\Phi^+(y,x)}{\partial y_1}\right| = \mathrm{O}\left[\exp(-\delta_0\exp\left(\tau\left|y_1\right|\right)\right],\ y\to\infty,\ \rho<\tau<2\rho,$$

$$\left|\frac{\partial\Phi^+(y,x)}{\partial y_2}\right| = \mathrm{O}\left[\exp(-\delta_0\exp\left(\tau\left|y_1\right|\right)\right],\ y\to\infty,\ \rho<\tau<2\rho.$$

Suppose $U(y)\in A(G_1)$ that in a domain G_1 satisfies the growth condition

$$\left|U(y)\right|\le C\exp\left[\exp\left(2\rho-\varepsilon\right)\left|y_1\right|\right],\ \varepsilon>0. \tag{2.16}$$

We choose τ the inequality $2\rho-\varepsilon<\tau<2\rho$ in (2.15).

Then the condition (2.15) is satisfied for the region G_1, therefore, the following integral formula holds

$$U(x)=\int\limits_{\partial G_1} M(y,x)U(y)ds_y,\ x\in G_1. \tag{2.17}$$

where

$$M(y,x)=\left(E\left(\Phi^+(y,x)u^0\right)D^*\left(\frac{\partial}{\partial x}\right)\right)D(t^T).$$

If $U(y)\in A(G_2)$ satisfies the growth condition (2.14) in G_2, then for $2\rho-\varepsilon<\tau<2\rho$, similarly we obtain the following integral formula

$$U(x)=\int\limits_{\partial G_2} M(y,x)U(y)ds_y,\ x\in G_2. \tag{2.18}$$

where

$$M(y,x)=\left(E\left(\Phi^{-}(y,x)u^{0}\right)D^{*}\left(\frac{\partial}{\partial x}\right)\right)D(t^{T}).$$

Here $\Phi^{-}(y,x)$ it is defined by the formula (2.4), in which $K(w)$ it is replaced by the function $K_2(w)$:

$$K_2(w)=K(w)\exp\left[-\delta i\tau\left(w-h_1\right)-\delta_1 i\rho\left(w-\frac{h}{2}\right)\right], \tag{2.19}$$

where

$$h_1=\frac{h}{2}+\frac{h}{4},\ \ \frac{h}{2}<y_2<h,\ \ \frac{h}{2}<x_2<h_1,\ \ \delta>0,\ \ \delta_1>0.$$

In the formulas obtained with this formula, the integrals (according to (2.9)) converge uniformly for $\delta\geq 0$, when $U(y)\in A_\rho(G)$. In these formulas we put $\delta=0$ and, combining the formulas obtained, we find

$$U(x)=\int\limits_{\partial G}M(y,x)U(y)ds_y,\quad x\in G,\ \ x_2\neq\frac{h}{2}, \tag{2.20}$$

where

$$M(y,x)=\left(E\left(\tilde{\Phi}(y,x)u^{0}\right)D^{*}\left(\frac{\partial}{\partial y}\right)\right)D(t^{T}).$$

(integrals over the cross section $y_2=\dfrac{h}{2}$ are mutually destroyed)

$$\tilde{\Phi}(y,x)=(\Phi^{+}(y,x))_{\delta=0}=(\Phi^{-}(y,x))_{\delta=0}.$$

Here, $\tilde{\Phi}(y,x)$ is determined by the formula (2.4), in which $K(w)$ is determined from (2.15), where $\delta=0$ is laid. According to the continuation principle, formula (2.20) is true for $\forall x\in G$. Under condition (2.16), formula (2.20) is true for $\forall\delta_1\geq 0$. Assuming $\delta_1=0$, we obtain the proof of the theorem. Theorem 2.2 is proved. □

In the formula (2.4), choosing

$$K(w)=\frac{1}{w-x_2+3h}\exp(\sigma w),$$
$$K(x_2)=\frac{1}{3h}\exp(\sigma x_2),\ 0<x_2<h,\ h=\frac{\pi}{\rho}, \tag{2.21}$$

we get

$$\Phi_\sigma(y,x)=-\frac{e^{-\sigma x_2}}{2\pi(3h)^{-1}}\int\limits_0^\infty \operatorname{Im}\frac{\exp(\sigma w)}{(w-x_2+3h)(w-x_2)}\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du. \tag{2.22}$$

Then the integral formula (2.6) has the following form:

$$U(x)=\int\limits_{\partial G}N_\sigma(y,x)U(y)ds_y,\quad x\in G, \tag{2.23}$$

where

$$N_\sigma(y,x)=\left(E\left(\Phi_\sigma(y,x)u^{0}\right)D^{*}\left(\frac{\partial}{\partial y}\right)\right)D(t^{T}).$$

Suppose that the boundary of the domain G consists of a hyper plane $y_2 = 0$ and a smooth curve S extending to infinity and lying in the strip

$$0 < y_2 < h,\ h = \frac{\pi}{\rho},\ \rho > 0.$$

We assume that S is given by the equation

$$y_2 = \psi(y_1),\ \ -\infty < y_1 < \infty,$$

where $\psi(y_1)$ satisfies the condition

$$|\psi'(y_1)| \leq M < \infty, M = const.$$

3. Regularization of the Cauchy problem

Formulation of the problem. Suppose that $U(y) \in A_\rho(G)$ and

$$U(y)|_S = f(y),\ y \in S. \tag{3.1}$$

Here, $f(y)-$ a given continuous vector-valued function on S.

It is required to restore the vector function $U(y)$ in the region G, based on its values $f(y)$ on S.

The following is valid

Theorem 3.1. *Let $U(y) \in A_\rho(G)$ it satisfy the inequality*

$$|U(y)| \leq 1,\ y \in T. \tag{3.2}$$

If

$$U_\sigma(x) = \int\limits_S N_\sigma(y,x)U(y)ds_y, \quad x \in G, \tag{3.3}$$

then the following estimate holds

$$|U(x) - U_\sigma(x)| \leq C_\rho(\lambda, x)\sigma e^{-\sigma x_2},\ \sigma > 1,\ x \in G. \tag{3.4}$$

Here and below functions bounded on compact subsets of the domain G, we denote by $C_\rho(\lambda, x)$.

Proof. Using the integral formula (2.23) and the equality (3.3), we obtain

$$U(x) = U_\sigma(x) + \int\limits_T N_\sigma(y,x)U(y)ds_y,\ x \in G.$$

Taking inequality (3.2) into account, we estimate the following

$$|U(x) - U_\sigma(x)| \leq \int\limits_T |U(y)|\,|N_\sigma(y,x)|\,ds_y \leq \int\limits_T |N_\sigma(y,x)|\,ds_y,\ x \in G.$$

To do this, we estimate the integrals $\int\limits_{y_2=0} |\Phi_\sigma(y,x)|\,ds_y$, $\int\limits_{y_2=0} \left|\frac{\partial \Phi_\sigma(y,x)}{\partial y_1}\right| ds_y$ and $\int\limits_{y_2=0} \left|\frac{\partial \Phi_\sigma(y,x)}{\partial y_2}\right| ds_y$ on the part T of the plane $y_2 = 0$.

Let $\sigma > 0$. Separating the imaginary part of (2.22), we obtain

$$\Phi_\sigma(y, x) = \frac{e^{\sigma(y_2 - x_2)}}{2\pi(3h)^{-1}} \left[\int_0^\infty \left(\frac{(\beta + \beta_1)\cos\sigma\alpha_1}{(\alpha_1^2 + \beta_1^2)(\alpha_1^2 + \beta^2)} + \right.\right.$$
$$\left.\left. + \frac{\left(-\alpha_1^2 + \beta_1\beta\right)}{(\alpha_1^2 + \beta_1^2)(\alpha_1^2 + \beta^2)} \frac{\sin\sigma\alpha_1}{\alpha_1} \right) u\, I_0(\lambda u)\, du \right], \tag{3.5}$$

where

$$\alpha_1^2 = u^2 + \alpha^2,\ \beta = y_2 - x_2,\ \beta_1 = y_2 - x_2 + 3h.$$

We estimate first $\int\limits_{y_2=0} |\Phi_\sigma(y, x)|\, ds_y$. Taking into account equality (3.5) and inequality

$$I_0(\lambda u) \le \exp(\lambda u), \tag{3.6}$$

we have

$$\int\limits_{y_2=0} |\Phi_\sigma(y, x)|\, ds_y \le C_\rho(\lambda, x)\sigma\, e^{-\sigma x_2},\ \sigma > 1,\ x \in G. \tag{3.7}$$

To estimate the integrals $\int\limits_{y_2=0} \left|\frac{\partial\Phi_\sigma(y, x)}{\partial y_1}\right| ds_y$ and $\int\limits_{y_2=0} \left|\frac{\partial\Phi_\sigma(y, x)}{\partial y_2}\right| ds_y$, we use equalities (2.11) and (2.12). For this, using equalities (2.21) and choosing

$$K(w_0) = \exp(\sigma w_0),\ \sigma > 0, \tag{3.8}$$

we obtain the following formulas

$$-\frac{2\pi e^{\sigma x_2}}{(3h)^{-1}} \frac{\partial\Phi_\sigma}{\partial y_1} = \frac{(y_1 - x_1)\mathrm{Re}\ \exp(\sigma w_0) + sign(y_1 - x_1)(y_2 - x_2)\mathrm{Im}\ \exp(\sigma w_0)}{r^2} -$$
$$-(y_1 - x_1)\lambda \int_0^\infty \frac{\sqrt{u^2 + \alpha^2}\mathrm{Re}\ \exp(\sigma w) - (y_2 - x_2)\mathrm{Im}\ \exp(\sigma w)}{u^2 + r^2} \cdot \frac{I_1(\lambda u)du}{\sqrt{u^2 + \alpha^2}},\ y \ne x, \tag{3.9}$$

and

$$-\frac{2\pi e^{\sigma x_2}}{(3h)^{-1}} \frac{\partial\Phi_\sigma}{\partial y_2} = \frac{(y_2 - x_2)\mathrm{Re}\ \exp(\sigma w_0) + (y_1 - x_1)\mathrm{Im}\ \exp(\sigma w_0)}{r^2} -$$
$$-\lambda \int_0^\infty \frac{(y_2 - x_2)\mathrm{Re}\ \exp(\sigma w) - \sqrt{u^2 + \alpha^2}\mathrm{Im}\ \exp(\sigma w)}{u^2 + r^2} I_1(\lambda u)du,\ \ y_1 \ne x_1. \tag{3.10}$$

Taking into account equality (3.9) and inequality

$$I_1(\lambda u) \le \lambda u \exp(\lambda u), \tag{3.10}$$

we get

$$\int\limits_{y_2=0} \left|\frac{\partial\Phi_\sigma(y, x)}{\partial y_1}\right| ds_y \le C_\rho(\lambda, x)\sigma\, e^{-\sigma x_2},\ \sigma > 1,\ x \in G. \tag{3.12}$$

Analogously, taking into account equality (3.10) and inequality (3,12), we estimate the following integral

$$\int\limits_{y_2=0} \left|\frac{\partial \Phi_\sigma(y,x)}{\partial y_2}\right| ds_y \le C_\rho(\lambda,x)\sigma e^{-\sigma x_2},\ \sigma>1,\ x\in G. \tag{3.13}$$

From the inequalities (3.7), (3.12), and (3.13), we obtain (3.4). Theorem 3.1 is proved. □

Corollary 3.1. The limiting equality

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x),$$

holds uniformly on each compact set in the domain G.

Theorem 3.2. *Let $U(y)\in A_\rho(G)$ satisfy condition (3.2) on a part of the plane $y_2=0$, and on a smooth curve S the inequality*

$$|U(y)|\le\delta,\ 0<\delta<1. \tag{3.14}$$

Then the following estimate holds

$$|U(x)|\le C_\rho(\lambda,x)\sigma\delta^{\frac{x_2}{h}},\ \sigma>1,\ x\in G. \tag{3.15}$$

Proof. Using the integral formula (2.23), we have

$$U(x)=\int\limits_{\partial G} N_\sigma(y,x)U(y)ds_y =$$

$$=\int\limits_S N_\sigma(y,x)U(y)ds_y + \int\limits_T N_\sigma(y,x)U(y)ds_y,\ x\in G.$$

Taking into account the boundary condition (3.2) and inequality (3.14), we obtain the estimate

$$\begin{aligned}|U(x)| &\le \int\limits_S |U(y)|\,|N_\sigma(y,x)|\,ds_y + \int\limits_T |U(y)|\,|N_\sigma(y,x)|\,ds_y \le \\ &\le \delta\int\limits_S |N_\sigma(y,x)|\,ds_y + \int\limits_T |N_\sigma(y,x)|\,ds_y,\ x\in G.\end{aligned} \tag{3.16}$$

First we estimate the first integral of inequality (3.16). To do this, we estimate the integrals $\int\limits_S |\Phi_\sigma(y,x)|\,ds_y$, $\int\limits_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_1}\right| ds_y$ and $\int\limits_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_2}\right| ds_y$ on a smooth curve S.

Taking into account equality (3.5) and inequality (3.6), we have

$$\int\limits_S |\Phi_\sigma(y,x)|\,ds_y \le C_\rho(\lambda,x)\sigma e^{\sigma(h-x_2)},\ \sigma>1,\ x\in G. \tag{3.17}$$

Using (3.9) and inequality (3.11), we have

$$\int\limits_S \left|\frac{\partial\Phi_\sigma(y,x)}{\partial y_1}\right| ds_y \le C_\rho(\lambda,x)\sigma e^{\sigma(h-x_2)},\ \sigma>1,\ x\in G. \tag{3.18}$$

Similarly using (3.10) and inequality (3.11), we obtain

$$\int_S \left| \frac{\partial \Phi_\sigma(y,x)}{\partial y_2} \right| ds_y \leq C_\rho(\lambda, x) \sigma\, e^{\sigma(h-x_2)},\ \sigma > 1,\ x \in G. \tag{3.19}$$

From (3.17) - (3.19), we obtain

$$\left| \int_S N_\sigma(y,x) U(y) ds_y \right| \leq C_\rho(\lambda, x) \sigma \delta\, e^{\sigma(h-x_2)},\ \sigma > 1,\ x \in G. \tag{3.20}$$

The following is known

$$\left| \int_T N_\sigma(y,x) U(y) ds_y \right| \leq C_\rho(\lambda, x) \sigma\, e^{-\sigma x_2},\ \sigma > 1,\ x \in G. \tag{3.21}$$

Now taking into account (3.16), (3.20) - (3.21), we have

$$|U(x)| \leq \frac{C_\rho(\lambda, x)\sigma}{2} (\delta\, e^{\sigma h} + 1) e^{-\sigma x_2},\ \sigma > 1,\ x \in G.$$

Choosing σ from equality

$$\sigma = \frac{1}{h} \ln \frac{1}{\delta}, \tag{3.22}$$

we obtain the inequality (3.15). Theorem 3.2 is proved. □

Let $U(y) \in A_\rho(G)$ and together with $U(y)$ on S it is given its approximation $f_\delta(y)$, respectively, with an error $0 < \delta < 1$, $\max_S |U(y) - f_\delta(y)| \leq \delta$.

We set

$$U_{\sigma(\delta)}(x) = \int_S N_\sigma(y,x) f_\delta(y) ds_y,\ x \in G. \tag{3.23}$$

The following is valid

Theorem 3.3. *Let $U(y) \in A_\rho(G)$ it satisfy condition (3.2) on a part of the plane $y_2 = 0$.*

Then the following estimate holds

$$\left| U(x) - U_{\sigma(\delta)}(x) \right| \leq C_\rho(\lambda, x) \sigma \delta^{\frac{x_2}{h}},\ \sigma > 1,\ x \in G. \tag{3.24}$$

Proof. From the integral formulas (2.23) and (3.23), we have

$$U(x) - U_{\sigma(\delta)}(x) = \int_S N_\sigma(y,x) \left\{ U(y) - f_\delta(y) \right\} ds_y + \int_T N_\sigma(y,x) U(y) ds_y.$$

Now, repeating the proof of Theorems 3.1 and 3.2, we obtain

$$\left| U(x) - U_{\sigma(\delta)}(x) \right| \leq \frac{C(x)\sigma}{2} (\delta\, e^{\sigma h} + 1)\, e^{-\sigma x_2}.$$

Hence, choosing σ from (3.22), we obtain (3.24). □

Corollary 3.2. The limiting equality

$$\lim_{\delta \to 0} U_{\sigma(\delta)}(x) = U(x),$$

holds uniformly on each compact set in the domain G.

Thus, the functional $U_{\sigma(\delta)}(x)$ determines the regularization of the solution of problem (2.23), (3.23).

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