Fundamental Journal of Mathematics and Applications, 2 (2) (2019) 123-129
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.635754

Three Equivalent n-Norms on the Space of p-Summable Sequences

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Article Info

Keywords: *Equivalence, n-norm, Semi-inner product g, p-summable sequence space*
2010 AMS: *46B20, 46A45, 46A99, 46B99*
Received: *22 October 2019*
Accepted: *08 December 2019*
Available online: *20 December 2019*

Abstract

Given a normed space, one can define a new n-norm using a semi-inner product g on the space, different from the n-norm defined by Gähler. In this paper, we are interested in the new n-norm which is defined using such a functional g on the space ℓ^p of p-summable sequences, where $1 \le p < \infty$. We prove particularly that the new n-norm is equivalent with the one defined previously by Gunawan on ℓ^p.

1. Introduction

On a normed space $(X, \|\cdot\|)$, let $g : X^2 \to \mathbb{R}$ be the functional defined by the formula

$$g(x,y) := \frac{1}{2}\|x\| \left[\tau_+(x,y) + \tau_-(x,y)\right],$$

with

$$\tau_\pm(x,y) := \lim_{t \to 0^\pm} \frac{\|x+ty\| - \|x\|}{t}.$$

Then, one may check that g satisfies the following properties:

(1) $g(x,x) = \|x\|^2$ for every $x \in X$;
(2) $g(\alpha x, \beta y) = \alpha\beta\, g(x,y)$ for every $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$;
(3) $g(x, x+y) = \|x\|^2 + g(x,y)$ for every $x, y \in X$;
(4) $|g(x,y)| \le \|x\|\,\|y\|$ for every $x, y \in X$.

Assuming that the g-functional is linear in the second argument then $[y,x] = g(x,y)$ is a *semi-inner product* on X. Note that all vector spaces in text are assumed to be over $\mathbb{R}$. For example, one may observe that the functional

$$g(x,y) := \|x\|_p^{2-p} \sum_k |x_k|^{p-1} \operatorname{sgn}(x_k)\, y_k, \quad x := (x_k), y := (y_k) \in \ell^p$$

is a semi-inner product on ℓ^p, $1 \le p < \infty$ [1].

Remark 1.1. *Note that not all vector spaces have the property that the g-functional is linear in the second argument. If the normed space is smooth, then the g-functional is linear in the second argument. A normed spaces with the property that the g-functional is linear in the second argument is referred to as normed spaces of (G)-type [2].*

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By using a semi-inner product g, Miličić [3] introduced the following orthogonality relation on X: x is said to be g-orthogonal to y, denoted by $x \perp_g y$, provided that $g(x,y) = 0$. For more recent works, see in [4, 5].
Recently, Nur and Gunawan in [6] defined a 2-norm on X by

$$\|x_1, x_2\|_g := \sup_{\|y_j\| \le 1, j=1,2} \begin{vmatrix} g(y_1,x_1) & g(y_2,x_1) \\ g(y_1,x_2) & g(y_2,x_2) \end{vmatrix}.$$

Similarly, we can define an n-norm (with $n \ge 2$) using the semi-inner product g on X. An *n-norm* on X is a mapping $\|\cdot,\dots,\cdot\| : X \times \cdots \times X \longrightarrow \mathbb{R}$ which satisfies the following four properties:

(1) $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent;
(2) $\|x_1,\dots,x_n\|$ is invariant under permutation;
(3) $\|\alpha x_1,\dots,x_n\| = |\alpha| \|x_1,\dots,x_n\|$ for every $x_1,\dots,x_n \in X$ and for every $\alpha \in \mathbb{R}$;
(4) $\|x_1,\dots,x_{n-1}, y+z\| \le \|x_1,\dots,x_{n-1},y\| + \|x_1,\dots,x_{n-1},z\|$ for every $x,y,z \in X$.

The pair $(X, \|\cdot,\dots,\cdot\|)$ is called an *n-normed space*.
The theory of 2-normed spaces was initially introduced by Gähler [7] in the 1960's. Meanwhile, the theory of n-normed spaces for $n \ge 2$ was developed in [8]-[10]. See [11]-[15] for recent results on this subject.
On the space ℓ^p of p-summable sequences, where $1 \le p < \infty$, the following n-norm

$$\|x_1,\dots,x_n\|_p := \left[\frac{1}{n!} \sum_{k_1} \cdots \sum_{k_n} \left(\text{abs} \begin{vmatrix} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n} \end{vmatrix} \right)^p \right]^{\frac{1}{p}} \tag{1.1}$$

is defined by Gunawan in [16]. As shown in [17, 18], this n-norm is equivalent with the one formulated by Gähler in [8]-[10], namely

$$\|x_1,\dots,x_n\|'_p := \sup_{\|y_j\|_{p'} \le 1, j=1,\dots,n} \begin{vmatrix} \sum_k x_{1k}y_{1k} & \cdots & \sum_k x_{1k}y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk}y_{1k} & \cdots & \sum_k x_{nk}y_{nk} \end{vmatrix}, \tag{1.2}$$

where p' denotes the dual exponent of p. Precisely, we have the following theorem.

Theorem 1.2. *[19] For every $x_1,\dots,x_n \in \ell^p$ $(1 \le p < \infty)$, we have*

$$(n!)^{\frac{1}{p}-1} \|x_1,\dots,x_n\|_p \le \|x_1,\dots,x_n\|'_p \le (n!)^{\frac{1}{p}} \|x_1,\dots,x_n\|_p.$$

In this article, we shall first prove that, on ℓ^p $(1 \le p < \infty)$, the new 2-norm $\|\cdot,\cdot\|_g$ is equivalent with the 2-norm $\|\cdot,\cdot\|_p$ which is defined in (1.1). Using this result, we can prove that the 2-normed space $(\ell^p, \|\cdot,\cdot\|_g)$ is complete. We then extend the result for all $n \ge 2$.

2. Main results

Before we discuss the equivalence between the two 2-norms on ℓ^p $(1 \le p < \infty)$, we need some definitions. Let $(X, \|\cdot\|)$ be a normed space. We define the g-orthogonal projection of a vector y on a subspace S of X as follows.

Definition 2.1. *[20] Let $y \in X$ and $S = span\{x_1,\dots,x_m\}$ be a subspace of X with $\Gamma(x_1,\dots,x_m) = \det[g(x_i,x_j)] \ne 0$. The g-orthogonal projection of y on S, which we denote by y_S, is defined by*

$$y_S := -\frac{1}{\Gamma(x_1,\dots,x_m)} \begin{vmatrix} 0 & x_1 & \cdots & x_m \\ g(x_1,y) & g(x_1,x_1) & \cdots & g(x_1,x_m) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_m,y) & g(x_m,x_1) & \cdots & g(x_m,x_m) \end{vmatrix},$$

and its g-orthogonal complement $y - y_S$ is given by

$$y - y_S = \frac{1}{\Gamma(x_1,\dots,x_m)} \begin{vmatrix} y & x_1 & \cdots & x_m \\ g(x_1,y) & g(x_1,x_1) & \cdots & g(x_1,x_m) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_m,y) & g(x_m,x_1) & \cdots & g(x_m,x_m) \end{vmatrix}.$$

Observe here that $x_i \perp_g y - y_S$ for every $i = 1,\dots,m$. Note that, if $S = \text{span}\{x\}$, then

$$y_S = \frac{g(x,y)}{\|x\|^2} x,$$

and $y - y_S$ is the g-orthogonal complement y on S. It is clear here that $x \perp_g y - y_S$.

Next, let $x_1,\ldots,x_n \in X$ be a set of n linearly independent vectors. We may construct a *left g-orthogonal sequence* $x_1^*,\ldots,x_n^*$ with $x_1^* := x_1$, and

$$x_i^* := x_i - (x_i)_{S_{i-1}}, \tag{2.1}$$

where $S_{i-1} = \text{span}\left\{x_1^*,\ldots,x_{i-1}^*\right\}$ for $i = 2,\ldots,n$. Observe here that $x_i^* \perp_g x_j^*$ for $i < j$ (see [15, 20]).
For $X = \ell^p$ $(1 \le p < \infty)$, we have relation for the n-norm $\|x_1,\ldots,x_n\|_p$ and the 'volume' of the n-dimensional parallelepiped spanned by $\{x_1,\ldots,x_n\}$ in ℓ^p, namely $V(x_1,\ldots,x_n) = \prod_{i=1}^{n} \|x_i^*\|_p$, as follows.

Theorem 2.2. *[19] Let $\{x_1,\ldots,x_n\}$ be a set of linearly independent vectors in ℓ^p $(1 \le p < \infty)$. Then we have*

$$(n!)^{\frac{1}{p}-1}\|x_1,\ldots,x_n\|_p \le V(x_{i_1},\ldots,x_{i_n}) \le (n!)^{\frac{1}{p}}\|x_1,\ldots,x_n\|_p$$

for any permutation $(i_1,\ldots,i_n)$ of $(1,\ldots,n)$.

Note that the value of $V(x_1,\ldots,x_n)$ may not be invariant under permutation of $(x_1,\ldots,x_n)$ because $g(\cdot,\cdot)$ may not be symmetry. The above theorem states that all possible values of $V(x_{i_1},\ldots,x_{i_n})$ lie between two multiples of $\|x_1,\ldots,x_n\|_p$, independent of the permutation.

2.1. The equivalence between two 2-norms

Let us consider Gunawan's definition and Gähler's definition of 2-norm on ℓ^p $(1 \le p < \infty)$, namely:

$$\|x_1,x_2\|_p = \left[\sum_{k_1}\sum_{k_2}\left(\text{abs}\begin{vmatrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2}\end{vmatrix}\right)^p\right]^{\frac{1}{p}}$$

and

$$\|x_1,x_2\|_p' := \sup_{\|y_j\|_{p'}\le 1,\, j=1,2}\begin{vmatrix} \sum_k x_{1k}y_{1k} & \sum_k x_{1k}y_{2k} \\ \sum_k x_{2k}y_{1k} & \sum_k x_{2k}y_{2k}\end{vmatrix}.$$

Meanwhile, Nur and Gunawan's 2-norm is given by

$$\|x_1,x_2\|_{g,p} = \sup_{\|y_j\|_p\le 1,\, j=1,2}\begin{vmatrix} \|y_1\|_p^{2-p}\sum_k |y_{1k}|^{p-1}\text{sgn}(y_{1k})x_{1k} & \|y_2\|_p^{2-p}\sum_k |y_{2k}|^{p-1}\text{sgn}(y_{2k})x_{1k} \\ \|y_1\|_p^{2-p}\sum_k |y_{1k}|^{p-1}\text{sgn}(y_{1k})x_{2k} & \|y_2\|_p^{2-p}\sum_k |y_{2k}|^{p-1}\text{sgn}(y_{2k})x_{2k}\end{vmatrix}.$$

Remark 2.3. *Using properties of determinants, the above 2-norm may be rewritten as*

$$\|x_1,x_2\|_{g,p} = \sup_{\|y_j\|_p\le 1,\, j=1,2}\frac{1}{2}\prod_{j=1}^{2}\|y_j\|_p^{2-p}\sum_{k_1}\sum_{k_2}\begin{vmatrix} |y_{1k_1}|^{p-1}sgn\left(y_{1k_1}\right) & |y_{1k_2}|^{p-1}sgn\left(y_{1k_2}\right) \\ |y_{2k_1}|^{p-1}sgn\left(y_{2k_1}\right) & |y_{2k_2}|^{p-1}sgn\left(y_{2k_2}\right)\end{vmatrix}\begin{vmatrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2}\end{vmatrix}.$$

For $p = 2$, we observe that

$$\|x_1,x_2\|_{g,2} = \sup_{\|y_j\|_2\le 1,\ j=1,2}\frac{1}{2}\sum_{k_1}\sum_{k_2}\begin{vmatrix} y_{1k_1} & y_{1k_2} \\ y_{2k_1} & y_{2k_2}\end{vmatrix}\begin{vmatrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2}\end{vmatrix}.$$

One may then verify that the three 2-norms $\|\cdot,\cdot\|_2$, $\|\cdot,\cdot\|_2'$ and $\|\cdot,\cdot\|_{g,2}$ are identical (see [6, 12]).

For other values of p, we have the following theorem.

Theorem 2.4. *For every $x_1,x_2 \in \ell^p$ $(1 \le p < \infty)$, we have*

$$2^{\frac{1}{p}-1}\|x_1,x_2\|_p \le \|x_1,x_2\|_{g,p} \le \|x_1,x_2\|_p' \le 2^{\frac{1}{p}}\|x_1,x_2\|_p.$$

Proof. For $j = 1,2$, let $y_j \in \ell^p$ with $\|y_j\|_p \le 1$. Take $u_j = (u_{jk})$ with $u_{jk} = \|y_j\|_p^{2-p}|y_{jk}|^{p-1}\text{sgn}(y_{jk})$. We observe that $u_j \in \ell^{p'}$ with $\|u_j\|_{p'} = \|y_j\|_p$. As a consequence, we have $\|x_1,x_2\|_{g,p} \le \|x_1,x_2\|_p'$. By using Theorem 1.2, we obtain

$$\|x_1,x_2\|_{g,p} \le \|x_1,x_2\|_p' \le 2^{\frac{1}{p}}\|x_1,x_2\|_p.$$

Next, assume that $\{x_1,x_2\}$ is linearly independent. Using the process in (2.1), we obtain the left g-orthogonal set $\left\{x_1^*,x_2^*\right\}$. Then, by Theorem 2.2, we have

$$2^{\frac{1}{p}-1}\|x_1,x_2\|_p \le V(x_1,x_2) = \|x_1^*\|_p\|x_2^*\|_p.$$

For $j=1,2$, let $y_j=\frac{x_j^*}{\|x_j^*\|_p}$, so that $\|y_j\|_p=1$. It follows from the properties of semi-inner product g and matrix determinants that

$$\begin{aligned}\begin{vmatrix} g(y_1,x_1) & g(y_2,x_1) \\ g(y_1,x_2) & g(y_2,x_2)\end{vmatrix} &= \begin{vmatrix} \frac{1}{\|x_1^*\|_p}g(x_1^*,x_1^*) & \frac{1}{\|x_2^*\|_p}g(x_2^*,x_1^*) \\ \frac{1}{\|x_1^*\|_p}g(x_1^*,x_2^*) & \frac{1}{\|x_2^*\|_p}g(x_2^*,x_2^*)\end{vmatrix} \\ &= \|x_1^*\|_p\,\|x_2^*\|_p = V(x_1,x_2) \\ &\geq 2^{\frac{1}{p}-1}\,\|x_1,x_2\|_p\,.\end{aligned}$$

By the definition of $\|\cdot,\cdot\|_{g,p}$, we conclude that $\|x_1,x_2\|_{g,p}\geq 2^{\frac{1}{p}-1}\|x_1,x_2\|_p$. Combining with the previous inequalities, we have

$$2^{\frac{1}{p}-1}\,\|x_1,x_2\|_p\leq\|x_1,x_2\|_{g,p}\leq\|x_1,x_2\|_p'\leq 2^{\frac{1}{p}}\,\|x_1,x_2\|_p.$$

Note that if $\{x_1,x_2\}$ is a linearly dependent set, then all the 2-norms are equal 0, and so we have the equalities. □

Corollary 2.5. *For $1\leq p<\infty$, the three 2-norms $\|\cdot,\cdot\|_{g,p}$, $\|\cdot,\cdot\|_p'$, and $\|\cdot,\cdot\|_p$ are pairwise equivalent.*

Since $(\ell^p,\|\cdot,\cdot\|_p)$ is a 2-Banach space [1], we obtain the following corollary.

Corollary 2.6. *For $1\leq p<\infty$, the 2-normed space $(\ell^p,\|\cdot,\cdot\|_{g,p})$ is a 2-Banach space.*

2.2. The equivalence between two n-norms

All results in above subsection can be extended to n-normed spaces for any $n\geq 2$. Suppose that g is a semi-inner product on $(X,\|\cdot\|)$. Consider the following mapping $\|\cdot,\ldots,\cdot\|_g$ on $X\times\cdots\times X$:

$$\|x_1,\ldots,x_n\|_g=\sup_{\|y_j\|\leq 1,\, j=1,\ldots,n}\begin{vmatrix} g(y_1,x_1) & \cdots & g(y_n,x_1) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n)\end{vmatrix}=\sup_{\|y_j\|\leq 1,\ j=1,\ldots,n}\det[g(y_j,x_i)]. \tag{2.2}$$

If $\|y_j\|\leq 1$ for $j=1,\ldots,n$, then $\det[g(y_j,x_i)]\leq n!\prod_{i=1}^{n}\|x_i\|$. Note that the factor $n!$ comes from the number of terms in the expansion of $\det[g(y_j,x_i)]$. The following fact tells us that $\|\cdot,\ldots,\cdot\|_g$ is a finite number.

Fact 2.7. *The inequality*

$$\|x_1,\ldots,x_n\|_g\leq n!\prod_{i=1}^{n}\|x_i\|$$

holds whenever $x_1,\ldots,x_n\in X$.

Moreover, we have the following result.

Proposition 2.8. *The mapping (2.2) defines an n-norm on X.*

Proof. It is obvious that, if $\{x_1,\ldots,x_n\}$ is linearly dependent, then we have $\|x_1,\ldots,x_n\|_g=0$. Conversely, if $\|x_1,\ldots,x_n\|_g=0$, then the rows of the matrix $[g(y_j,x_i)]$ are linearly dependent for every $y_1,\ldots,y_n\in X$ with $\|y_j\|\leq 1,\ j=1,\ldots,n$. This happens only if $x_1,\ldots,x_n$ are linearly dependent.

Next, by using the properties of supremum and matrix determinants, we obtain the invariance of $\|x_1,\ldots,x_2\|_g$ under permutation. Furthermore, we have $\|\alpha x_1,\ldots x_n\|_g=|\alpha|\,\|x_1,\ldots,x_n\|_g$ for $\alpha\in\mathbb{R}$.

Finally, for arbitrary $x_0,x_1,\ldots x_n\in X$, we obtain

$$\begin{aligned}\|x_0+x_1,\ldots,x_n\|_g &= \sup_{\|y_j\|\leq 1,\ j=1,\ldots,n}\begin{vmatrix} g(y_1,x_0+x_1) & \cdots & g(y_n,x_0+x_1) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n)\end{vmatrix} \\ &\leq \sup_{\|y_j\|\leq 1,\, j=1,\ldots,n}\begin{vmatrix} g(y_1,x_0) & \cdots & g(y_n,x_0) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n)\end{vmatrix}+\sup_{\|y_j\|\leq 1,\, j=1,\ldots,n}\begin{vmatrix} g(y_1,x_1) & \cdots & g(y_n,x_1) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n)\end{vmatrix} \\ &= \|x_0,\ldots,x_n\|_g+\|x_1,\ldots,x_n\|_g.\end{aligned}$$

This completes the proof. □

The following theorem holds for an inner product space $(X,\langle\cdot,\cdot\rangle)$.

Theorem 2.9. *If $(X,\langle\cdot,\cdot\rangle)$ is a real inner product space, then the two n-norms $\|\cdot,\dots,\cdot\|_g$ in (2.2) and $\|\cdot,\dots,\cdot\|_s$ given by*

$$\|x_1,\dots,x_n\|_s := \begin{vmatrix} \langle x_1,x_1\rangle & \cdots & \langle x_n,x_1\rangle \\ \vdots & \ddots & \vdots \\ \langle x_1,x_n\rangle & \cdots & \langle x_n,x_n\rangle \end{vmatrix}^{\frac{1}{2}}$$

are identical.

Proof. On the inner product space X, the functional $g(\cdot,\cdot)$ is identical with the inner product $\langle\cdot,\cdot\rangle$. Therefore,

$$\|x_1,\dots,x_n\|_g = \sup_{\|y_j\|\le 1,\, j=1,\dots,n} \begin{vmatrix} \langle y_1,x_1\rangle & \cdots & \langle y_n,x_1\rangle \\ \vdots & \ddots & \vdots \\ \langle y_1,x_n\rangle & \cdots & \langle y_n,x_n\rangle \end{vmatrix}.$$

Now, applying the generalized Cauchy-Schwarz inequality [21] and Hadamard's inequality [22], we get

$$\|x_1,\dots,x_n\|_g \le \sup_{\|y_j\|\le 1,\, j=1,\dots,n} \|x_1,\dots,x_n\|_s\|y_1,\dots,y_n\|_s \le \|x_1,\dots,x_n\|_s.$$

Conversely, suppose that $\{x_1,\dots,x_n\}$ is linearly independent. Using the Gram-Schmidt process, we get the orthogonal set $\{x_1',\dots,x_n'\}$. Because the determinant of the Gram matrix of a linearly independent set being equal to the Gram matrix of the associated orthogonal set (obtained using Gram-Schmidt process), we have $\|x_1,\dots,x_n\|_s = \|x_1',\dots,x_n'\|_s = \|x_1'\|\cdots\|x_n'\|$. For $j=1,\dots,n$, let $y_j = \frac{x_j'}{\|x_j'\|}$, so that $\|y_j\|=1$. Then, by the properties of the inner product and matrix determinants, we obtain

$$\begin{aligned} \begin{vmatrix} \langle y_1,x_1\rangle & \cdots & \langle y_n,x_1\rangle \\ \vdots & \ddots & \vdots \\ \langle y_1,x_n\rangle & \cdots & \langle y_n,x_n\rangle \end{vmatrix} &= \begin{vmatrix} \langle y_1,x_1'\rangle & \cdots & \langle y_n,x_1'\rangle \\ \vdots & \ddots & \vdots \\ \langle y_1,x_n'\rangle & \cdots & \langle y_n,x_n'\rangle \end{vmatrix} = \frac{1}{\|x_1'\|\cdots\|x_n'\|} \begin{vmatrix} \langle x_1',x_1'\rangle & \cdots & \langle x_n',x_1'\rangle \\ \vdots & \ddots & \vdots \\ \langle x_1',x_n'\rangle & \cdots & \langle x_n',x_n'\rangle \end{vmatrix} \\ &= \|x_1'\|\cdots\|x_n'\| = \|x_1,\dots,x_n\|_s. \end{aligned}$$

Thus, $\|x_1,\dots,x_n\|_g \ge \|x_1,\dots,x_n\|_s$. Hence we conclude that $\|x_1,\dots,x_n\|_g = \|x_1,\dots,x_n\|_s$ whenever $\{x_1,\dots,x_n\}$ is linearly independent. If $\{x_1,\dots,x_n\}$ is linearly dependent, then $\|x_1,\dots,x_n\|_g = \|x_1,\dots,x_n\|_s = 0$. □

Remark 2.10. *Note that, in an inner product space, we have the well-known Hadamard's inequality [22]*

$$\|x_1,\dots,x_n\|_g = \|x_1,\dots,x_n\|_s \le \|x_1\|\cdots\|x_n\|,$$

which is better than that in Fact **??**.

For $X=\ell^p$ $(1\le p<\infty)$, we rewrite the formula in (2.2) as

$$\|x_1,\dots,x_n\|_{g,p} = \sup_{\|y_j\|_p\le 1,\, j=1,\dots,n} \begin{vmatrix} g(y_1,x_1) & \cdots & g(y_n,x_1) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n) \end{vmatrix}.$$

Substituting $g(y_j,x_i) = \|y_j\|_p^{2-p}\sum_k |y_{jk}|^{p-1}\text{sgn}(y_{jk})x_{ik}$ and using the properties of determinants, we have

$$\begin{aligned} \|x_1,\dots,x_n\|_{g,p} &= \sup_{\|y_j\|_p\le 1,\, j=1,\dots,n} \begin{vmatrix} \|y_1\|_p^{2-p}\sum_k |y_{1k}|^{p-1}\text{sgn}(y_{1k})x_{1k} & \cdots & \|y_n\|_p^{2-p}\sum_k |y_{nk}|^{p-1}\text{sgn}(y_{nk})x_{1k} \\ \vdots & \ddots & \vdots \\ \|y_1\|_p^{2-p}\sum_k |y_{1k}|^{p-1}\text{sgn}(y_{1k})x_{nk} & \cdots & \|y_n\|_p^{2-p}\sum_k |y_{nk}|^{p-1}\text{sgn}(y_{nk})x_{nk} \end{vmatrix} \\ &= \sup_{\|y_j\|_p\le 1,\, j=1,\dots,n} \prod_{j=1}^n \|y_j\|_p^{2-p}\sum_{k_1}\cdots\sum_{k_n}\prod_{j=1}^n |y_{jk_j}|^{p-1}\text{sgn}\left(y_{jk_j}\right) \begin{vmatrix} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n} \end{vmatrix}. \end{aligned} \tag{2.3}$$

Corollary 2.11. *For $p=2$, the three n-norms $\|\cdot,\dots,\cdot\|_2$ in (1.1), $\|\cdot,\dots,\cdot\|_2'$ in (1.2) and $\|\cdot,\dots,\cdot\|_{g,2}$ in (2.3) are identical.*

For $p\ne 2$, we have the following generalization of Theorem 2.4.

Theorem 2.12. *For every $x_1,\dots,x_n\in\ell^p$ $(1\le p<\infty)$, we have*

$$(n!)^{\frac{1}{p}-1}\|x_1,\dots,x_n\|_p \le \|x_1,\dots,x_n\|_{g,p} \le \|x_1,\dots,x_n\|_p' \le (n!)^{\frac{1}{p}}\|x_1,\dots,x_n\|_p.$$

Proof. For each $j=1,\dots,n$, let $y_j\in\ell^p$ with $\|y_j\|_p\le 1$. Then take $u_j=(u_{jk})$ with $u_{jk}=\|y_j\|_p^{2-p}|y_{jk}|^{p-1}\mathrm{sgn}(y_{jk})$. We observe that $u_j\in\ell^{p'}$ with $\|u_j\|_{p'}=\|y_j\|_p\le 1$. As a consequence, we have

$$\|x_1,\dots,x_n\|_{g,p}\le\|x_1,\dots,x_n\|_p'.$$

By using Theorem 1.2, we obtain

$$\|x_1,\dots,x_n\|_{g,p}\le\|x_1,\dots,x_n\|_p'\le(n!)^{\frac{1}{p}}\|x_1,\dots,x_n\|_p.$$

Conversely, suppose that $\{x_1,\dots,x_n\}$ is a linearly independent set. Using $x_1^*=x_1$ and so forth as in (2.1), we obtain the left g -orthogonal set $\{x_1^*,\dots,x_n^*\}$. Then, it follows from Theorem 2.2 that

$$(n!)^{\frac{1}{p}-1}\|x_1,\dots,x_n\|_p\le V(x_1,\dots,x_n)=\|x_1^*\|_p\cdots\|x_n^*\|_p.$$

For $j=1,\dots,n$, let $y_j=\frac{x_j^*}{\|x_j^*\|_p}$, so that $\|y_j\|_p=1$. Next, using the properties of matrix determinants and the semi-inner product g, we have

$$\begin{aligned}\begin{vmatrix} g(y_1,x_1) & \cdots & g(y_n,x_1)\\ \vdots & \ddots & \vdots\\ g(y_1,x_n) & \cdots & g(y_n,x_n)\end{vmatrix} &= \begin{vmatrix} \frac{1}{\|x_1^*\|_p}g(x_1^*,x_1^*) & \cdots & \frac{1}{\|x_n^*\|_p}g(x_n^*,x_1^*)\\ \vdots & \ddots & \vdots\\ \frac{1}{\|x_1^*\|_p}g(x_1^*,x_n^*) & \cdots & \frac{1}{\|x_n^*\|_p}g(x_n^*,x_n^*)\end{vmatrix}\\ &= \|x_1^*\|_p\cdots\|x_n^*\|_p=V(x_1,\dots,x_n)\\ &\ge (n!)^{\frac{1}{p}-1}\|x_1,\dots,x_n\|_p,\end{aligned}$$

whence $\|x_1,\dots,x_n\|_{g,p}\ge(n!)^{\frac{1}{p}-1}\|x_1,\dots,x_n\|_p$. Combining with the previous inequalities, we obtain

$$(n!)^{\frac{1}{p}-1}\|x_1,\dots,x_n\|_p\le\|x_1,\dots,x_n\|_{g,p}\le\|x_1,\dots,x_n\|_p'\le(n!)^{\frac{1}{p}}\|x_1,\dots,x_n\|_p.$$

If $\{x_1,\dots,x_n\}$ is linearly dependent, then all the n-norms vanish and so we have the equalities. □

Corollary 2.13. *For $1\le p<\infty$, the three n-norms $\|\cdot,\dots,\cdot\|_{g,p}$, $\|\cdot,\dots,\cdot\|_p'$ and $\|\cdot,\dots,\cdot\|_p$ are equivalent.*

Knowing that the space $(\ell^p,\|\cdot,\dots,\cdot\|_p)$ is an n-Banach space in [16], we have a generalization of Corollary 2.6 as follows.

Corollary 2.14. *For $1\le p<\infty$, the space $(\ell^p,\|\cdot,\dots,\cdot\|_{g,p})$ is an n-Banach space.*

3. Concluding remarks

In this paper, a new n-norm is defined using a semi-inner product g on ℓ^p for $1\le p<\infty$. Accordingly, on the space ℓ^p $(1\le p<\infty)$, we have three different n-norms, namely Gähler's n-norm $\|\cdot,\dots,\cdot\|_p'$ defined in [8]-[10], Gunawan's n-norm $\|\cdot,\dots,\cdot\|_p$ defined in [16], and $\|\cdot,\dots,\cdot\|_{g,p}$ defined here in (2.3). In Corollary 2.13, we have just seen that the three n-norms on ℓ^p are equivalent. As expected, the case where $p=2$ is special. Here, the three n-norms on ℓ^2 are identical.
In addition to the above three n-norms, we also have a formula for another n-norm using the semi-inner product g on ℓ^p $(1\le p<\infty)$, namely

$$\|x_1,\dots,x_n\|_{g,p}^{\circ}=\sup_{\|y_1,\dots,y_n\|_p\le 1}\begin{vmatrix} g(y_1,x_1) & \cdots & g(y_n,x_1)\\ \vdots & \ddots & \vdots\\ g(y_1,x_n) & \cdots & g(y_n,x_n)\end{vmatrix}.$$

Since $g(y_j,x_i)=\|y_j\|_p^{2-p}\sum_k|y_{jk}|^{p-1}\mathrm{sgn}(y_{jk})x_{ik}$, we obtain

$$\begin{aligned}\|x_1,\dots,x_n\|_{g,p}^{\circ} &= \Bigg[\sup_{\|y_1,\dots,y_n\|_p\le 1}\frac{1}{n!}\prod_{j=1}^{n}\|y_j\|_p^{2-p}\times\\ &\quad\times\sum_{k_1}\cdots\sum_{k_n}\begin{vmatrix} |y_{1k_1}|^{p-1}\mathrm{sgn}\left(y_{1k_1}\right) & \cdots & |y_{1k_n}|^{p-1}\mathrm{sgn}\left(y_{1k_n}\right)\\ \vdots & \ddots & \vdots\\ |y_{nk_1}|^{p-1}\mathrm{sgn}\left(y_{nk_1}\right) & \cdots & |y_{nk_n}|^{p-1}\mathrm{sgn}\left(y_{nk_n}\right)\end{vmatrix}\begin{vmatrix} x_{1k_1} & \cdots & x_{1k_n}\\ \vdots & \ddots & \vdots\\ x_{nk_1} & \cdots & x_{nk_n}\end{vmatrix}\Bigg].\end{aligned}$$

Note that, for $p=2$, we have $\|x_1,\dots,x_n\|_{g,2}=\|x_1,\dots,x_n\|_{g,2}^{\circ}$. For other values of p, we can show that

$$\|x_1,\dots,x_n\|_{g,p}\le(n!)^{2-\frac{1}{p}}\|x_1,\dots,x_n\|_{g,p}^{\circ}.$$

Indeed, assuming that $x_1,\dots,x_n$ are linearly independent, let $x_1^*,\dots,x_n^*$ be the vectors obtained from $x_1,\dots,x_n$ through the process in (2.1). By taking $y_j = \frac{x_j^*}{\sqrt[n]{\|x_1^*,\dots,x_n^*\|_p}}$ $(j=1,\dots,n)$, we obtain $\|y_1,\dots,y_n\|_p = 1$. Next, using the properties of matrix determinants and the semi-inner product g, we have

$$\begin{vmatrix} g(y_1,x_1) & \cdots & g(y_n,x_1) \\ \vdots & \ddots & \vdots \\ g(y_1,x_n) & \cdots & g(y_n,x_n) \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt[n]{\|x_1^*,\dots,x_n^*\|_p}} g(x_1^*,x_1^*) & \cdots & \frac{1}{\sqrt[n]{\|x_1^*,\dots,x_n^*\|_p}} g(x_n^*,x_1^*) \\ \vdots & \ddots & \vdots \\ \frac{1}{\sqrt[n]{\|x_1^*,\dots,x_n^*\|_p}} g(x_1^*,x_n^*) & \cdots & \frac{1}{\sqrt[n]{\|x_1^*,\dots,x_n^*\|_p}} g(x_n^*,x_n^*) \end{vmatrix}$$
$$= \frac{\|x_1^*\|_p^2 \cdots \|x_n^*\|_p^2}{\|x_1^*,\dots,x_n^*\|_p}.$$

Since $\|x_1,\dots,x_n\|_p \le (n!)^{1-\frac{1}{p}} \|x_1^*\|_p \cdots \|x_n^*\|_p$ by Theorem 2.2 and $\|x_1^*,\dots,x_n^*\|_p = \|x_1,\dots,x_n\|_p$, we obtain

$$\|x_1,\dots,x_n\|_{g,p}^{\circ} \ge (n!)^{\frac{2}{p}-2} \|x_1,\dots,x_n\|_p.$$

Moreover, using Theorem 2.12, we have

$$\|x_1,\dots,x_n\|_{g,p} \le (n!)^{2-\frac{1}{p}} \|x_1,\dots,x_n\|_{g,p}^{\circ}.$$

It follows from this inequality that the convergence of a sequence in $\|\cdot,\dots,\cdot\|_{g,p}^{\circ}$ implies the convergence in $\|\cdot,\dots,\cdot\|_{g,p}$, and hence also in $\|\cdot,\dots,\cdot\|_p$. Unfortunately, up to now, we do not know if the converse is true.

Acknowledgement

This work is supported by ITB Research and Innovation Program 2019.

References

[1] J. R. Giles, *Classes of semi-inner product spaces*, Trans. Amer. Math. Soc., **129**–3 (1967), 436–446.
[2] S. S. Dragomir, *Semi-inner Products and Applications*, Nova Science Publishers, 2004.
[3] P. M. Miličić, *Sur le g-angle dans un espace norme*, Mat. Vesnik, **45** (1993), 43–48.
[4] M. Nur, H. Gunawan, *A new orthogonality and angle in a normed space*, Aequationes Math., **93** (2019), 547–555.
[5] M. Nur, H.Gunawan, O. Neswan, *A formula for the g-angle between two subspaces of a normed space*, Beitr. Algebra Geom., **59**-1 (2018), 133–143.
[6] M. Nur, H. Gunawan, *A note on the formula of the g-angle between two subspaces in a normed space*, (Feb. 2019) arXiv:1809.01909v2 [math.FA].
[7] S. Gähler, *Lineare 2-normierte räume*, Math. Nachr., **28** (1964), 1–43.
[8] S. Gähler, *Untersuchungen über verallgemeinerte m-metrische räume. I"*, Math. Nachr., **40** (1969), 165–189.
[9] S. Gähler, *Untersuchungen über verallgemeinerte m-metrische räume. II*", Math. Nachr., **40** (1969), 229–264.
[10] S. Gähler, *Untersuchungen über verallgemeinerte m-metrische räume. III*, Math. Nachr., **41** (1970), 23–26.
[11] S. Ekariani, H. Gunawan, M. Idris, *A contractive mapping theorem on the n-normed space of p-summable*, J. Math. Anal., **4**1 (2013), 1–7.
[12] S. M. Gozali, H. Gunawan, O. Neswan, *On n-norms and bounded n-linear functionals in a Hilbert space*, Ann. Funct. Anal., **1** (2010), 72–79.
[13] H. Gunawan, *On n-inner products, n-norms, and the Cauchy-Schwarz inequality*, Sci. Math. Japon., **55** (2002), 53–60.
[14] H. Gunawan, H. Mashadi, *On n-normed spaces*, Int. J. Math. Math. Sci. **27** (2001), 631–639.
[15] H. Gunawan, W. Setya-Budhi, M. Mashadi, S.Gemawati, *On volumes of n-dimensional parallelepipeds in ℓ^p spaces*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat., **16** (2005), 48–54.
[16] H. Gunawan, *The space of p-summable sequences and its natural n-norms*, Bull. Austral. Math. Soc., **64** (2001), 137–147.
[17] Ş. Konca, M. Idris, *Equivalence among three 2-norms on the space of p-summable sequences*, J. Inequal. Spec. Funct., **7**(4), (2016) 218–224.
[18] A. Mutaqin, H. Gunawan, *Equivalence of n-norms on the space of p-summable sequences*, J. Indones. Math. Soc., **16** (2010), 39–49.
[19] R. A. Wibawa-Kusumah, H. Gunawan, *Two equivalent n-norms on the space of p-summable sequences*, Period. Math. Hungar., **67**–1 (2013), 63–69.
[20] P. M. Miličić, *On the Gram-Schmidt projection in normed spaces*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat., **4** (1993), 89–96.
[21] S. Kurepa, *On the Buniakowsky-Cauchy-Schwarz inequality*, Glas. Mat. III Ser., **21**(1) (1966), 147–158.
[22] F. R. Gantmacher, *The Theory of Matrices*, AMS Chelsea Publishing, **1**, 2000.