Journal of Universal Mathematics
Vol.2 No.2 pp.166-174 (2019)
ISSN-2618-5660

ON WEAK BIHARMONIC GENERALIZED ROTATIONAL SURFACE IN $\mathbb{E}^4$

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Abstract. In the present paper we consider weak biharmonic rotational surfaces in Euclidean 4-space $\mathbb{E}^4$. We have proved that the general rotational surface of parallel mean curvature vector field is weak biharmonic then either it is minimal or a constant mean curvature. Further, we show that if Vranceanu surface of constant mean curvature is weak-biharmonic then it is a Clifford torus in $\mathbb{E}^4$.

1. Introduction

The biharmonic submanifolds, i.e., the submanifolds satisfying the condition $\Delta\overrightarrow{H} = 0$ have been investigated in [15], [13], [14] and [18]. It is known that in many cases they reduce to minimal submanifolds, but in pseudo-Euclidean spaces they do not (see [9]). Further, biharmonic hypersurfaces in Riemannian manifolds considered by Y-L, Ou in [23]. For spherical case see also [6]. Recently, the researchers considered biharmonic submanifolds in different ambient spaces [2] and [20]. See also [24] for some recent progress of biharmonic submanifolds. The proper-biharmonic (i.e. non-harmonic) submanifolds of a real space form were extensively studied in [3], [4] and [8].

There is another possibility to introduce $\Delta^D\overrightarrow{H}$ by means of the normal curvature Δ, and to define the submanifolds with *harmonic mean curvature vector* (or shortly *weak biharmonic submanifolds*), as those satisfying $\Delta^D\overrightarrow{H} = 0$. The submanifold is said to be 2−parallel if the third fundamental form $\alpha_3 = \overline{\nabla}h$ of M is parallel, i.e., $\overline{\nabla}\alpha_3 = 0$ holds identically. The first results about 2−parallel submanifolds were obtained for curves in $\mathbb{E}^m$. In [5] all curves with harmonic mean curvature vector in $\mathbb{E}^m$ are classified. In [19] the second named author and at all found some results related with weak biharmonic submanifolds and 2−parallelity.

The rotational embeddings are introduced first by N.H. Kuiper in 1970 [21]. In [10], Cole studied with the general theory of rotations 4−dimensional Euclidean space in $\mathbb{E}^4$. Later, Moore considered the *general rotational surface* M in $\mathbb{E}^4$ [22]. However, the rotational surfaces in $\mathbb{E}^4$ with constant curvatures are studied in [26] and [11]. A special case of Moore consideration is the general rotational surface (see, [1], [16], [17] and [27]). Moreover, *Vranceanu surface* are the interesting examples of general rotational surface in $\mathbb{E}^4$ [25].

Key words and phrases. Biharmonic, Rotational surface, Weak biharmonic.

In the present paper we give the necessary and sufficient conditions for general rotational surfaces to become weak biharmonic. Further, we obtain some results for weak biharmonic Vranceanu surface. We also give some examples.

2. Preliminaries

In the present section we recall definitions and results of [7]. Let M be a $n-$dimensional Euclidean submanifold of $\mathbb{E}^m$. Consider the orthonormal frame field $\{e_1, ..., e_n, e_{n+1}, ..., e_m\}$ in $\mathbb{E}^m$ such that $e_1, e_2, ...,$ e_n, are tangent to the M and $e_{n+1}, e_{n+2} ...,$ e_m normal to M. Then the Gaussian and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + h(e_i, e_j);\ 1 \leq i, j \leq n, \tag{2.1}$$

$$\widetilde{\nabla}_{e_i} e_\alpha = -A_{e_\alpha} e_i + D_{e_i} e_\alpha;\ n+1 \leq \alpha \leq m, \tag{2.2}$$

where h is the *second fundamental form*, D is the normal connection and A_{e_α} the *shape operator* in the direction of e_α.

The *mean curvature vector field* $\overrightarrow{H}$ of M is defined by

$$\overrightarrow{H} = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \tag{2.3}$$

The norm of the mean curvature vector $\alpha = \left\| \overrightarrow{H} \right\|$ is called *mean curvature* of M. Recall that a submanifold is said to be *minimal* if its mean curvature vanishes identically. The submanifolds with constant mean curvature are called *$CMC-$submanifolds*.

The Laplacian on the normal bundle of M is given by

$$\Delta^D = -\sum_{i=1}^{n} \left(D_{e_i} D_{e_i} - D_{\nabla_{e_i} e_i} \right). \tag{2.4}$$

where D is the normal connection of M.

3. Weak Biharmonic Submanifolds in Euclidean Spaces

Let M be a $n-$dimensional smooth submanifold in $m-$dimensional Euclidean space $\mathbb{E}^m$. For the local orthonormal frame field $\{e_1, ..., e_n, e_{n+1}, ..., e_m\}$ define $\overrightarrow{H} = \alpha e_{n+1}$. We can compute the Laplacian of $\overrightarrow{H}$ with respect to the normal connection D of M by

$$\begin{aligned} \Delta^D \overrightarrow{H} &= \sum_{i=1}^{n} \left(D_{\nabla_{e_i} e_i} \overrightarrow{H} - D_{e_i} D_{e_i} \overrightarrow{H} \right) \\ &= \sum_{i=1}^{n} \left(D_{\nabla_{e_i} e_i} (\alpha e_{n+1}) - D_{e_i} D_{e_i} (\alpha e_{n+1}) \right) \qquad (3.1) \\ &= \sum_{i=1}^{n} \left\{ (\Delta \alpha) e_{n+1} + \alpha D_{\nabla_{e_i} e_i} e_{n+1} - 2 e_i [\alpha] D_{e_i} e_{n+1} - \alpha D_{e_i} D_{e_i} e_{n+1} \right\}. \end{aligned}$$

Hence, multiplying both sides of equation (3.1) by e_{n+1} one can get

$$\left\langle \Delta^D \overrightarrow{H}, e_{n+1} \right\rangle = \Delta \alpha + \alpha \left\| D e_{n+1} \right\|^2. \tag{3.2}$$

We give the following well-known definitions;

Definition 3.1. Let M be a $n-$dimensional smooth submanifold in $m-$dimensional Euclidean space $\mathbb{E}^m$. If the condition

$$D\overrightarrow{H} = 0, \tag{3.3}$$

holds then the submanifold M is called $H-$parallel [7].

The complete classification of $H-$parallel surfaces in Euclidean $m-$space, $m \geq 4$, was obtained by [7] and Yau [28] in the following result;

Theorem 3.2. *A surface M of a Euclidean $m-$space $\mathbb{E}^m$ is $H-$parallel if and only if it is one of the following surfaces:*
i) a minimal surface of $\mathbb{E}^m$;
ii) a minimal surface of a hypersphere of $\mathbb{E}^m$;
iii) a surface of $\mathbb{E}^3$ with constant mean curvature;
iv) a surface of constant mean curvature lying in a hypersphere of an affine $4-$subspace of $\mathbb{E}^m$.

Definition 3.3. Let M be a $n-$dimensional smooth submanifold in $m-$dimensional Euclidean space $\mathbb{E}^m$. If the condition

$$\Delta^D \overrightarrow{H} = 0, \tag{3.4}$$

holds then the submanifold M is said to have *harmonic mean curvature vector field* [5]. These kind of submanifolds shortly called *weak biharmonic* [19].

We obtain the following result.

Theorem 3.4. *Let M be a n-dimensional smooth CMC-submanifold in m-dimensional Euclidean space $\mathbb{E}^m$. If M is weak biharmonic submanifold then either it is minimal or a $H-$parallel submanifold of $\mathbb{E}^m$.*

Proof. Suppose that M is a weak biharmonic submanifolds of $\mathbb{E}^m$ then from the equation (3.2)

$$\Delta\alpha + \alpha \left\| De_{n+1} \right\|^2 = 0,$$

holds. Since the mean curvature is a constant function on M then $\Delta\alpha = 0$ implies $\alpha \left\| De_{n+1} \right\| = 0$. Consequently, we have two possible cases; $\alpha = 0$, or $\left\| De_{n+1} \right\| = 0$ and $\alpha \neq 0$. In the first case M is a minimal submanifold. Further, substituting $\overrightarrow{H} = \alpha e_{n+1}$ into the second condition one can show that M is $H-$parallel. This completes the proof of the theorem. ∎

4. Weak Biharmonic General Rotational Surfaces in $\mathbb{E}^4$

Let M be a general rotational surface in $\mathbb{E}^4$ given with the parametrization

$$x(u, v) = (f(u)\cos cv, f(u)\sin cv, g(u)\cos dv, g(u)\sin dv), \tag{4.1}$$

where $\gamma(u) = (f(u), g(u))$ is the meridian curve of the rotation. The orthonormal frame field of M is given by

$$
\begin{aligned}
e_1 &= \frac{\partial}{\psi(u)\partial u}, \\
e_2 &= \frac{\partial}{\phi(u)\partial v}, \\
e_3 &= \frac{1}{\psi(u)}(g'(u)\cos cv, g'(u)\sin cv, -f'(u)\cos dv, -f'(u)\sin dv), \\
e_4 &= \frac{1}{\phi(u)}(-dg(u)\sin cv, dg(u)\cos cv, cf(u)\sin dv, -cf(u)\cos dv).
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
\psi(u) &= \sqrt{(f'(u))^2 + (g'(u))^2}, \\
\phi(u) &= \sqrt{c^2 f^2(u) + d^2 g^2(u)}.
\end{aligned}
\tag{4.3}
$$

With respect to this frame the Gaussian and Weingarten formulas (2.1)-(2.2) of M look like

$$
\begin{aligned}
\tilde{\nabla}_{e_1} e_1 &= \frac{\kappa}{\psi^3} e_3, \\
\tilde{\nabla}_{e_1} e_2 &= \frac{\rho}{\psi\phi^2} e_4, \\
\tilde{\nabla}_{e_2} e_1 &= \frac{\lambda}{\psi\phi^2} e_2 + \frac{\rho}{\psi\phi^2} e_4, \\
\tilde{\nabla}_{e_2} e_2 &= -\frac{\lambda}{\psi\phi^2} e_1 + \frac{\beta}{\psi\phi^2} e_3,
\end{aligned}
\tag{4.4}
$$

and

$$
\begin{aligned}
\tilde{\nabla}_{e_1} e_3 &= \frac{\kappa}{\psi^3} e_1, \\
\tilde{\nabla}_{e_1} e_4 &= -\frac{\rho}{\psi\phi^2} e_2, \\
\tilde{\nabla}_{e_2} e_3 &= -\frac{\beta}{\psi\phi^2} e_2 + \frac{\delta}{\psi\phi^2} e_4, \\
\tilde{\nabla}_{e_2} e_4 &= -\frac{\rho}{\psi\phi^2} e_1 - \frac{\delta}{\psi\phi^2} e_3.
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
\kappa &= f''g' - f'g'', \\
\lambda &= c^2 ff' + d^2 gg', \\
\beta &= c^2 f'g - d^2 fg', \\
\rho &= cd\,(f'g - fg'), \\
\delta &= cd\,(ff' + gg')
\end{aligned}
\tag{4.6}
$$

are the smooth functions on M.

Consequently, by the use of (4.4) we obtain the shape operator matrices as follows;

$$A_{e_3} = \begin{pmatrix} \frac{\kappa}{\psi^3} & 0 \\ 0 & \frac{\beta}{\psi\phi^2} \end{pmatrix}$$
$$A_{e_4} = \begin{pmatrix} 0 & \frac{\rho}{\psi\phi^2} \\ \frac{\rho}{\psi\phi^2} & 0 \end{pmatrix}.$$

So, the Gaussian curvature and mean curvature vector of M are given by

$$\begin{aligned} K &= det\,(A_{e_3}) + det\,(A_{e_4}) \\ &= \tfrac{1}{\psi^2\phi^2}\left(\kappa\beta - \tfrac{\rho^2}{\phi^2}\right). \end{aligned} \tag{4.7}$$

$$\begin{aligned} \overrightarrow{H} &= \tfrac{1}{2}\left(tr\,(A_{e_3})\, e_3 + tr\,(A_{e_4})\, e_4\right) \\ &= \tfrac{1}{2\psi}\left(\tfrac{\kappa}{\psi^2} + \tfrac{\beta}{\phi^2}\right) e_3, \end{aligned} \tag{4.8}$$

respectively. So, we get

$$\alpha = \left\|\overrightarrow{H}\right\| = \frac{\phi^2\kappa + \psi^2\beta}{2\psi^3\phi^2}.$$

Definition 4.1. The normal curvature of the surface $M \subset \mathbb{E}^m$ is defined by

$$K_N = \left\{ \sum_{1=\alpha<\beta}^{m-2} \left\langle R^{\perp}(e_1, e_2)N_\beta, N_\alpha \right\rangle^2 \right\}^{\frac{1}{2}}, \tag{4.9}$$

(see,[12]), where $R^{\perp}$ is the curvature tensor of the normal bundle of M defined by

$$R^{\perp}(e_1, e_2)N_\beta = h(e_1, A_{N_\beta} e_2) - h(e_2, A_{N_\beta} e_1).$$

The normal curvature of the general rotational surface M is given by

$$\begin{aligned} K_N &= h^3_{12}(h^4_{22}h^4_{11}) + h^4_{12}(h^3_{22}h^3_{11}) \\ &= \frac{\rho}{\psi^2\phi^2}\left(\frac{\kappa}{\psi^2} - \frac{\beta}{\phi^2}\right). \end{aligned}$$

We have the following well-known result.

Proposition 1. [16] Let M be a generalized rotational surface given with the parametrization (4.1). If M is a minimal surface with vanishing normal curvature then it is a part of the plane with the parametrization

$$x(u,v) = (ucosv, usinv, c_0ucosv, \varepsilon c_0usinv),\ u > 0,\ \varepsilon = \frac{c}{d} = \pm 1. \tag{4.10}$$

We obtain the following results;

Theorem 4.2. *Let M be a generalized rotational surface given with the parametrization (4.1). If M is $H-$parallel then either it is minimal or a $CMC-$surface given with the parametrization*

$$x(u,v) = \left(r_0\cos\left(\frac{u}{r_0}\right)\cos cv, r_0\cos\left(\frac{u}{r_0}\right)\sin cv, r_0\sin\left(\frac{u}{r_0}\right)\cos dv, r_0\sin\left(\frac{u}{r_0}\right)\sin dv\right) \tag{4.11}$$

which is a minimal surface in $\mathbb{S}^3(r_0) \subset \mathbb{R}^4$.

Proof. Differentiating (4.8) with respect to e_1, e_2 and using (4.5) a straight-forward computation gives

$$\begin{aligned} D_{e_1}\overrightarrow{H} &= \tfrac{\alpha'}{\psi}e_3 \\ D_{e_2}\overrightarrow{H} &= \tfrac{\alpha\delta}{\psi\phi^2}e_4. \end{aligned} \tag{4.12}$$

Assume that M is a $H-$parallel surface then (4.12) implies that either $\alpha = 0$ or $\alpha' = 0$ and $\delta = 0$. For the first case M is minimal surface satisfying

$$\phi^2\kappa + \psi^2\beta = 0,$$

where ϕ, ψ, κ and β are smooth functions defined in (4.3) and (4.6). For the second case M is a $CMC-$surface satisfying $ff' + gg' = 0$, i.e., $f^2 + g^2 = a^2$, where a is a positive real number. Hence, the meridian curve γ is an open part of a circle parametrized by

$$\gamma(u) = \left(a\cos\left(\frac{u}{a}\right), a\sin\left(\frac{u}{a}\right)\right)$$

(see,[16], pp. 77). Therefore, M is an open part of the surface given by (4.11). $\square$

Similarly, Differentiating (4.12) with respect to e_1, e_2 and using (4.5) we obtain

$$\begin{aligned} D_{e_1}D_{e_1}\overrightarrow{H} &= \tfrac{1}{\psi}\left(\tfrac{\alpha'}{\psi}\right)'e_3 \\ D_{e_2}D_{e_2}\overrightarrow{H} &= -\alpha\left(\tfrac{\delta}{\psi\phi^2}\right)^2 e_3 \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} D_{\nabla_{e_1}e_1}\overrightarrow{H} &= 0 \\ D_{\nabla_{e_2}e_2}\overrightarrow{H} &= -\tfrac{\lambda}{\psi\phi^2}D_{e_1}\overrightarrow{H} \\ &= -\tfrac{\lambda\alpha'}{\psi^2\phi^2}e_3. \end{aligned} \tag{4.14}$$

The Laplacian of the mean curvature vector can be expressed as

$$\Delta^D\overrightarrow{H} = D_{\nabla_{e_1}e_1}H + D_{\nabla_{e_2}e_2}H - D_{e_1}D_{e_1}H - D_{e_2}D_{e_2}H. \tag{4.15}$$

So, substituting (4.13) and (4.14) into (4.15) we obtain the Laplacian of the mean curvature vector $\overrightarrow{H}$ as follows

$$\Delta^D\overrightarrow{H} = \left\{\frac{\alpha\delta^2 + \phi^2\alpha'\left(\frac{\phi^2\psi'}{\psi} - \lambda\right) - \alpha''\phi^4}{\psi^2\phi^4}\right\}e_3 \tag{4.16}$$

We obtain the following results.

Theorem 4.3. *Let M be a generalized rotational surface given with the parametrization (4.1). Then M is weak biharmonic if and only if*

$$\alpha\delta^2 + \phi^2\alpha'\left(\frac{\phi^2\psi'}{\psi} - \lambda\right) - \alpha''\phi^4 = 0, \tag{4.17}$$

holds, where α is the mean curvature and δ, ψ, λ are the smooth functions on M defined as before.

Corollary 4.4. Let M be a generalized rotational surface of constant mean curvature. If M is weak biharmonic then either it is minimal or a $CMC-$surface given with the parametrization (4.11).

Proof. Let M be a generalized rotational surface of constant mean curvature. If M is weak biharmonic then by (4.17) $\alpha\delta^2 = 0$. So we have the possibilities; $\alpha = 0$ or $ff' + gg' = 0$. These imply that either M is minimal or a $CMC-$surface given with the parametrization (4.11). □

Definition 4.5. The Vranceanu surface in $\mathbb{E}^4$ is defined by the following parametrization;

$$f(u) = r(u)\cos u,\ g(u) = r(u)\sin u,\ a = b = 1, \tag{4.18}$$

where $r(u)$ is a real valued non-zero function [25]. If $r(u)$ is a real constant then the Vranceanu surface turns into a Clifford torus i.e., it is the product of two plane circles with the same radius (see, [27]).

For Vranceanu surface one can get the following results;

Proposition 2. Let M be a Vranceanu surface in $\mathbb{E}^4$ given with the parametrization (4.18). Then the Gaussian curvature and mean curvature vector of M are given by

$$K = K_N = \frac{(r')^2 - rr''}{\left(r^2 + (r')^2\right)^2} \tag{4.19}$$

and

$$\overrightarrow{H} = \frac{rr'' - 3(r')^2 - 2r^2}{2\left(r^2 + (r')^2\right)^{3/2}} e_3. \tag{4.20}$$

respectively.

Proposition 3. Let M be a Vranceanu surface given with the parametrization (4.18). Then the Laplacian of the mean curvature vector is given by

$$\Delta^D \overrightarrow{H} = \left\{\eta^2\varphi^2\alpha - \eta\left(\eta' + \eta\varphi\right)\alpha' - \eta^2\alpha''\right\} e_3 \tag{4.21}$$

where

$$\eta = \frac{1}{\sqrt{r^2 + (r')^2}},\ \varphi = \frac{r'}{r} \tag{4.22}$$

are smooth functions on M.

Corollary 4.6. Let M be a Vranceanu rotational surface of constant mean curvature. If M is weak biharmonic then it is a Clifford torus in $\mathbb{E}^4$.

5. Conclusion

Harmonic surfaces are one of the important subject in differential geometry. They are the generalization of minimal surfaces. Recently, weak biharmonic surfaces are considered with some researchers. In the present study we consider general rotational surfaces in $\mathbb{E}^4$ satisfying the weak biharmonic condition. We also give some results related with Vranceanu surfaces in $\mathbb{E}^4$. It is possible to consider other kind of surfaces in higher dimensional Euclidean spaces satisfying weak biharmonic property.

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