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Research Article

Intuitionistic fuzzy initial value problems - an application

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Abstract

In this paper, by using the properties of α and β cuts of intuitionistic fuzzy numbers, we have firstly proposed a method to find the general solution of the second order initial value problem with intuitionistic fuzzy initial values under intuitionistic Zadeh's extension principle interpretation. Then we have given some numerical examples for the proposed method.

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1. Introduction

To utilize uncertainty in engineering and mathematical models, L. A. Zadeh coined the fuzzy set idea in 1965 [36]. In this idea, every element in a set is given with a function $\mu(x) : X \to [0, 1]$, called membership function, to explain the full membership, non-membership and partial membership of the elements to the set. Since this approach was considered as one of the powerful tools to handle vagueness, it has been applied to various fields of science and engineering [1–3, 14, 16–24, 26, 27, 33, 34] In fuzzy set theory, $1 - \mu$ is considered as the non-membership function of an element to the set. That is why, the sum of the membership function and non-membership function of an element is always one. However, due to inadequate or incomplete information in models, there may be uncertainty in the membership or non-membership of an element. Hence, some extensions of fuzzy set theory were introduced [4, 25, 28]. One of these extensions is Atanassov's intuitionistic fuzzy set theory [4].

In 1986, Atanassov [4] introduced the intuitionistic fuzzy set concept. In his concept, every element in a set is accompanied with a membership function $\mu(x) : X \to [0, 1]$ and a non-membership function $\nu(x) : X \to [0, 1]$ such that the sum of both is less than or equal to 1. Hence the difference $1 - (\mu + \nu)$, called hesitation degree, is used to express the lack of knowledge and imprecision in a model. Later, Atanassov unveil some important intrinsic properties of intuitionistic fuzzy sets in his further researches [5–12].

As a tool to explain vagueness, fuzzy initial value problems have a significant importance in physics, biology, engineering and the other fields of science. Hence many researches

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were done in this topic. For example, in population models [1, 14], civil engineering [24], computational biology [21, 26] and etc. [13, 22, 24, 31].

Recently intuitionistic fuzzy initial value problems (IFIVPs) have been studied as well [29, 30, 32]. In these papers, mostly first order intuitionistic initial value problems were studied. Although the second order linear differential equations have a variety of applications such as the motion of a particle, the vibration of a spring and the flow of electrons in an electric circuit [35], we have not found any studies in second order intuitionistic fuzzy initial value problems in our literature search.

Generally in the application of fuzzy numbers, the arithmetic operations on fuzzy numbers are performed either by Zadeh's extension principle or by α-cuts of fuzzy numbers [15, 22]. In this paper we will propose a method to find the solution of second order intuitionistic fuzzy initial value problems under intuitionistic Zadeh's Extension Principle [11] by performing interval arithmetic operations on α and β cuts of intuitionistic fuzzy numbers.

This article is organized as follows. In Section 2, we introduce some basic definitions and theorems which we will use in further sections. In Section 3, we introduced our proposed method and give some numerical examples. Finally conclusions are given in Section 4.

2. Preliminaries

Definition 2.1. [4] Let $A \subseteq X$ and let $\mu_A(x) : X \to [0,1]$, $\nu_A(x) : X \to [0,1]$ be two functions such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. The set

$$\bar{A}^i = \{(x, \mu_A(x), \nu_A(x)) : x \in X, \mu_A(x), \nu_A(x) : X \to [0,1]\}$$

is called an intuitionistic fuzzy set of X.
Here $\mu_A(x)$ is called membership function and $\nu_A(x)$ is called non-membership function.

We will denote set of all intuitionistic fuzzy sets of X by $IF(X)$.

Definition 2.2. [4] Let $\bar{A}^i \in IF(X)$. The set

$$A(\alpha, \beta) = \{x \in X : \alpha,\ \beta \in [0,1];\ \mu_A(x) \geq \alpha,\ \nu_A(x) \leq \beta,\ 0 \leq \alpha + \beta \leq 1\}$$

is called the (α, β)-cut of the intuitionistic fuzzy set $\bar{A}^i$.

Definition 2.3. [4] Let $\bar{A}^i \in IF(X)$. The α-cut of $\bar{A}^i$ is defined as follows:
For $\alpha \in (0,1]$

$$A(\alpha) = \{x \in A : \mu_A(x) \geq \alpha\},$$

and for $\alpha = 0$

$$A(0) = cl\Big(\bigcup_{\alpha \in (0,1]} A(\alpha)\Big).$$

Here and after we will use "cl" to denote the closure of a set.

Definition 2.4. [4] Let $\bar{A}^i \in IF(X)$. The β-cut of $\bar{A}^i$ is defined as follows:
For $\beta \in [0,1)$

$$A^*(\beta) = \{x \in A : \nu_A(x) \leq \beta\},$$

and for $\beta = 1$

$$A^*(1) = cl\Big(\bigcup_{\beta \in [0,1)} A^*(\beta)\Big).$$

Theorem 2.5. [4] *Let $\bar{A}^i \in IF(X)$. Then $A(\alpha, \beta) = A(\alpha) \cap A^*(\beta)$ holds.*

Definition 2.6. An intuitionistic fuzzy set $\bar{A}^i \in IF(\mathbb{R}^n)$ satisfying the following properties is called an intuitionistic fuzzy number in $\mathbb{R}^n$

(1) $\bar{A}^i$ is a normal set, i.e., $\exists\, x_0 \in \mathbb{R}^n$ such that $\mu_A(x_0) = 1$ and $\nu_A(x_0) = 0$
(2) $A(0)$ and $A^*(1)$ are bounded sets in $\mathbb{R}^n$.
(3) $\mu_A : \mathbb{R}^n \to [0,1]$ is an upper semi-continuous function, i.e., $\forall k \in [0,1], (\{x \in A : \mu_A(x) < k\})$ is an open set.
(4) $\nu_A : \mathbb{R}^n \to [0,1]$ is a lower semi-continuous function, i.e., $\forall k \in [0,1]\{x \in A : \nu_A(x) > k\}$ is an open set.
(5) The membership function $\mu_A(x)$ is quasi-concave, i.e., $\forall \lambda \in [0,1], \forall x,\ y \in \mathbb{R}^n$

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min(\mu_A(x), \mu_A(y)).$$

(6) The non-membership function $\nu_A(x)$ is quasi-convex; i.e.,$\forall \lambda \in [0,1], \forall x,\ y \in \mathbb{R}^n$

$$\nu_A(\lambda x + (1-\lambda)y) \leq \max(\nu_A(x), \nu_A(y)); \forall \lambda \in [0,1].$$

We will denote the set of all intuitionistic fuzzy numbers of $\mathbb{R}^n$ by $IF_N(\mathbb{R}^n)$.

Remark 2.7. In some articles [29,30] the non-membership function $\nu_A(x)$ is called a *fuzzy concave* function in the definition of intuitionistic fuzzy numbers. However, in our view, it is more suitable to address it as *quasi-convex* because the inequality $\nu_A(x)(\lambda x_1 + (1-\lambda)x_2) \leq \max(\nu_A(x_1), \nu_A(x_2))$ is already an equivalence for quasi-convex functions.

Definition 2.8. [29] A triangular intuitionistic fuzzy number (TIFN) $\bar{A}^i \in IF_N(\mathbb{R})$ is defined with the following membership and non-membership functions:

$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1};\ a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2};\ a_2 \leq x \leq a_3 \\ 0;\ \text{otherwise,} \end{cases}$$

and

$$\nu_A(x) = \begin{cases} \frac{a_2-x}{a_2-a_1^*};\ a_1^* \leq x \leq a_2 \\ \frac{x-a_2}{a_3^*-a_2};\ a_2 \leq x \leq a_3^* \\ 1;\ \text{otherwise.} \end{cases}$$

Here $a_1^* \leq a_1 \leq a_2 \leq a_3 \leq a_3^*$ and it is denoted by $\bar{A}^i = (a_1, a_2, a_3; a_1^*, a_2, a_3^*)$.

Now we will give fundamental theorems characterizing α and β-cuts of intuitionistic fuzzy numbers based on the characterization and stacking theorems in [15, 22] for fuzzy numbers.

Theorem 2.9. *Let $\bar{A}^i \in IF_N(\mathbb{R}^n)$ and $\alpha, \beta \in [0,1]$ such that its α and β cuts given by $A(\alpha) = \{x \in \mathbb{R}^n : \mu_A(x) \geq \alpha\}$ and $A^*(\beta) = \{x \in \mathbb{R}^n : \nu_A(x) \leq \beta\}$. Then the followings hold:*

(1) *$A(\alpha)$ and $A^*(\beta)$ are non-empty compact and convex sets in $\mathbb{R}^n$*
(2) *If $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ then $A(\alpha_2) \subseteq A(\alpha_1)$.*
(3) *If $0 \leq \beta_1 \leq \beta_2 \leq 1$ then $A^*(\beta_1) \subseteq A^*(\beta_2)$.*
(4) *If (α_n) is a non-decreasing sequence converging to α then*

$$\bigcap_{n=1}^{\infty} A(\alpha_n) = A(\alpha).$$

(5) *If (β_n) is a non-increasing sequence converging to β then*

$$\bigcap_{n=1}^{\infty} A^*(\beta_n) = A^*(\beta).$$

(6) *If (α_n) is a non-increasing sequence converging to 0 then*

$$cl(\bigcup_{n=1}^{\infty} A(\alpha_n)) = A(0).$$

(7) *If* (β_n) *is a non-decreasing sequence converging to* 1 *then*

$$cl(\bigcup_{n=1}^{\infty} A^*(\beta_n)) = A^*(1).$$

Corollary 2.10. *Let* $\bar{A}^i \in IF_N(\mathbb{R})$*. Then* $A(\alpha)$ *and* $A^*(\beta)$ *are closed and bounded intervals such that*

$$A(\alpha) = [A_1(\alpha), A_2(\alpha)],$$

and

$$A^*(\beta) = [A_1^*(\beta), A_2^*(\beta)].$$

Here

$$\begin{aligned} A_1(\alpha) &= \inf\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}, \\ A_2(\alpha) &= \sup\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}, \\ A_1^*(\beta) &= \inf\{x \in \mathbb{R} : \nu_A(x) \leq \beta\}, \\ A_2^*(\beta) &= \sup\{x \in \mathbb{R} : \nu_A(x) \leq \beta\}. \end{aligned}$$

Now we will give Zadeh's extension principle for intuitionistic fuzzy environment.

Definition 2.11. [11] Let X and Y be two sets and $f : X \to Y$ be a function. Let $\bar{A}^i$ be an intuitionistic fuzzy set in X. Then $f(\bar{A}^i)$ is an intuitionistic fuzzy set in Y such that for every $y \in Y$

$$\mu_{f(\bar{A}^i)}(y) = \begin{cases} \sup\{\mu_A(x) : f(x) = y\}; & y \in f(X) \\ 0; & y \notin f(X), \end{cases},$$

and

$$\nu_{f(\bar{A}^i)}(y) = \begin{cases} \inf\{\nu_A(x) : f(x) = y\}; & y \in f(X) \\ 1; & y \notin f(X). \end{cases}$$

3. A method for IFIVPs

In this section, we will propose a method to find the solution of second order intuitionistic fuzzy initial value problems under intuitionistic Zadeh's extension principle interpretation [11]. Under this interpretation, firstly we will solve the crisp initial value problem. Then, we will get the solution of the intuitionistic fuzzy initial value problem from the crisp solution with the help of intuitionistic Zadeh's extension principle. To prevent the switching of endpoints of α and β cuts, we will apply Heaviside (step) function during the interval operations on α and β cuts.

3.1. The method

Now, we will consider the following type of intuitionistic fuzzy initial value problem:

$$\begin{cases} y''(x) + a_1 y'(x) + a_2 y(x) = \sum_{j=1}^{r} \bar{b}_j^i g_j(x) \\ y(0) = \bar{\gamma}_0^i; y'(0) = \bar{\gamma}_1^i. \end{cases} \tag{3.1}$$

Here a_1 and a_2 are crisp constants and g_j $(j = 1, ..., r)$ are continuous functions on the interval $[0, \infty)$. The initial conditions $\bar{\gamma}_0^i$, $\bar{\gamma}_1^i$ and forcing coefficients $\bar{b}_j^i$ $(j = 1, \ldots, r)$ are intuitionistic fuzzy numbers.

Theorem 3.1. *Let* $\bar{Y}^i(x)$ *be the solution of the intuitionistic initial value problem in (3.1) obtained by intuitionistic Zadeh's extension principle. Let* α *and* β *cuts of* $\bar{Y}^i(x)$, $\bar{b}_j^i$ $(j : 1, .., r)$ *and* $\bar{\gamma}_k^i$ $(k : 0, 1)$ *be given by* $[Y_1(x,\alpha), Y_2(x,\alpha)]$, $[Y_1^*(x,\beta), Y_2^*(x,\beta)]$; $[b_{j1}(\alpha), b_{j2}(\alpha)]$, $[b_{j1}^*(\beta), b_{j2}^*(\beta)]$ *and* $[\gamma_{k1}(\alpha), \gamma_{k2}(\alpha)]$, $[\gamma_{k1}^*(\beta), \gamma_{k2}^*(\beta)]$*, respectively. Then the* α *and* β *cuts of the solution can be determined as follows:*

$$\begin{cases} Y_1(x,\alpha) = \sum_{k=0}^{1}[\gamma_{k2}(\alpha) - (\gamma_{k2}(\alpha) - \gamma_{k1}(\alpha))\theta(A_k(x))]A_k(x) \\ \quad + \sum_{j=1}^{r}[b_{j2}(\alpha) - (b_{j2}(\alpha) - b_{j1}(\alpha))\theta(B_j(x))]B_j(x), \\ Y_2(x,\alpha) = \sum_{k=0}^{1}[\gamma_{k1}(\alpha) + (\gamma_{k2}(\alpha) - \gamma_{k1}(\alpha))\theta(A_k(x))]A_k(x) \\ \quad + \sum_{j=1}^{r}[b_{j1}(\alpha) + (b_{j2}(\alpha) - b_{j1}(\alpha))\theta(B_j(x))]B_j(x), \\ Y_1^*(x,\beta) = \sum_{k=0}^{1}[\gamma_{k2}^*(\beta) - (\gamma_{k2}^*(\beta) - \gamma_{k1}^*(\beta))\theta(A_k(x))]A_k(x) \\ \quad + \sum_{j=1}^{r}[b_{j2}^*(\beta) - (b_{j2}^*(\beta) - b_{j1}^*(\beta))\theta(B_j(x))]B_j(x), \\ Y_2^*(x,\beta) = \sum_{k=0}^{1}[\gamma_{k1}^*(\beta) + (\gamma_{k2}^*(\beta) - \gamma_{k1}^*(\beta))\theta(A_k(x))]A_k(x) \\ \quad + \sum_{j=1}^{r}[b_{j1}^*(\beta) + (b_{j2}^*(\beta) - b_{j1}^*(\beta))\theta(B_j(x))]B_j(x). \end{cases} \tag{3.2}$$

Here $A_k(x)$ and $B_j(x)$ are continuous functions of x and θ is Heaviside function.

Proof. We will firstly solve the following crisp initial value problem related to the intuitionistic fuzzy initial value problem in (3.1)

$$y''(x) + a_1 y'(x) + a_2 y(x) = \sum_{j=1}^{r} b_j g_j(x), \tag{3.3}$$

$$y(0) = \gamma_0;\; y'(0) = \gamma_1. \tag{3.4}$$

Here a_1, a_2, γ_0, γ_1 and $b_j (j = 1, \ldots, r)$ are real (crisp) numbers. The general solution of the differential equation in (3.3) can be written as:

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) + \sum_{j=1}^{r} b_j G_j(x), \tag{3.5}$$

where c_1 and c_2 are arbitrary constants and $G_j(x)$ $(j = 1, ..., r)$ are the particular solutions for each of the following differential equations:

$$y''(x) + a_1 y'(x) + a_2 y(x) = g_j(x); (j = 1, ..., r).$$

Here $y_1(x)$ and $y_2(x)$ are linearly independent complementary solutions satisfying (3.3). Let us next obtain the arbitrary constants c_1 and c_2 in (3.5).

$$c_1 y(0) + c_2 y_2(0) + \sum_{j=1}^{r} b_j G_j(0) = \gamma_0,$$

and

$$c_1 y_1'(0) + c_2 y_2'(0) + \sum_{j=1}^{r} b_j G_j'(0) = \gamma_1,$$

Here after, we will use the following notations for the sake of shortness.

$$\begin{aligned} W &= \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}; \\ w_{11} &= y_1(0), w_{12} = y_2(0), w_{21} = y_1'(0), w_{22} = y_2'(0); \\ \overrightarrow{c} &= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \overrightarrow{\gamma} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}, \overrightarrow{\Delta}_j = \begin{pmatrix} \Delta_{0j} \\ \Delta_{1j} \end{pmatrix}, \\ \Delta_{0j} &= G_j(0), \Delta_{1j} = G_j'(0);\; j = 1, ..., r. \end{aligned}$$

According to these notations, we can write (3.6) in the matrix form:

$$W\overrightarrow{c} = \overrightarrow{\gamma} - \sum_{j=1}^{r} b_j \overrightarrow{\Delta}_j. \tag{3.6}$$

Using Cramer's rule, we obtain c_1 and c_2 as follows:

$$c_j = \frac{|W_{1j}|}{|W|} - \frac{|W_{2j}|}{|W|};\; j = 1, 2.$$

Here

$$\begin{aligned}
|W| &= \begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix} = w_{11}w_{22} - w_{21}w_{12}; \\
|W_{11}| &= \begin{vmatrix} \gamma_0 & w_{12} \\ \gamma_1 & w_{22} \end{vmatrix} = \gamma_0 w_{22} - \gamma_1 w_{12}; \\
|W_{12}| &= \begin{vmatrix} w_{11} & \gamma_0 \\ w_{21} & \gamma_1 \end{vmatrix} = \gamma_1 w_{11} - \gamma_0 w_{21}; \\
|W_{21}| &= \begin{vmatrix} \sum_{j=1}^r b_j \Delta_{0j} & w_{12} \\ \sum_{j=1}^r b_j \Delta_{1j} & w_{22} \end{vmatrix} = \sum_{j=1}^r b_j(\Delta_{0j} w_{22} - \Delta_{1j} w_{12}); \\
|W_{22}| &= \begin{vmatrix} w_{11} & \sum_{j=1}^r b_j \Delta_{0j} \\ w_{21} & \sum_{j=1}^r b_j \Delta_{1j} \end{vmatrix} = \sum_{j=1}^r b_j(\Delta_{1j} w_{11} - \Delta_{0j} w_{21}).
\end{aligned}$$

Thus, $c_1$and c_2 can be rewritten as

$$\begin{aligned}
c_1 &= \frac{|W_{11}| - |W_{21}|}{|W|} = \frac{\gamma_0 w_{22} - \gamma_1 w_{12} - \sum_{j=1}^r b_j(\Delta_{0j} w_{22} - \Delta_{1j} w_{12})}{|W|}, \\
c_2 &= \frac{|W_{12}| - |W_{22}|}{|W|} = \frac{\gamma_1 w_{11} - \gamma 0 w_{21} - \sum_{j=1}^r b_i(\Delta_{1j} w_{11} - \Delta_{0j} w_{21})}{|W|}.
\end{aligned}$$

To simplify the results above, c_1 and c_2 can be rewritten in the following form, respectively:

$$c_1 = \gamma_0 f_{22} - \gamma_1 f_{12} + \sum_{j=1}^r b_j(\Delta_{1j} f_{12} - \Delta_{0j} f_{22}),$$

and

$$c_2 = \gamma_1 f_{11} - \gamma_0 f_{21} + \sum_{j=1}^r b_j(\Delta_{0j} f_{21} - \Delta_{1j} f_{11}).$$

where $f_{ij} = \frac{w_{ij}}{|W|}; i, j = 1, 2.$
Substituting c_1 and c_2 into (3.5), we obtain the solution as:

$$\begin{aligned}
Y(x) = {} & \gamma_0 \left(f_{22} y_1(x) - f_{21} y_2(x)\right) + \gamma_1 (f_{11} y_2(x) - f_{12} y_1(x)) + \\
& \sum_{j=1}^r b_j(G_j(x) + y_1(x)(\Delta_{1j} f_{12} - \Delta_{0j} f_{22}) \\
& + y_2(x)(\Delta_{0j} f_{21} - \Delta_{1j} f_{11})).
\end{aligned}$$

Next we will use the following notations for the sake of simplicity

$$A_0(x) = f_{22} y_1(x) - f_{21} y_2(x), \tag{3.7}$$

$$A_1(x) = f_{11} y_2(x) - f_{12} y_1(x), \tag{3.8}$$

$$B_j(x) = G_j(x) + y_1(x)(\Delta_{1j} f_{12} - \Delta_{0j} f_{22}) + y_2(x)(\Delta_{0j} f_{21} - \Delta_{1j} f_{11}), \tag{3.9}$$

where $j = 1, ..., r$. Thus the solution of the crisp initial value problem can be written as:

$$Y(x) = \gamma_0 A_0 + \gamma_1 A_1 + \sum_{j=1}^r b_j B_j. \tag{3.10}$$

By Zadeh's Extension Principle for the intuitionistic fuzzy sets we can write the solution of the fuzzy initial value problem as follows:

$$\bar{Y}^i(x) = \bar{\gamma}_0^i A_0 + \bar{\gamma}_1^i A_1 + \sum_{j=1}^r \bar{b}_j^i B_j. \tag{3.11}$$

In terms of α and β cuts of the intuitionistic fuzzy numbers we obtain that

$$\begin{cases} [Y_1(x,\alpha), Y_2(x,\alpha)] = \sum_{k=0}^{1}[\gamma_{k1}(\alpha), \gamma_{k2}(\alpha)]A_k(x) \\ \qquad + \sum_{j=1}^{r}[b_{j1}(\alpha), b_{j2}(\alpha)]B_j(x), \\ \\ [Y_1^*(x,\beta), Y_2^*(x,\beta)] = \sum_{k=0}^{1}[\gamma_{k1}^*(\beta), \gamma_{k2}^*(\beta)]A_k(x) \\ \qquad + \sum_{j=1}^{r}[b_{j1}^*(\beta), b_{j2}^*(\beta)]B_j(x). \end{cases} \tag{3.12}$$

Here $\gamma_{j1}(\alpha)$, $b_{j1}(\alpha)$, $Y_1(\alpha)$; $\gamma_{j1}^*(\beta)$, $b_{j1}^*(\beta)$, $Y_1^*(\beta)$ are lower bounds for α-cuts and β-cuts, respectively; and $\gamma_{j2}(\alpha)$, $b_{j2}(\alpha)$, $Y_2(\alpha)$; $\gamma_{j2}^*(\beta)$, $b_{j2}^*(\beta)$, $Y_2^*(\beta)$ are upper bounds for α-cuts and β-cuts, respectively.

Using the Heaviside function and interval arithmetics we can write the α and β cuts of the solution $\bar{Y}^i(x)$ as follows:

$$\begin{cases} Y_1(x,\alpha) = \sum_{k=0}^{1}[\gamma_{k2}(\alpha) - (\gamma_{k2}(\alpha) - \gamma_{k1}(\alpha))\theta(A_k(x))]A_k(x) \\ \qquad + \sum_{j=1}^{r}[b_{j2}(\alpha) - (b_{j2}(\alpha) - b_{j1}(\alpha))\theta(B_j(x))]B_j(x), \\ \\ Y_2(x,\alpha) = \sum_{k=0}^{1}[\gamma_{k1}(\alpha) + (\gamma_{k2}(\alpha) - \gamma_{k1}(\alpha))\theta(A_k(x))]A_k(x) \\ \qquad + \sum_{j=1}^{r}[b_{j1}(\alpha) + (b_{j2}(\alpha) - b_{j1}(\alpha))\theta(B_j(x))]B_j(x), \\ \\ Y_1^*(x,\beta) = \sum_{k=0}^{1}[\gamma_{k2}^*(\beta) - (\gamma_{k2}^*(\beta) - \gamma_{k1}^*(\beta))\theta(A_k(x))]A_k(x) \\ \qquad + \sum_{j=1}^{r}[b_{j2}^*(\beta) - (b_{j2}^*(\beta) - b_{j1}^*(\beta))\theta(B_j(x))]B_j(x), \\ \\ Y_2^*(x,\beta) = \sum_{k=0}^{1}[\gamma_{k1}^*(\beta) + (\gamma_{k2}^*(\beta) - \gamma_{k1}^*(\beta))\theta(A_k(x))]A_k(x) \\ \qquad + \sum_{j=1}^{r}[b_{j1}^*(\beta) + (b_{j2}^*(\beta) - b_{j1}^*(\beta))\theta(B_j(x))]B_j(x). \end{cases} \tag{3.13}$$

Hence the proof is completed. □

Now let us consider the following numerical examples

Example 3.2. Consider the following second order intuitionistic fuzzy initial value problem:

$$\begin{aligned} y'' + 3y' + 2y &= \bar{b}_1^i x^2 + \bar{b}_2^i \cos(x), \\ \bar{\gamma}_0^i &= (0,1,2;-1,1,3), \\ \bar{\gamma}_1^i &= (4,5,7;3,5,8) \end{aligned}$$

where $\bar{b}_1^i = (3,4,5;2,4,6)$, $\bar{b}_2^i = (19,20,21;18,20,22)$.
Let us first solve the crisp initial value problem:

$$\begin{cases} y'' + 3y' + 2y = 4x^2 + 20\cos(x), \\ \quad y(0) = 1, \ y\prime(0) = 5. \end{cases}$$

The general solution of the differential equation in this initial value problem is

$$Y(x) = c_1 e^{-2x} + c_2 e^{-x} + 2x^2 - 6x + 7 + 2\cos(x) + 6\sin(x).$$

Thus, we can obtain proper solutions for $g_1(x) = x^2$ and $g_2(x) = \cos(x)$ as follows:

$$\begin{aligned} G_1(x) &= \frac{1}{4}(2x^2 - 6x + 7), \\ G_2(x) &= \frac{1}{20}(2\cos(x) + 6\sin(x)). \end{aligned}$$

The functions $A_0(x)$, $A_1(x)$, $B_1(x)$ and $B_2(x)$ are as follows:

$$\begin{aligned}
A_0(x) &= -e^{-2x} + 2e^{-x}, \\
A_1(x) &= -e^{-2x} + e^{-x}, \\
B_1(x) &= \frac{e^{-2x}}{4} - 2e^{-x} + \frac{1}{4}(7 - 6x + 2x^2), \\
B_2(x) &= \frac{2}{5}e^{-2x} - \frac{e^{-x}}{2} + \frac{1}{20}(2\cos(x) + 6\sin(x)).
\end{aligned} \tag{3.14}$$

According to (3.17) the α and β cuts of the solution can be found as follows:

$$\begin{aligned}
Y_1(x,\alpha) = {} & [7 - 2\alpha - (3 - 3\alpha)\theta(A_0(x)]A_0(x) + \\
& [2 - \alpha - (2 - 2\alpha)\theta(A_1(x))]A_1(x) + \\
& [5 - \alpha - (2 - 2\alpha)\theta(B_1(x))]B_1(x) + \\
& [21 - \alpha - (2 - 2\alpha)\theta(B_2(x))]B_2(x),
\end{aligned}$$

$$\begin{aligned}
Y_2(x,\alpha) = {} & [4 + \alpha + (3 - 3\alpha)\theta(A_0(x)]A_0(x) + \\
& [\alpha + (2 - 2\alpha)\theta(A_1(x))]A_1(x) + \\
& [3 + \alpha + (2 - 2\alpha)\theta(B_1(x))]B_1(x) + \\
& [19 + \alpha + (2 - 2\alpha)\theta(B_2(x))]B_2(x),
\end{aligned}$$

$$\begin{aligned}
Y_1^*(x,\beta) = {} & [5 + 3\beta - 4\beta\theta(A_0(x)]A_0(x) + \\
& [1 + 2\beta - 4\beta\theta(A_1(x))]A_1(x) + \\
& [4 + 2\beta - 4\beta\theta(B_1(x))]B_1(x) + \\
& [20 + 2\beta - 4\beta\theta(B_2(x))]B_2(x)
\end{aligned}$$

and

$$\begin{aligned}
Y_2^*(x,\beta) = {} & [5 - 2\beta + 5\beta\theta(A_0(x)]A_0(x) + \\
& [1 - 2\beta + 4\beta)\theta(A_1(x))]A_1(x) + \\
& [4 - 2\beta + 4\beta\theta(B_1(x))]B_1(x) + \\
& [20 - 2\beta + 4\beta\theta(B_2(x))]B_2(x).
\end{aligned}$$

Here $\theta(x)$ is the Heaviside function.

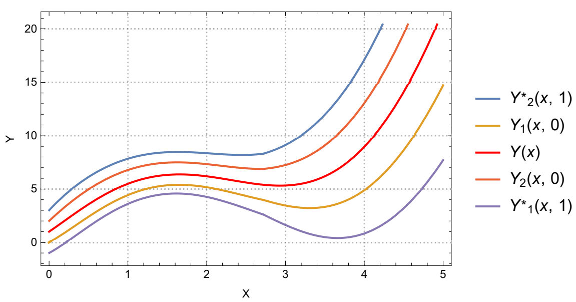

Figure 1. Solutions for $\alpha = 0$ and $\beta = 1$ of the intuitionistic fuzzy solution for Example 3.2.

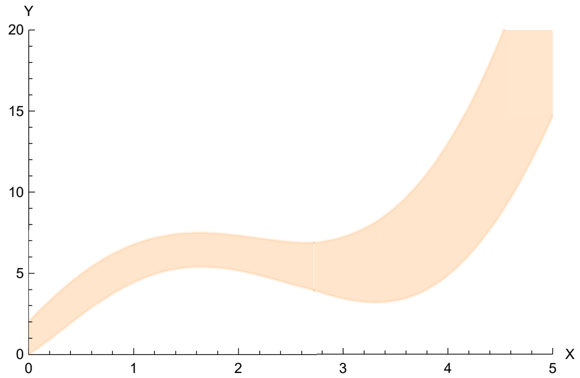

Figure 2. The region of $A(\alpha)$ of the intuitionistic fuzzy solution for Example 3.2.

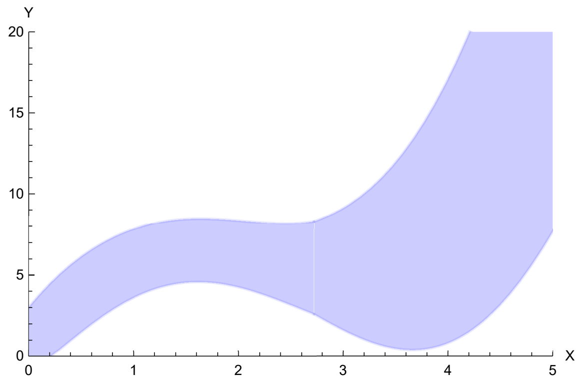

Figure 3. The region of $A^*(\beta)$ of the intuitionistic fuzzy solution for Example 3.2.

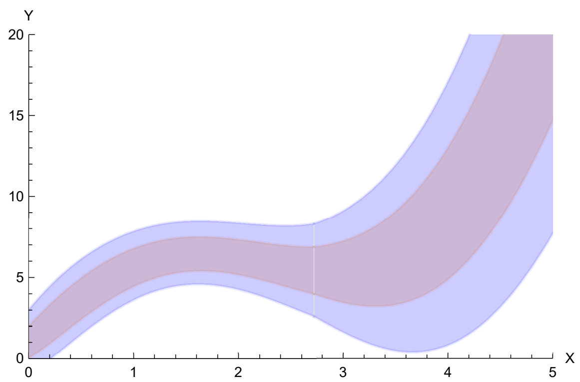

Figure 4. The intersection of $A(\alpha)$ and $A^*(\beta)$ of the intuitionistic fuzzy solution for Example 3.2.

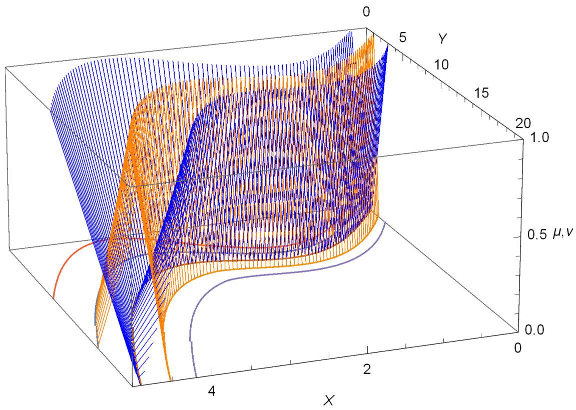

Figure 5. Membership and non-membership functions of intuitionistic fuzzy solution for Example 3.2.

Table 1. Values of $Y_1(x,\alpha), Y_2(x,\alpha), Y_1^*(x,\beta)$ and $Y_2^*(x,\beta)$ at $x = 3$ for Example 3.2

	$Y_1^*(x,\beta)$	$Y_1(x,\alpha)$	$Y(x)$	$Y_2(x,\alpha)$	$Y_2^*(x,\beta)$
0	5.32651	3.4505	5.32651	7.24984	5.32651
0.1	4.95604	3.6381	5.32651	7.0575	5.70645
0.2	4.58557	3.8257	5.32651	6.86517	6.08638
0.3	4.2151	4.0133	5.32651	6.67284	6.46631
0.4	3.84463	4.2009	5.32651	6.48051	6.84625
0.5	3.47415	4.38851	5.32651	6.28817	7.22618
0.6	3.10368	4.57611	5.32651	6.09584	7.60612
0.7	2.73321	4.76371	5.32651	5.90351	7.98605
0.8	2.36274	4.95131	5.32651	5.71118	8.36598
0.9	1.99226	5.13891	5.32651	5.51885	8.74592
1.0	1.62179	5.32651	5.32651	5.32651	9.12585

Example 3.3. Consider the second order linear intuitionistic fuzzy initial value problem:

$$\begin{aligned} y'' + 3y' + 2y &= \bar{b}_1^i x + \bar{b}_2^i sin(x), \\ \bar{\gamma}_0^i &= (0,1,2;-1,1,3), \\ \bar{\gamma}_1^i &= (4,5,6;3,5,7), \end{aligned}$$

where the forcing coefficients are $\bar{b}_1^i = (-5,-4,-3;-6,-4,-2)$ and $\bar{b}_2^i = (-21,-20,-19;-22,-20,-18)$.
With the proposed method we obtain α and β cuts of the intuitionistic solution as follows:

$$\begin{aligned} Y_1(x,\alpha) = \; & [2 - 2\alpha - (2-2\alpha)\theta(A_0(x)]A_0(x) + \\ & [6 - \alpha - (2-2\alpha)\theta(A_1(x))]A_1(x) + \\ & [-3 - \alpha - (2-2\alpha)\theta(B_1(x))]B_1(x) + \\ & [-19 - \alpha - (2-2\alpha)\theta(B_2(x))]B_2(x), \end{aligned}$$

$$\begin{aligned} Y_2(x,\alpha) = \; & [4 + \alpha + (2-2\alpha)\theta(A_0(x)]A_0(x) + \\ & [\alpha + (2-2\alpha)\theta(A_1(x))]A_1(x) + \\ & [-5 + \alpha + (2-2\alpha)\theta(B_1(x))]B_1(x) + \\ & [-21 + \alpha + (2-2\alpha)\theta(B_2(x))]B_2(x), \end{aligned}$$

$$\begin{aligned} Y_1^*(x,\beta) &= [5+2\beta-3\beta\theta(A_0(x)]A_0(x)+ \\ &\quad [1+2\beta-4\beta\theta(A_1(x))]A_1(x)+ \\ &\quad [-4+2\beta-4\beta\theta(B_1(x))]B_1(x)+ \\ &\quad [-20+2\beta-4\beta\theta(B_2(x))]B_2(x) \end{aligned}$$

and

$$\begin{aligned} Y_2^*(x,\beta) &= [5-2\beta+4\beta\theta(A_0(x)]A_0(x)+ \\ &\quad [1-2\beta+4\beta)\theta(A_1(x))]A_1(x)+ \\ &\quad [-4-2\beta+4\beta\theta(B_1(x))]B_1(x)+ \\ &\quad [-20-2\beta+4\beta\theta(B_2(x))]B_2(x). \end{aligned}$$

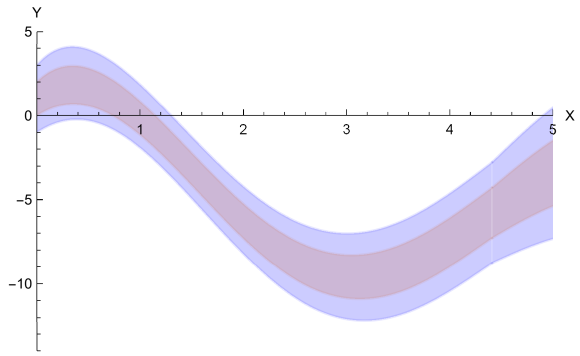

Figure 6. The intersection of $A(\alpha)$ and $A^*(\beta)$ of the intuitionistic fuzzy solution for Example 3.3.

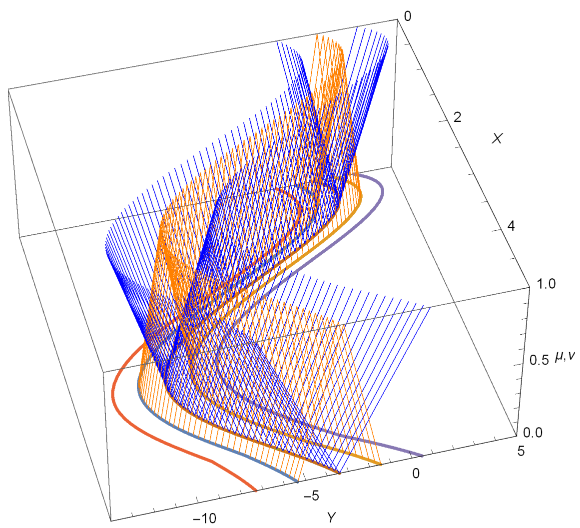

Figure 7. Membership and non-membership functions of intuitionistic fuzzy solution for Example 3.3.

Note that all numerical results and graphics are obtained by using Wolfram: Mathematica 11.

4. Conclusions

In this paper, we proposed an algorithm to solve a second order initial value problem in intuitionistic fuzzy environment. In Theorem 3.1. we have given the solutions of the second order intuitionistic fuzzy initial value problems with the help of Zadeh's Extension Principle for intuitionistic fuzzy numbers. To prevent the switching of endpoints of α and β cuts, we used Heaviside function during the interval operations on α and β cuts. Based on the method proposed in Theorem 3.1 we have given some numerical examples.

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