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On Directed Baire Spaces

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Abstract

We study directed Baire spaces and their relevant topological properties. A characterization of directed Baire spaces is given using point finite family of $G_\delta-$sets. Further, we prove that the product of directed Baire space with a metric hereditarily directed Baire space is a downward-directed Baire space. Finally, it is established that the product of a Baire space with a hereditarily metric Volterra space is again a Volterra space.

1. Introduction

A topological space X is a Baire space (resp. second category) if intersection of any sequence of dense open subsets of X is dense (resp. non-empty). It follows from the definition that the intersection of countably many dense $G_\delta-$sets of Baire space (resp. second category) X must be dense (resp. non-empty) in X [1]. The properties of Baire spaces and characterizations are studied in [2]. A family $\mathscr{B}$ of non-empty open subsets of a topological space is said to be *pseudo base* [3] ($\pi-$*base*) if every non-empty open set contains at least one member of $\mathscr{B}$. A space X is called a $P-$space [4] if every countable intersection of open subsets of X is open. A directed set [5](or a directed preorder or a filtered set) is a non-empty set Δ together with a reflexive and transitive binary relation $\leq$ (that is, a preorder), with the additional property that every pair of elements has an upper bound. In other words, for any a and b in Δ, there must exists a $c \in \Delta$ with $a \leq c$ and $b \leq c$. In this article, we consider only the directed set in which every two elements of it are comparable. A space X is a directed Baire space if intersection of family of dense $G_\delta-$subsets $\{D_\alpha \mid \alpha \in \Delta\}$ of X is dense and weakly directed Baire space if intersection of family of dense $G_\delta-$subsets $\{D_\alpha \mid \alpha \in \Delta\}$ of X is non-empty where Δ is a directed set. A space X is called downward-directed Baire space if intersection of the family of decreasing dense $G_\delta-$subsets $\{D_\alpha \mid \alpha \in \Delta\}$ of X is dense, where Δ is a directed set. The following Example 1.1 shows the existence of directed Baire spaces.

Example 1.1. *Consider $X = [0,\infty)$ with the topology having $\mathscr{B} = \{[0,a) \mid a \neq 0 \in X\}$ as its basis. In this space, the intersection of any family of dense $G_\delta-$subsets of X is dense.*

By definition itself it is clear that every directed Baire space is Baire, but there are Baire spaces which are not directed Baire, refer Example 1.2.

Example 1.2. *Consider $\mathbb{R}$ with usual metric. Since $\mathbb{R}$ is a complete metric space, it is a Baire space and hence second category. Since $\mathbb{Q}\cup\{\alpha\}$ is countable and each singleton sets of $\mathbb{R}$ is closed, $\mathbb{Q}\cup\{\alpha\}$ is an $F_\sigma-$set and its complement is a dense $G_\delta-$subset of $\mathbb{R}$ for every irrational $\alpha \in \mathbb{R}$. Hence there exists a family of dense $G_\delta-$sets such that their intersection is not dense (in particular empty set) namely $(\mathbb{Q}\cup\{\alpha\})^c$ where α runs over irrationals. Hence $\mathbb{R}$ with usual metric is neither directed Baire space nor weakly directed Baire.*

Also, by definition itself it is clear that every directed Baire space is weakly directed Baire, but the converse does not hold as shown by Example 1.3.

Example 1.3. *Consider $\mathbb{R}$ with the topology obtained from the basis $\mathscr{B} = \{(a,b) \mid a,b \in \mathbb{R}\}\cup\{0\}$. Since every dense set contains $\{0\}$, the intersection of every family of dense $G_\delta-$sets is non-empty. For every irrational $\alpha \in \mathbb{R}$ the set $(\mathbb{Q}-\{0\})\cup\{\alpha\}$ is countable, and each*

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singleton sets of $\mathbb{R}$ is closed, the above defined set is an $F_\sigma -$set and its complement is a dense $G_\delta -$subset. Hence there is a family of dense $G_\delta -$sets such that their intersection is not dense (which equals $\{0\}$) namely, $(\mathbb{Q}-\{0\}\cup\{\alpha\})^c$ where α runs over irrationals. Hence this topological space is weakly directed Baire, Baire and second category but not directed Baire.

Example 1.4. *There is a space which is weakly directed Baire and hence second category but not Baire and directed Baire. Let $X=\mathbb{Q}\cup(1,2)$ where $\mathbb{Q}$ denotes the set of the rational numbers in $(0,1)$ of the real line. Topologize X by the subbasis $\{\{(a,b)\mid a,b\in(1,2)\}\cup\mathbb{Q}\}$. Then X is not Baire because the open set $\mathbb{Q}$ is of first category. But X is weakly directed Baire in itself as the open subset $(1,2)$ is.*

Example 1.5. *Consider (X,τ) where $X=[0,\infty)$ and the topology τ has $\{[a,\infty)-F\mid a\in X$ and F is a finite subset of $X\}$ as its basis. By its construction, it is of first category, so it is none of the Baire, second category, directed Baire and weakly directed Baire.*

Example 1.6. *There is a space which is second category but not weakly directed Baire, Baire and directed Baire. Topologize $X=\mathbb{Q}\cup(0,2)$ by the subbasis $\{\{(a,b)\mid a,b\in(0,2)\}\cup\mathbb{Q}\}$. Since $\mathbb{Q}$ is a first category set, X fails to be a Baire space and so X is not a directed Baire space. But X is of second category in itself as the open subset $(1,2)$ is. Since $\mathbb{Q}$ is countable, $\{r_1,r_2,r_3\ldots\}$ be the sequential arrangements of $\mathbb{Q}$ and I is the collection of all irrationals in $(0,2)$. Define $H_i^\alpha=\{r_i\}\cup\{I-\{\alpha\}\}$ where α is irrational in $(0,2)$. $\{H_i^\alpha\mid i\in\mathbb{N}$ and $\alpha\in I\}$ is the collection of dense $G_\delta -$sets in X whose intersection is empty.*

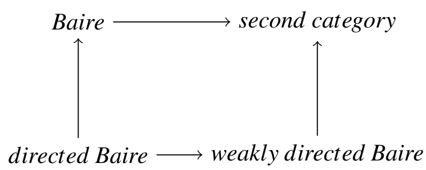
Theorem 1.7. *Every compact $p-$space X is a directed Baire space.*

Proof. Suppose $\{U_\alpha\mid\alpha\in\Delta\}$ is a family of dense $G_\delta -$subsets of X, and take U as an open subset of X. It is enough to show that $U\cap(\bigcap_\alpha U_\alpha)\neq\emptyset$. Since in a $p-$space, every $G_\delta -$sets are open, it is possible to construct a non-empty open subset V_α of X such that $V_\alpha\subset U\cap U_\alpha$. Defining recursively we have non-empty open subsets $\{V_\alpha\}$ of X such that $V_{\alpha+1}\subset V_\alpha\cap U_{\alpha+1}$, for each α and $V_{\alpha+1}$ is the successor of V_α. Suppose $\bigcap_\alpha V_\alpha=\emptyset$. Then define $W_\alpha=X-cl(V_\alpha)$, so that $\{W_\alpha\}$ is an open cover of X. As X is compact, we find a finite sub-cover $\{W_{\alpha_1},W_{\alpha_2},\ldots W_{\alpha_n}\}$ such that $\{W_{\alpha_k}\}$ covers X. Since $W_\alpha=X-cl(V_\alpha)$ we have $cl(V_\alpha)\subset int(X-W_\alpha)=X-cl(W_\alpha)$. Hence $V_\alpha\subset X-cl(W_\alpha)$ for every α. But $\emptyset=\bigcap_{k=1}^{n}(X-cl(W_{\alpha_k}))\supset\bigcap_{n=1}^{n}V_{\alpha_k}=V_{\alpha_n}$, which is a contradiction. Thus, $\emptyset\neq\bigcap_\alpha V_\alpha\subset U\cap(\bigcap_\alpha U_\alpha)$. □

Corollary 1.8. *Every submaximal compact space is a directed Baire space.*

Proof. Since in submaximal spaces, every dense set is open, the proof follows. □

Since every compact $p-$spaces are also Baire spaces, by Theorem 1.7, there are spaces which are Baire, directed Baire, weakly directed Baire and second category. Example 1.2 shows that there are Baire spaces which are not weakly directed Baire space, Example 1.4 shows that there are weakly directed Baire space which are neither Baire nor directed Baire spaces. The following arrow diagram shows the relation between the four spaces namely, Baire, second category, directed Baire and weakly directed Baire.



Hence Baire, second category, directed Baire and weakly directed Baire are independent concepts.

2. Characterization for directed Baire space

The hereditary property of (resp. weakly) directed Baire spaces need not be true for arbitrary spaces. Here, we prove that for some classes of subspaces, these properties are hereditary. This gives us a new characterization for directed Baireness of spaces.

Theorem 2.1. *In a directed Baire space X, if $H\subset X$ and $A\subset H$ where A is a G_δ set implies that $int(\overline{A})$ is dense in H, then H is a directed Baire space.*

Proof. Since $int(\overline{A})$ is dense in H, we have $\overline{H}\subset\overline{(H\cap int\overline{A})}$. For, $x\in\overline{H}$ implies $U_x\cap H\neq\emptyset$ for every neighborhood U_x of x. Therefore $y\in U_x\cap H$ is a neighborhood of y in H, Since $int(\overline{A})$ is dense in H, $(U_x\cap H)\cap int(\overline{A})\neq\emptyset$ so that $U_x\cap(H\cap int(\overline{A}))\neq\emptyset$. Hence $\overline{H}\subset\overline{(H\cap int\overline{A})}$. Since $A\subset H$, $\overline{A}\subset\overline{H}\subset\overline{H\cap int(\overline{A})}$. But $\overline{A}\subset\overline{H}\subset\overline{H\cap int(\overline{A})}\subset\overline{A}$. Therefore, $\overline{A}=\overline{H}=\overline{H\cap int(\overline{A})}$.
Let $\{D_\alpha:\alpha\in\Delta\}$ be a family of dense $G_\delta -$subsets in H. Then $\overline{H\cap D_\alpha}=\overline{H}$ for every α. Define $A^+=A\cup(X-\overline{H})$, and $D_\alpha^+=D_\alpha\cup(X-\overline{H})$ for every α. Then A^+ and D_α^+ are dense $G_\delta -$sets in X for every α and X is directed Baire $A^+\cap(\bigcap_\alpha D_\alpha)$ is dense in X. Hence $(A\cap(\bigcap_\alpha D_\alpha))\cup(X-\overline{H})$ is dense in X.
Now $\overline{A}=\overline{H}$ implies $\overline{int(\overline{A})}\subset\overline{(A\cap(\bigcap_\alpha D_\alpha))}$. Suppose not, there is an element $a\in int(\overline{A})$ with $a\notin\overline{(A\cap(\bigcap_\alpha D_\alpha))}$. That is, there exists U_a such that $U_a\cap(A\cap(\bigcap_\alpha D_\alpha))=\emptyset$. For every $a\in int(\overline{A})$, there exists V_a such that $V_a\cap(X-\overline{H})=\emptyset$. Since $a\in int(\overline{A})$, there exists an open set V_a, $V_a\subset\overline{A}$ so that $V_a\subset\overline{H}$. Take $W_a=U_a\cap V_a$. Then $W_a\cap((A\cap(\bigcap_\alpha D_\alpha))\cup(X-\overline{H}))=\emptyset$, which is a contradiction. $\overline{H}\subset\overline{(H\cap int(\overline{A}))}\subset\overline{(int(\overline{A}))}\subset\overline{A\cap(\bigcap_\alpha D_\alpha)}\subset\overline{H\cap(\bigcap_\alpha D_\alpha)}\subset\overline{H}$. Therefore, $\bigcap_\alpha D_\alpha$ is dense in H. □

Remark 2.2. *Observe that H satisfies the hypothesis of Theorem 2.1 if H is open or a regular closed or a dense G_δ subset of X.*

Corollary 2.3. *X is a directed Baire space if and only if each non-empty open subspace is a weakly directed Baire space.*

Proof. If X is directed Baire, then every non-empty open subspace is also directed Baire and hence weakly directed Baire. Conversely, suppose that each non-empty open subspace is weakly directed Baire. Let $\{D_\alpha \mid \alpha \in \Delta\}$ be a family of dense $G_\delta-$subsets of X. If O is a non-empty open subset of X, then $O \cap D_\alpha$ are $G_\delta-$subsets of O which are dense in O. Then $\bigcap_\alpha (O \cap D_\alpha) \neq \emptyset$ so that $O \cap (\bigcap_\alpha D_\alpha) \neq \emptyset$. Therefore, $\bigcap_\alpha D_\alpha$ is dense in X. □

Remark 2.4. *X is directed Baire if and only if each non-empty open subspaces U of X cannot be written as the union of any family of nowhere dense $F_\sigma-$sets in U.*

Theorem 2.5. *If $\mathscr{O}$ is a family of open subsets of X whose union is dense in X, then the following hold.*
(a) If there is some non-empty $U_1 \in \mathscr{O}$ such that U_1 is weakly directed Baire, then X is weakly directed Baire.
(b) If each member of $\mathscr{O}$ is directed Baire, then X is directed Baire.

Proof. (a) Let U_1 be a weakly directed Baire set in $\mathscr{O}$ and $\{O_\alpha \mid \alpha \in \Delta\}$ be a family of dense $G_\delta-$subsets of X. Then $U_1 \cap O_\alpha$ are dense $G_\delta-$sets in U_1. Since U_1 is weakly directed Baire, $\bigcap_\alpha (U_1 \cap O_\alpha) \neq \emptyset$ in U_1 which implies that $U_1 \cap (\bigcap_\alpha O_\alpha) \neq \emptyset$ so that $\bigcap_\alpha O_\alpha \neq \emptyset$. Hence X is a weakly directed Baire space.
(b) Let V_1 be a non-empty open subset of X and $\{O_\alpha \mid \alpha \in \Delta\}$ be a family of dense $G_\delta-$subsets of X. Since $\cup\{U_1 \mid U_1 \in \mathscr{O}\}$ is dense in X, $V_1 \cap U_1 \neq \emptyset$ for some $U_1 \in \mathscr{O}$. By hypothesis, $V_1 \cap U_1$ is a weakly directed Baire subspace of U_1. Now $O_\alpha \cap (U_1 \cap V_1)$ are dense $G_\delta-$sets of $U_1 \cap V_1$, $\bigcap_\alpha O_\alpha \cap (U_1 \cap V_1) \neq \emptyset$. Therefore, $(\bigcap_\alpha O_\alpha) \cap V_1 \neq \emptyset$. Hence X is directed Baire. □

Now we characterize directed Baire spaces in terms of point finite $G_\delta-$cover of X. A family $\mathscr{U} = \{U_\alpha \mid \alpha \in \mathscr{I}\}$ is said to be point finite in a topological space X if every point of X lies in only finite members of $\mathscr{U}$, and it is locally finite at $x \in X$ if every neighborhood of x intersects only finite members of $\mathscr{U}$.

Theorem 2.6. *A space X is directed Baire if and only if every point finite $G_\delta-$cover of X is locally finite at a dense set of points.*

Proof. Let $\mathscr{W} = \{U_\alpha \mid \alpha \in \Delta\}$ be a point finite $G_\delta-$cover of X and U be a non-empty open subset of X. Assume that $\mathscr{W}$ is not locally finite at any point of U. If $\mathscr{V} = \{V_\alpha\}$, $V_\alpha = U_\alpha \cap U$, then each open set in $\mathscr{W}$ intersects many members of $\mathscr{V}$. Put $\mathscr{F} = \{F_\alpha \mid F_\alpha \subset \Delta$ and $\Delta - F_\alpha$ is finite$\}$. Let $\mathscr{J}$ be the index set of the family $\mathscr{F}$. Now for each $J \in \mathscr{J}$, define $X_J = Bd(\cup\{V_\beta \mid \beta \in F_J\})$. Each X_J is closed and $int(X_J) = \emptyset$, so that each X_J is nowhere dense. Let $x \in U$. Since $\mathscr{W} = \{U_\alpha\}$ is point finite, there exists a $J' \in \mathscr{J}$ such that x belongs to the members of $\{V_\alpha \mid \alpha \in \Delta - F_{J'}\}$, but no other members of $\mathscr{V}$. So $x \notin \cup\{V_\beta \mid \beta \in F_{J'}\}$. If V is an open set containing x, then V intersects some members of $\{V_\beta \mid \beta \in F_{J'}\}$, since $\mathscr{V} = \{V_\alpha\}$ is not locally finite at any point of U. Since $x \notin \cup\{V_\beta \mid \beta \in F_{J'}\}$, $x \in X_{J'}$. Hence $U = \cup(U \cap X_J)$, which is a contradiction, by Remark 2.4. Conversely, let U be a non-empty open subset of X. Suppose X is not directed Baire, $U = \cup X_\alpha$, where $int(\overline{X_\alpha}) = \emptyset$ for each α in the index set Δ, by Remark 2.4. Let $U_0 = X$ and for each $\alpha \in \Delta$ define $U_\alpha = U - \bigcup_{\beta \leq \alpha} \overline{X_\beta}$. Let $\mathfrak{U} = \{U_\alpha\}$ where $\alpha \in \Delta$, which is a point finite $G_\delta-$cover of X. Then $\mathfrak{U}$ is locally finite at some x in U. Let O be an open set of x such that $x \in O \subset U$. Since $int(X_\alpha) = \emptyset$, $O \nsubseteq \bigcup_{\beta \leq \alpha} \overline{X_\beta}$ for each α. Thus, O must intersect every member of $\mathfrak{U}$, which is a contradiction to locally finiteness of the point finite $G_\delta-$cover $\mathfrak{U}$. □

Blumberg [6] showed that for every real valued function f defined on the real line $\mathbb{R}$, there exists a dense subset D of $\mathbb{R}$ such that $f|_D$ is continuous. We will say that space X has Blumberg's property with, respect to Y if for every function $f : X \to Y$, there exists a dense subset D of X such that $f|_D$ is continuous. It is known [7] that for a metric space X, X is a Baire space if and only if X has Blumberg's property with respect to the reals. In Theorem 2.7, the similar result is proved for directed Baire space.

Theorem 2.7. *Let Y contain an infinite discrete subset $D = \{y_\alpha \mid \alpha \in \Delta\}$. If X satisfies Blumberg's property with respect to Y, then X is a directed Baire space.*

Proof. Let $\mathbb{D} = \{y_\alpha \mid \alpha \in \Delta\}$ be a infinite discrete subset of Y. If X is not a directed Baire space, then there is an open set U in X such that $U = (\bigcup_\alpha U_\alpha)$. Define a function $f : X \to Y$ as follows: let $f(x) = y_{\alpha_0}$ for each $x \in X - U$, where $y_{\alpha_0} \in D$ and let $f(x) = y_\beta$ for each $x \in U$, where $\beta = \min\{\alpha \mid x \in U_\alpha\}$. From the construction of the function f, $f|_D$ is not continuous for every dense subset D of X. □

3. Product of directed Baire spaces

A directed Baire space in which every closed subspace is also directed Baire space is called a hereditarily directed Baire space. We discuss the product of directed Baire spaces. The following Lemma 3.1 is useful in the sequel.

Lemma 3.1. *Let Y be a topological space, (A,d) be a metric space and C be a dense $G_\delta-$subset of $Y \times A$. Then given any finite subset F of A, $\varepsilon > 0$ and non-empty open set O of Y, there exists a finite subset A' of A and a dense $G_\delta-$subset C_Y of O such that*
(i) for each $z \in F$, there exists $a \in A'$ with $d(z,a) < \varepsilon$
(ii) $C_Y \times A' \subseteq C$.

Proof. For the given finite subset F of A, define an open subset $V = \cup B(z,\varepsilon)$ of A where union runs over the points of F. Since C is dense in $Y \times A$, $(O \times V) \cap C \neq \emptyset$. Then $C_Y = P_X((O \times V) \cap C)$ and $A' \subset P_Y((O \times V) \cap C)$ are the requirements. □

Theorem 3.2. *If X is a directed Baire space and Y is a metrizable hereditarily directed Baire space, then $X \times Y$ is a downward-directed Baire space.*

Proof. Let $\{D_\alpha \mid \alpha \in \Delta\}$ be a family of decreasing dense $G_\delta-$sets in $X\times Y$. We prove that $\bigcap_\alpha D_\alpha$ is dense in $X\times Y$. Let G and H be any non-empty open sets in X and Y, respectively. To prove $[\bigcap_\alpha D_\alpha]\cap(G\times H)\neq\emptyset$. Let $\{s_\alpha \mid \alpha\in\Delta\}$ be a net in $[0,\infty)$ with usual metric, which converges to 0.
Let α_1 be the least member of Δ. Since D_{α_1} is dense $(G\times H)\cap D_{\alpha_1}\neq\emptyset$. Define a dense $G_\delta-$set of G, $X_{\alpha_1}=P_X((G\times H)\cap D_{\alpha_1})$ and $Z_{\alpha_1}=\{y\}$, where $y\in P_Y((G\times H)\cap D_{\alpha_1})$.
By Lemma 3.1, for any finite subset Z_{α_1} of Y, non-empty open set G in X, dense $G_\delta-$set D_β of $X\times Y$ and $s_\beta>0$, we can find a finite subset Y_β of Y and a dense G_δ subset X_β of G such that
(i) for each $z\in Z_{\alpha_1}$, one can find $y\in Y_\beta$ with $d(z,y)<s_\beta$
(ii) $X_\beta\times Y_\beta\subseteq D_\beta$. Then we define
(iii) $Z_\beta=Z_{\alpha_1}\cup Y_\beta$, where β is the successor of α_1.
Continuing in this way, we reached a family of dense $G_\delta-$subsets $\{X_\alpha\mid\alpha\in\Delta\}$ of G. Since X is a directed Baire space, $\bigcap_\alpha X_\alpha\neq\emptyset$. Choose $x\in\bigcap_\alpha X_\alpha$ and define, for each $\alpha\in\Delta$, the dense $G_\delta-$subsets $\{W_\alpha\mid\alpha\in\Delta\}$ of Y so that $\{x\}\times W_\alpha=(\{x\}\times Y)\cap D_\alpha$.
Let $Z^+=\bigcup_\alpha Z_\alpha\subset H$. Since Y is hereditarily directed Baire, $\overline{Z^+}$ is directed Baire which implies $W_\alpha\cap Z^+$ is dense in Z^+ for each $\alpha\in\Delta$. For, let $z\in Z^+$, $\alpha\in\Delta$ and $\varepsilon>0$ be given. Since the net (s_α) converges to 0, for the neighborhood $[0,\varepsilon)$ of 0, we can find $\delta\in\Delta$ such that $0\le s_\alpha<\varepsilon$ for every $\alpha>\delta$. Choose $\rho\in\Delta$ sufficiently large so that $\rho>\alpha$, $s_\rho<\varepsilon$ and $z\in Z_\rho$. There is an element $y\in Y_{\rho_1}$ with $d(y,z)<s_{\rho_1}<s_\delta<\varepsilon$ and $(x,y)\in D_{\rho_1}\cap(\{x\}\times Y)$, which implies that $y\in W_{\rho_1}\subset W_\alpha$, where ρ_1 is the successor of ρ. Thus, $y\in B(z,\varepsilon)\cap(W_\alpha\cap Z^+)$ so that $B(z,\varepsilon)\cap(W_\alpha\cap Z^+)$ is non-empty. Choosing $y\in(\bigcap_\alpha W_\alpha)\cap H\cap\overline{Z^+}$, we get that $(x,y)\in[\bigcap_\alpha D_\alpha]\cap(G\times H)$. □

Theorem 3.3. *Let X and Y be directed Baire spaces. If either of the space has a countable pseudo base, their product is directed Baire.*

Proof. Assume that $X\times Y$ is not directed Baire. We can find an open set $G\times H$ in the product space such that $(G\times H)\cap(\bigcap_\alpha D_\alpha)=\emptyset$ where $\{D_\alpha\mid\alpha\in\Delta\}$ is a family of dense $G_\delta-$ sets in $X\times Y$. Since D_α are $G_\delta-$sets, $D_\alpha=\bigcap_{n=1}^{\infty}D_\alpha^n$ where D_α^n are open in $X\times Y$. Since D_α is dense, each D_α^n is also dense.
Let $\{V_k\}$ be a countable pseudo base for Y. Now for each n,k and α, define $h_\alpha^{n,k}=D_\alpha^n\cap(U\times V_k)$. Also, define $H_\alpha^{n,k}=P_X(h_\alpha^{n,k})$ so that $H_\alpha^{n,k}$ are open. Also, D_α^n is dense in $G\times H$ implies $D_\alpha^n\cap(G\times V_k)$ is dense in $G\times V_k$ which implies $h_\alpha^{n,k}$ is dense in $G\times V_k$. For any open set U_1 in G, $U_1\times V_k$ is an open set in $U\times V_k$. Therefore, $(U_1\times V_k)\cap h_\alpha^{n,k}\neq\emptyset$ implies $U_1\cap P_X(h_\alpha^{n,k})\neq\emptyset$ which implies $U_1\cap H_\alpha^{n,k}\neq\emptyset$. Therefore, each $H_\alpha^{n,k}$ is dense in G. Since X is directed Baire, G will become directed Baire, by Remark 2.4.
Since G is directed Baire, $\bigcap_{n,k}[G\cap H_\alpha^{n,k}]$ are dense in G and so $\bigcap_{n,k}[G\cap H_\alpha^{n,k}]\neq\emptyset$. Therefore, there exists some $a\in G$ with $a\in\bigcap_{n,k}[G\cap H_\alpha^{n,k}]$ which gives $a\in H_\alpha^{n,k}$ for every n,k.
Define $D_\alpha^n(a)=\{b\in H\mid(a,b)\in D_\alpha^n\}$. For each V_k, $(a,b)\in D_\alpha^n\cap(G\times V_k)$ for all n,k implies $(a,b)\in\bigcap_n[D_\alpha^n\cap(G\times V_k)]$ which gives that $(a,b)\in D_\alpha\cap(G\times V_k)$. Therefore, there is some $b\in V_k$ with $(a,b)\in D$ so that $b\in V_k$ such that $b\in D_\alpha^n(a)$. Therefore, $D_\alpha^n(a)\cap V_k\neq\emptyset$. Therefore, $D_\alpha^n(a)$ is dense in H. Also, $D_\alpha^n(a)$ is an open set.
Since Y is directed Baire, H is also directed Baire and hence $\bigcap_{n,\alpha}D_\alpha^n(a)\neq\emptyset$. Therefore, we can find $z\in H$ with $z\in\bigcap_{n,\alpha}D_\alpha^n(a)$ and hence $(a,z)\in\bigcap_{n,k}D_\alpha^n=D_\alpha$ which is not possible. Thus, $\bigcap_\alpha D_\alpha\neq\emptyset$ and so $G\times H$ is a weakly directed Baire space. □

4. Product of Volterra spaces

In 1993, the class of Volterra spaces was introduced by Gauld and Piotrowski [8]. A topological space (X,τ) is said to be *Volterra* [8, 9] (resp. *weakly Volterra* [8]) if the intersection of any two dense $G_\delta-$ sets in X is dense (resp. non-empty). By the definition itself, every Baire space is Volterra and every space of second category is weakly Volterra. Is there exists a Baire space X whose square X^2 is not Baire ? The first space with such properties, constructed under the Continuum Hypothesis, is due to Oxtoby [3]. This example was improved to an absolute one by Cohen [10] relying on forcing. Finally, Fleissner and Kunen [11] constructed a metrizable Baire space X whose square X^2 is not Baire in ZFC by direct combinatorial arguments. Gauld, Greenwood and piotrowski [12], using stationary sets in the result of Flessner proved that there exists a metric Baire space whose square is not even Weakly Volterra. Spadaro [13] proved that the product of a hereditarily volterra space and a hereditarily Baire space may fail to be weakly volterra. In [14], Moors proved that "The Product of a Baire space with a hereditarily Baire metric space is Baire". In that proof, he use Choquet game [15]-[17] played on X to get a non-empty subset for any given sequence of dense open sets in X.

Theorem 4.1. *If X is Baire and Y is metrizable hereditarily Volterra, then $X\times Y$ is a Volterra space.*

Proof. Suppose that C and D are two dense $G_\delta-$sets in $X\times Y$. Let G and H be non-empty open sets in X and Y, respectively. To prove $(C\cap D)\cap(G\times H)\neq\emptyset$. Since C and D are dense $G_\delta-$sets, $C=\bigcap_{n=1}^{\infty}C_n$ and $D=\bigcap_{n=1}^{\infty}D_n$, where $\{C_n\}$ and $\{D_n\}$ are decreasing sequence of open dense sets in $X\times Y$.
Denseness of C gives that $(G\times H)\cap C\neq\emptyset$. Define a dense $G_\delta-$set of G, $C_1=P_X((G\times H)\cap C)$ and $Z_1^C=\{b^C\}$, where $b^C\in P_Y((G\times H)\cap C)$.
Also, since D is dense, $(G\times H)\cap D\neq\emptyset$. Define a dense $G_\delta-$set of G, $D_1=P_X((G\times H)\cap D)$ and $Z_1^D=\{b^D\}$, where $b^D\in P_Y((G\times H)\cap C)$.
Also, define $Z_1=Z_1^C\cup Z_1^D$.
By Lemma 3.1, for a finite set Z_1 of Y, non-empty open set G in X, dense $G_\delta-$set C of $X\times Y$ and $\frac{1}{2}>0$, there is a finite subset Z_2^C of Y and a dense G_δ subset C_2 of G such that
(i) for every $a\in Z_1$, there is some $b\in Z_2^C$ with $d(a,b)<\frac{1}{2}$
(ii) $C_2\times Z_2^C\subseteq C$.

Also, by Lemma 3.1, for a finite set Z_1 of Y, non-empty open set G in X, dense $G_\delta-$set D of $X\times Y$, and $\frac{1}{2}>0$ there is a finite subset Z_2^D of Y and a dense G_δ subset D_2 of G such that
(i) for every $a\in Z_1$, there is some $b\in Z_2^D$ with $d(a,b)<\frac{1}{2}$
(ii) $D_2\times Z_2^D\subseteq D$.
Define $Z_2=Z_1\cup Z_2^C\cup Z_2^D$.
Continuing in this way, for every $n\in\mathbb{D}$, by Lemma 3.1, given any finite subset Z_{n-1} of Y, non-empty open set G in X, dense $G_\delta-$set C of $X\times Y$ and $\frac{1}{n}>0$, there is a finite subset Z_n^C of Y and a dense G_δ subset C_n of G such that
(i) for every $a\in Z_{n-1}$, there is some $b\in Z_n^C$ with $d(a,b)<\frac{1}{n}$
(ii) $C_n\times Z_n^C\subseteq C$.
Also, given any finite subset Z_{n-1} of Y, non-empty open set G in X, dense $G_\delta-$set D of $X\times Y$ and $\frac{1}{n}>0$, there is a finite subset Z_n^D of Y and a dense G_δ subset D_n of G such that
(i) for every $a\in Z_{n-1}$, there is some $b\in Z_n^D$ with $d(a,b)<\frac{1}{n}$
(ii) $D_n\times Z_n^D\subseteq D$.
Define $Z_n=Z_{n-1}\cup Z_n^C\cup Z_n^D$.
The countable collection $\{C_n\mid n\in\mathbb{D}\}\cup\{D_n\mid n\in\mathbb{D}\}$ of dense $G_\delta-$subsets can be enumerated as a sequence of dense $G_\delta-$sets $\{H_i\mid i\in\mathbb{D}\}$ of G. Since every H_i is a dense $G_\delta-$set, $H_i=\bigcap_{j=1}^{\infty}H_i^j$ where H_i^j is a dense open set in G. Since a countable union of countable set the family $\{H_i^j\mid i,j\in\mathbb{D}\}$ also can be enumerated as a sequence of dense open sets $\{O_m\mid m\in\mathbb{D}\}$. Since X is a Baire space, the open subset G is also a Baire space. Therefore, $\bigcap_{m=1}^{\infty}O_m\neq\emptyset$.
Choose $s\in\bigcap_{m=1}^{\infty}O_m$ and define, $C(s)=\{t\in H\mid(s,t)\in C\}$ and $D(s)=\{t\in H\mid(s,t)\in D\}$. Now $C(s)=(\bigcap_m C_m)(s)=\bigcap_m[C_m(s)]$, because $t\in(\bigcap_m C_m)(s)\Leftrightarrow(s,t)\in\bigcap_m C_m\Leftrightarrow(s,t)\in C_m$ for all $m\Leftrightarrow t\in C_m(s)$ for all $m\Leftrightarrow t\in\bigcap_m[C_m(s)]$. Therefore, $C(s)$ is a $G_\delta-$set. Similarly, $D(s)$ is also a $G_\delta-$set.
Let $S=\bigcup_{n=1}^{\infty}Z_n\subset H$. Since Y is hereditarily Volterra, $\overline{S}$ is Volterra and hence $C(s)\cap S$ and $D(s)\cap S$ are dense in S.
For, $a\in Z$, and $\varepsilon>0$ be given. Choose $N\in\mathbb{N}$ sufficiently large so that $1/N<\varepsilon$ and $a\in Z_{N-1}$. There is some $t\in Z_N^C$ such that $d(t,a)<1/N<\varepsilon$ and $C_N\times Z_N^C\subseteq C$. Hence $(s,t)\in(C_N\times Y)\cap(\{s\}\times Y)\subset C$, which implies that $t\in C(s)$. Thus, $t\in B(a,\varepsilon)\cap C(s)\cap Z\neq\emptyset$. Similarly, $D(s)\cap Z$ is also dense in Z. Choosing $t\in C(s)\cap D(s)\cap H\cap\overline{Z}$, we get $(s,t)\in C\cap D\cap(G\times H)$. Hence $C\cap D$ is dense in the product space. □

In Theorem 4.1 above, the hereditary property cannot be dropped, since Fleissner and Kunen [11] constructed a metrizable Baire space X whose square X^2 is not Baire. Since Spadaro [13], shows that the product of a hereditarily volterra space and a hereditarily Baire space may fail to be weakly volterra the metrizability of the Volterra space cannot be dropped in the above theorem.
Piotrowski raised a question that, "*Whether* $X\times[0,1]$ *is Volterra or not*? for any Volterra space X". As a partial answer to this question, in Corollary 4.2 below, we consider a subfamily of Volterra spaces consisting of metrizable hereditarily Volterra space, and proved that cartesian product of X and $[0,1]$ is again a Volterra space.

Corollary 4.2. *If* (X,τ) *is a metrizable hereditarily Volterra space, then* $X\times[0,1]$ *is also a Volterra space.*

Proof. It is well known that, a subset A of a complete metric space (M,d) is complete if and only if A is a closed subset of M and consequently, $[0,1]$ is complete. Since $[0,1]$ is Baire, $X\times[0,1]$ is Volterra, by Baire Category Theorem and Theorem 4.1. □

5. Conclusion

In this paper, we have introduced the concepts of directed Baire and weakly directed Baire spaces. Since every compact p-spaces are also Baire spaces, we have proved that there are spaces which are Baire, directed Baire, weakly directed Baire and second category. Also, it is shown that there are Baire spaces which are not weakly directed Baire space and there are weakly directed Baire space which are neither Baire nor directed Baire spaces, by giving examples. Hence we have proved that the concepts namely, Baire, second category, directed Baire and weakly directed Baire are independent. We have proved that the product of directed Baire spaces is also Directed Baire if either of the space has a countable pseudo base. Also, we have provided partial answer for the question raised by Piotrowski regarding product of Volterra spaces. The results of this article can also be applied on generalized topological spaces and ideal topological spaces by some suitable modifications. We hope that this work will provide the basis for further study on directed Baire spaces.

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