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Research Article

Generalization of functions of bounded Mocanu variation with respect to 2k-symmetric conjugate points

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Abstract

In this paper, by using convolution we generalize the class of analytic functions of bounded Mocanu variation with respect to 2k-symmetric conjugate points and study some of its basic properties. Our results generalize many research works in the literature.

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Keywords. bounded radius rotation, bounded boundary rotation, bounded Mocanu variation, 2k-symmetric conjugate points

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f defined on the unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in E). \tag{1.1}$$

Also, suppose that S, K, S^*, and C denote the subclasses of $\mathcal{A}$ which are univalent, close-to-convex, starlike, and convex in E respectively. We denote by $P_m(\gamma)$ the class of functions $p(z)$ analytic in the unit disc E satisfying the properties $p(0) = 1$ and, for $z = re^{i\theta}, m \geq 2$,

$$\int_0^{2\pi} \left| Re \frac{p(z) - \gamma}{1 - \gamma} \right| d\theta \leq m\pi, \qquad (0 \leq \gamma < 1). \tag{1.2}$$

The class $P_m(\gamma)$ for $\gamma = 0$ and $0 \leq \gamma < 1$ has been introduced and investigated by Pinchuk [13], and Padmanabhan and Parvatham [12] (see also [11]), respectively. We note that $P_m(0) = P_m$, and $P_2(\gamma) = P(\gamma)$ is the class of analytic function with positive real part greater than γ. For $m = 2$ and $\gamma = 0$, we have the class P of functions with positive real part.

We can rewrite (1.2) as

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\gamma) z e^{-it}}{1 - z e^{-it}} d\mu(t),$$

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where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2 \qquad \text{and} \qquad \int_0^{2\pi} |d\mu(t)| \leq m.$$

Also, for $p \in P_m(\gamma)$, we can write from (1.2)

$$p(z) = (\frac{m}{4} + \frac{1}{2})p_1(z) - (\frac{m}{4} - \frac{1}{2})p_2(z), \qquad p_1, p_2 \in P_2(\gamma), z \in E.$$

It is known [7] that $P_m(\gamma)$ is a convex set. Also $p \in P_m(\gamma)$ is in $P_2(\gamma) = P(\gamma)$ for $|z| < r_1$, where

$$r_1 = \frac{1}{2}[m - \sqrt{m^2 - 4}].$$

We say that $f \in \mathcal{A}$ is subordinate to $F \in \mathcal{A}$, and we write $f(z) \prec F(z)$ (or simply $f \prec F$), if there exists a function

$$\omega \in \Omega := \{\omega \in \mathcal{A} : |\omega(z)| \leq |z| \quad (z \in E)\},$$

such that $f(z) = F(\omega(z))$. In particular, if F is univalent in E, we have the following equivalence

$$f(z) \prec F(z) \qquad \Longleftrightarrow \qquad [f(0) = F(0) \wedge f(E) \subset F(E)].$$

Recently Mocanu introduced the class $\mathcal{M}(\alpha)$ of functions $f \in \mathcal{A}$ such that $\frac{f(z)f'(z)}{z} \neq 0$ for $z \in E$ and

$$Re\left\{\alpha\frac{zf'(z)}{f(z)} + (1-\alpha)\frac{(zf'(z))'}{f'(z)}\right\} > 0 \qquad (z \in E).$$

In particular, $S^* := \mathcal{M}(1)$, $K := \mathcal{M}(0)$ are the well-known classes of starlike functions and convex functions, respectively. Also, Wang et al. [17] (see also [18]) introduced the class $\mathcal{K}_{sc}^{(k)}(\alpha, \varphi)$ of functions $f \in \mathcal{A}$ such that

$$\alpha\frac{zf'(z)}{f_{2k}(z)} + (1-\alpha)\frac{(zf'(z))'}{f'_{2k}(z)} \prec \varphi(z), \qquad (z \in E),$$

where $\varphi(z) \in P$, $k \geq 2$ is a fixed positive integer and $f_{2k}(z)$ is defined by the following equality

$$f_{2k}(z) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}[\varepsilon^{-\upsilon}f(\varepsilon^{\upsilon}z) + \varepsilon^{\upsilon}\overline{f(\varepsilon^{\upsilon}\overline{z})}], \qquad (\varepsilon = \exp(\frac{2\pi i}{k})).$$

Also, Noor et al. [5] (see also, [1], [6], [7], [8], [9]) introduced and investigated class $R_s^k(\gamma)$ of analytic functions of bounded radius rotation of order γ with respect to symmetrical points if and only if

$$\frac{2zf'(z)}{f(z) - f(-z)} \in P_k(z), \qquad (z \in E).$$

Motivated by the aforementioned classes, and [1], [2], [3], [15], [16], we now introduce and investigate the following classes $\mathcal{M}_{\lambda,\mu}^k(\Phi, \xi, h)$ and $\mathcal{CM}_{\lambda,\mu}^k(\Phi, \xi, \mathbf{h})$ associated with functions of bounded variation with respect to 2k- symmetric conjugate points.

Let h be convex and symmetric with respect to the real axis with $h(0) = 1$, $\mu \geq 1$, and define

$$\mathcal{K}_\mu(h) := \{\mu q_1 + (1-\mu)q_2 : q_1, q_2 \prec h\}.$$

We note that the class $\mathcal{P} := \mathcal{K}_1\left(\frac{1+z}{1-z}\right)$ is the well-known class of Carathéodory functions.

It is easy to verify that

(i) $\mathcal{K}_\mu(h)$ is convex set,

(ii) if $1 \leq \mu \leq \lambda$ then $\mathcal{K}_\mu(h) \subset \mathcal{K}_\lambda(h)$,

(iii) Let $h'(0) \neq 0$ and $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{K}_\mu(h)$ then for $z = re^{i\theta}$,

$$|a_n| \leq (2\mu - 1)|h'(0)|, \qquad (n \geq 1), \tag{1.3}$$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq \frac{1 + [(2\mu-1)^2|h'(0)|^2 - 1]r^2}{1 - r^2}, \tag{1.4}$$

$$\frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})| d\theta \leq \frac{(2\mu-1)|h'(0)|}{(1-r)^2}. \tag{1.5}$$

Throughout this paper we assume that $\phi, \varphi, \xi \in \mathcal{A}$ and ϕ, φ, ξ are symmetric with respect to the real axis.

Definition 1.1. Let $\lambda \in R$ and $\Phi = (\phi, \varphi)$. We denote by $\mathcal{M}^k_{\lambda,\mu}(\Phi, \xi, h)$ the class of functions $f \in \mathcal{A}$ such that

$$(1-\lambda)\frac{(\xi * \phi) * f}{(\xi * \varphi) * f_{2k}} + \lambda \frac{\phi * f}{\varphi * f_{2k}} \in \mathcal{K}_\mu(h), \tag{1.6}$$

where * denotes the Hadamard product (or convolution) and $f_{2k}(z)$ is defined by

$$f_{2k}(z) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}[\varepsilon^{-\upsilon} f(\varepsilon^\upsilon z) + \varepsilon^\upsilon \overline{f(\varepsilon^\upsilon \overline{z})}], \qquad (\varepsilon = \exp(\frac{2\pi i}{k})).$$

Moreover, let us define

$$\mathcal{M}^k_{\lambda,\mu}(\Phi, h) := \mathcal{M}^k_{\lambda,\mu}(\Phi, \xi_1, h), \qquad \mathcal{M}^k_{\lambda,\mu}(\varphi, h) := \mathcal{M}^k_{\lambda,\mu}((\varphi_1, \varphi_2), h),$$

$$\mathcal{W}^k_\mu(\Phi, h) := \mathcal{M}^k_{1,\mu}(\Phi, z, h), \qquad \mathcal{W}^k_\mu(\varphi, h) := \mathcal{W}^k_\mu((z\varphi', \varphi), h),$$

where

$$\xi_1(z) = z + \sum_{n=2}^{\infty} \frac{z^n}{n}, \qquad \varphi_1 = z\varphi'(z), \qquad \varphi_2 = z\varphi_1', \qquad (z \in E). \tag{1.7}$$

Definition 1.2. Let $\mathbf{m} = (\mu_1, \mu_2)$ with $\mu_1, \mu_2 \geq 1$ and let h_1, h_2 be convex analytic functions that are symmetric with respect to the real axis so that $h_1(0) = h_2(0) = 1$. Suppose that $\mathbf{h} = (h_1, h_2)$. We say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{CM}^k_{\lambda,\mu}(\Phi, \xi, \mathbf{h})$ if there exists $g \in \mathcal{W}^k_{\mu_2}(\varphi, h_2)$ such that

$$(1-\lambda)\frac{(\xi * \phi) * f}{(\xi * \varphi) * g_{2k}} + \lambda \frac{\phi * f}{\varphi * g_{2k}} \in \mathcal{K}_{\mu_1}(h_1), \tag{1.8}$$

where $g_{2k}(z)$ is defined by

$$g_{2k}(z) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}[\varepsilon^{-\upsilon} g(\varepsilon^\upsilon z) + \varepsilon^\upsilon \overline{g(\varepsilon^\upsilon \overline{z})}], \qquad (\varepsilon = \exp(\frac{2\pi i}{k})).$$

Moreover, suppose that

$$\mathcal{CM}^k_{\lambda,\mu}(\Phi, \mathbf{h}) := \mathcal{CM}^k_{\lambda,\mu}(\Phi, \xi_1, \mathbf{h}), \qquad \mathcal{CM}^k_{\lambda,\mu}(\varphi, \mathbf{h}) := \mathcal{CM}^k_{\lambda,\mu}(((\varphi_2, \varphi_1), \mathbf{h})),$$

$$\mathcal{CW}^k_\mu(\Phi, h) := \mathcal{CM}^k_{1,\mu}(\Phi, z, h), \qquad \mathcal{CW}^k_\mu(\varphi, \mathbf{h}) := \mathcal{CM}^k_{1,\mu}((z\varphi', \varphi), \mathbf{h}),$$

where ξ_1, φ_1 and φ_2 are defined by (1.7).

These general classes of functions reduce to the well-known classes by judicious choices of the parameters. In particular, the class $\mathcal{M}^k_{\lambda,\mu}(\varphi, h)$ contains the functions $f \in \mathcal{A}$ such that

$$(1-\lambda)\frac{z(\varphi * f)'(z)}{(\varphi * f_{2k})(z)} + \lambda\left(\frac{(z(\varphi * f)')'(z)}{(\varphi * f_{2k})'(z)}\right) \in \mathcal{K}_\mu(h).$$

The classes

$$R^k_\mu(h) := \mathcal{M}^k_{1,\mu}(\Phi, \xi, h), \qquad V^k_\mu(h) := \mathcal{M}^k_{0,\mu}(\Phi, \xi, h)$$

are the general classes of bounded radius rotation functions with respect to 2k-symmetric conjugate points and bounded boundary rotation functions with respect to 2k-symmetric conjugate points, respectively.

In our investigation we need the following lemmas.

Lemma 1.3 (see [4]). *Let q be a convex analytic function in E. Also suppose that p is an analytic function in the unit disc and $P : E \mapsto \mathbb{C}$ be a function such that $ReP(z) > 0$ for $z \in E$. Then*

$$p(z) + P(z)zp'(z) \prec q(z) \Rightarrow p(z) \prec q(z).$$

Lemma 1.4 (see [4]). *Let $\beta, \gamma \in \mathbb{C}$ and h is convex (univalent) function in E with*

$$h(0) = 1 \qquad \text{and} \qquad Re(\beta h(z) + \gamma) > 0, \qquad (z \in E).$$

If p is analytic in E with $p(0) = 1$, then subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)$$

implies that

$$p(z) \prec h(z).$$

Lemma 1.5. *Let p and ψ be analytic functions in E with $p(0) = 1$ and $Re\psi(z) > 0$ for $z \in E$. If*

$$p(z) + \psi(z)zp'(z) \in \mathcal{K}_m(h),$$

then $p(z) \in \mathcal{K}_m(h)$.

Proof. From the definition of $\mathcal{K}_m(h)$, there exist two analytic functions q_1, q_2 with $q_1 \prec h$ and $q_2 \prec h$ such that

$$p(z) + \psi(z)zp'(z) = mq_1(z) + (1 - m)q_2(z). \tag{1.9}$$

Suppose that p_1 and p_2 are the solutions of the Cauchy problems

$$y(z) + \psi(z)zy'(z) = q_1(z), \qquad y(0) = 1, \tag{1.10}$$

and

$$y(z) + \psi(z)zy'(z) = q_2(z), \qquad y(0) = 1, \tag{1.11}$$

respectively. In the view of (1.10) and (1.11) we rewrite (1.9) as

$$p(z) + \psi(z)zp'(z) = m[p_1(z) + \psi(z)zp_1'(z)] + (1 - m)[p_2(z) + \psi(z)zp_2'(z)],$$

or equivalently,

$$[p(z) - mp_1(z) - (1 - m)p_2(z)] + z\psi(z)[p'(z) - mp_1'(z) - (1 - m)p_2'(z)] = 0. \tag{1.12}$$

Now if we define $\eta(z) = p(z) - mp_1(z) - (1 - m)p_2(z)$, then $\eta(0) = 0$ and (1.12) yields

$$\eta(z) + \psi(z)z\eta'(z) = 0, \qquad \eta(0) = 0. \tag{1.13}$$

But it is clear that Cauchy problem (1.13) has only the solution $\eta(z) = 0$. Hence $p(z) = mp_1(z) + (1 - m)p_2(z)$. For completing the proof we show that $p_1, p_2 \prec h$. From the equation (1.9) we can write

$$p_1(z) + \psi(z)zp_1'(z) \prec h(z).$$

Since $Re\psi(z) > 0$, applying Lemma 1.3 we obtain $p_1(z) \prec h(z)$. Similarly we have $p_2(z) \prec h(z)$ and this means that $p \in \mathcal{K}_m(\gamma)$ and the proof is complete. □

Lemma 1.6. *Let $\eta, f \in \mathcal{A}$ with $\eta(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $f(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Also suppose that η is symmetric with respect to the real axis. Then*

$$(\eta * f_{2k})(z) = (\eta * f)_{2k}(z).$$

Proof. By the definition of f_{2k} we have

$$f_{2k}(z) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}[\varepsilon^{-\upsilon}f(\varepsilon^{\upsilon}z) + \varepsilon^{\upsilon}\overline{f(\varepsilon^{\upsilon}\overline{z})}]$$
$$= z + \sum_{n=2}^{\infty}\left[\frac{1}{2k}\sum_{\upsilon=0}^{k-1}\left(b_n\varepsilon^{\upsilon(n-1)} + \overline{b_n}\varepsilon^{\upsilon(1-n)}\right)z^n\right].$$

But η is symmetric with respect to the real axis, so $\overline{a_n} = a_n$ for all $n \geq 2$ and it yields

$$(\eta * f_{2k})(z) = z + \sum_{n=2}^{\infty}\left[\frac{1}{2k}\sum_{\upsilon=0}^{k-1}\left(a_nb_n\varepsilon^{\upsilon(n-1)} + a_n\overline{b_n}\varepsilon^{\upsilon(1-n)}\right)z^n\right]$$
$$= z + \sum_{n=2}^{\infty}\left[\frac{1}{2k}\sum_{\upsilon=0}^{k-1}\left(a_nb_n\varepsilon^{\upsilon(n-1)} + \overline{a_nb_n}\varepsilon^{\upsilon(1-n)}\right)z^n\right]$$
$$= (\eta * f)_{2k}(z).$$

Hence the proof is complete. □

For $\alpha < 1$, we denote by $R(\alpha)$ the class of all analytic functions $f \in \mathcal{A}$ such that

$$f(z) * \frac{z}{(1-z)^{2(1-\alpha)}} \in S^*(\alpha).$$

The class $R(\alpha)$ is the class of prestarlike functions of order α introduced and investigated by Ruscheweyh [14].

Lemma 1.7 (see [14]). *Let $f \in \mathcal{R}(\alpha)$, $g \in S^*(\alpha)$. Then*

$$\frac{f * (qg)}{f * g}(E) \subseteq \overline{co}\,\{q(E)\},$$

for $q \in \mathcal{A}$.

2. Basic properties of $\mathcal{M}^k_{\lambda,\mu}(\Phi, \xi, h)$ and $\mathcal{CM}^k_{\lambda,\mu}(\Phi, \xi, \mathbf{h})$

Theorem 2.1. *Let $f \in \mathcal{M}^k_{\lambda,\mu}(\Phi, \xi, h)$. Then the function*

$$\psi(z) = f_{2k}(z) \tag{2.1}$$

belongs to $\mathcal{M}^k_{\lambda,\mu}(\Phi, \xi, h)$ in E.

Proof. Let $f \in \mathcal{M}^k_{\lambda,\mu}(\Phi, \xi, h)$. Then from Definition 1.1 we have

$$(1-\lambda)\frac{(\xi * \phi) * f}{(\xi * \varphi) * f_{2k}} + \lambda\frac{\phi * f}{\varphi * f_{2k}} \in \mathcal{K}_\mu(h), \qquad \text{for} \qquad z \in E,$$

or

$$(1-\lambda)\frac{(\xi * \phi * f)(z)}{(\xi * \varphi * f_{2k})(z)} + \lambda\frac{(\phi * f)(z)}{(\varphi * f_{2k})(z)} \in \mathcal{K}_\mu(h), \qquad \text{for} \qquad z \in E. \tag{2.2}$$

Replacing z by $\varepsilon^{\upsilon}z\,(\upsilon = 0, 1, 2, ..., k-1)$ in (2.2) leads to

$$(1-\lambda)\frac{(\xi * \phi * f)(\varepsilon^{\upsilon}z)}{(\xi * \varphi * f_{2k})(\varepsilon^{\upsilon}z)} + \lambda\frac{(\phi * f)(\varepsilon^{\upsilon}z)}{(\varphi * f_{2k})(\varepsilon^{\upsilon}z)} \in \mathcal{K}_\mu(h), \qquad \text{for} \qquad z \in E. \tag{2.3}$$

We note that

$$(\xi * \varphi * f_{2k})(\varepsilon^{\upsilon}z) = \varepsilon^{\upsilon}(\xi * \varphi * f_{2k})(z), \qquad (\varphi * f_{2k})(\varepsilon^{\upsilon}z) = \varepsilon^{\upsilon}(\varphi * f_{2k})(z),$$
$$\overline{(\xi * \varphi * f_{2k})(\varepsilon^{\upsilon}\overline{z})} = \varepsilon^{-\upsilon}(\xi * \varphi * f_{2k})(z), \qquad \overline{(\varphi * f_{2k})(\varepsilon^{\upsilon}\overline{z})} = \varepsilon^{-\upsilon}(\varphi * f_{2k})(z). \tag{2.4}$$

Thus, in the view of (2.3) and (2.4) we obtain

$$(1-\lambda)\frac{(\xi*\phi*f)(\varepsilon^{\upsilon}z)}{\varepsilon^{\upsilon}(\xi*\varphi*f_{2k})(z)}+\lambda\frac{(\phi*f)(\varepsilon^{\upsilon}z)}{\varepsilon^{\upsilon}(\varphi*f_{2k})(z)}\in\mathcal{K}_{\mu}(h) \tag{2.5}$$

and

$$(1-\lambda)\frac{\overline{(\xi*\phi*f)(\varepsilon^{\upsilon}\overline{z})}}{\varepsilon^{-\upsilon}(\xi*\varphi*f_{2k})(z)}+\lambda\frac{\overline{(\phi*f)(\varepsilon^{\upsilon}\overline{z})}}{\varepsilon^{-\upsilon}(\varphi*f_{2k})(z)}\in\mathcal{K}_{\mu}(h). \tag{2.6}$$

Since $\mathcal{K}_{\mu}(h)$ is a convex set, summing (2.5) and (2.6) leads to

$$\begin{aligned}(1-\lambda)&\frac{\frac{1}{2}[\varepsilon^{\upsilon}(\xi*\phi*f)(\varepsilon^{\upsilon}z)+\varepsilon^{-\upsilon}\overline{(\xi*\phi*f)(\varepsilon^{\upsilon}\overline{z})}]}{(\xi*\varphi*f_{2k})(z)}\\ &+\lambda\frac{\frac{1}{2}[\varepsilon^{\upsilon}(\phi*f)(\varepsilon^{\upsilon}\overline{z})+\varepsilon^{-\upsilon}\overline{(\phi*f)(\varepsilon^{\upsilon}z)}]}{(\varphi*f_{2k})(z)}\in\mathcal{K}_{\mu}(h).\end{aligned} \tag{2.7}$$

Putting $\upsilon=0,1,2,...,k-1$ in (2.7) and summing the resulting equations, yields

$$\begin{aligned}(1-\lambda)&\frac{\frac{1}{2k}\sum_{\upsilon=0}^{k-1}[\varepsilon^{\upsilon}(\xi*\phi*f)(\varepsilon^{\upsilon}z)+\varepsilon^{-\upsilon}\overline{(\xi*\phi*f)(\varepsilon^{\upsilon}\overline{z})}]}{(\xi*\varphi*f_{2k})(z)}\\ &+\lambda\frac{\frac{1}{2k}\sum_{\upsilon=0}^{k-1}[\varepsilon^{\upsilon}(\phi*f)(\varepsilon^{\upsilon}\overline{z})+\varepsilon^{-\upsilon}\overline{(\phi*f)(\varepsilon^{\upsilon}z)}]}{(\varphi*f_{2k})(z)}\in\mathcal{K}_{\mu}(h),\end{aligned}$$

and hence $\psi\in\mathcal{M}^{k}_{\lambda,\mu}(\Phi,\xi,h)$ in E. □

Putting $\lambda=0,1$ on the Theorem 2.1 we have the following results for the classes $R^{k}_{\mu}(h)$ and $V^{k}_{\mu}(h)$.

Corollary 2.2. *Let $f\in R^{k}_{\mu}(h)$. Then the function $\psi(z)=f_{2k}(z)$ belongs to $R^{k}_{\mu}(h)$ in E.*

Corollary 2.3. *Let $f\in V^{k}_{\mu}(h)$. Then the function $\psi(z)=f_{2k}(z)$ belongs to $V^{k}_{\mu}(h)$ in E.*

Theorem 2.4. *Let $0<\alpha\le 1$, $h_2=\frac{1+(1-2\alpha)z}{1-z}$ and $\mu_2=1$. Then*

$$\mathcal{CM}^{k}_{\lambda,\mu}(\varphi,\boldsymbol{h})\subseteq\mathcal{CM}^{k}_{\mu}(\varphi,\boldsymbol{h}).$$

Proof. Let $f\in\mathcal{CM}^{k}_{\lambda,\mu}(\varphi,\mathbf{h})$. Then by Definition 1.2 there exists a function $g\in\mathcal{W}^{k}_{1}(\varphi,h_2)$ such that

$$(1-\lambda)\frac{z(\varphi*f)'(z)}{(\varphi*g_{2k})(z)}+\lambda\left(\frac{(z(\varphi*f)')'(z)}{(\varphi*g_{2k})'(z)}\right)\in\mathcal{K}_{\mu_1}(h_1).$$

In the view of $g\in\mathcal{W}^{k}_{1}(\varphi,h_2)$ and applying Theorem 2.1 we know that $g_{2k}\in\mathcal{W}^{k}_{1}(\varphi,h_2)$, i.e,

$$q(z)=\frac{z(\varphi*g_{2k})'(z)}{(\varphi*g_{2k})(z)}\in\mathcal{K}_{1}(h_2). \tag{2.8}$$

Or, equivalently $q(z)\prec h_2(z)$.
By setting

$$p(z)=\frac{z(\varphi*f)'(z)}{(\varphi*g_{2k})(z)},$$

we get

$$\begin{aligned}zp'(z)&=z\frac{(z(\varphi*f)')'(z)(\varphi*g_{2k})(z)-z(\varphi*g_{2k})'(z)(\varphi*f)'(z)}{(\varphi*g_{2k})^2(z)}\\ &=z\frac{(z(\varphi*f)')'(z)}{(\varphi*g_{2k})(z)}-\frac{z(\varphi*f)'(z)}{(\varphi*g_{2k})(z)}q(z)\\ &=\frac{(z(\varphi*f)')'(z)}{(\varphi*g'_{2k})(z)}q(z)-\frac{z(\varphi*f)'(z)}{(\varphi*g_{2k})(z)}q(z).\end{aligned} \tag{2.9}$$

Therefore in the view of $f \in \mathcal{CM}^k_{\lambda,\mu}(\varphi, \mathbf{h})$ and (2.9) we conclude that

$$(1-\lambda)\frac{z(\varphi * f)'(z)}{(\varphi * g_{2k})(z)} + \lambda\left(\frac{(z(\varphi * f)')'(z)}{(\varphi * g_{2k})'(z)}\right) = p(z) + \lambda\frac{zp'(z)}{q(z)} \in \mathcal{K}_{\mu_1}(h_1).$$

Now from the relation (2.8) it is clear that $Req(z) > 0$, so applying Lemma 1.5, we get $p(z) \in \mathcal{K}_{\mu_1}(h_1)$ and the proof is complete. □

Theorem 2.5. *Let* $\Psi \in \mathcal{R}(\alpha)$, $0 < \alpha \leq 1$, $h_2(z) = \frac{1+(1-2\alpha)z}{1-z}$ *and* $\mu_2 = 1$. *Then*

$$f \in \mathcal{CW}^k_\mu(\varphi, \boldsymbol{h}) \Longrightarrow f \in \mathcal{CW}^k_\mu(\Psi * \varphi, \boldsymbol{h}).$$

Proof. Let $f \in \mathcal{CW}^k_\mu(\varphi, \mathbf{h})$. Then by Definition 1.2 there exists a function $g \in \mathcal{W}^k_1(\varphi, h_2)$ such that

$$\frac{z(\varphi * f)'(z)}{(\varphi * g_{2k})(z)} = \mu_1 q_1 + (1-\mu_1)q_2, \tag{2.10}$$

where $q_1, q_2 \prec h_1$. In the view of $g \in \mathcal{W}^k_1(\varphi, h_2)$ and applying Theorem 2.1 we know that $g_{2k} \in \mathcal{W}^k_1(\varphi, h_2)$, i.e,

$$q(z) = \frac{z(\varphi * g_{2k})'(z)}{(\varphi * g_{2k})(z)} \in \mathcal{K}_1(h_2). \tag{2.11}$$

Or, equivalently $\varphi * g_{2k}$ is starlike of order α. Set $T(z) = \varphi * g_{2k}$, then by using the properties of convolution we can rewrite (2.10) as

$$\frac{z(\Psi * \varphi * f)'(z)}{(\Psi * \varphi * g_{2k})(z)} = \mu_1\frac{(\Psi * q_1 T)(z)}{(\Psi * T)(z)} + (1-\mu_1)\frac{(\Psi * q_2 T)(z)}{(\Psi * T)(z)}. \tag{2.12}$$

Now applying Lemma 1.7 leads to $\frac{(\Psi * q_1 T)(z)}{(\Psi * T)(z)} \prec q_1(z)$ and $\frac{(\Psi * q_2 T)(z)}{(\Psi * T)(z)} \prec q_2(z)$. Hence from (2.12) we conclude the result. □

By using similar argument in the proof of Theorem 2.4 we obtain the following result and we omit its proof.

Theorem 2.6.

$$\mathcal{M}^k_{\lambda,1}(\varphi, h) \subseteq \mathcal{W}^k_1(\varphi, h). \tag{2.13}$$

Theorem 2.7. *Let* $0 < \lambda \leq 1$ *and* $f \in \mathcal{M}^k_{\lambda,\mu}(\varphi, h)$. *Then there exists a function* $k \in \mathcal{K}_\mu(h)$ *such that*

$$f_{2k}(z) = \left[\frac{1}{\lambda}\int_0^z u^{\frac{1-\lambda}{\lambda}} \exp\left(\frac{1}{\lambda}\int_0^u \frac{h(t)-1}{t}dt\right) du\right]^\lambda * \Psi, \tag{2.14}$$

where $\Psi * \varphi = \frac{z}{1-z}$ *and*

$$h(z) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}[k(\varepsilon^\upsilon z) + \overline{k(\varepsilon^\upsilon \overline{z})}]. \tag{2.15}$$

Proof. Since $f \in \mathcal{M}^k_{\lambda,\mu}(\varphi, h)$, there exists a function $k \in \mathcal{K}_\mu(h)$ such that

$$(1-\lambda)\frac{z(\varphi * f)'(z)}{(\varphi * f_{2k})(z)} + \lambda\left(\frac{(z(\varphi * f)')'(z)}{(\varphi * f_{2k})'(z)}\right) = k(z) \tag{2.16}$$

By using similar arguments given in the proof of Theorem 2.4 to (2.16) we obtain

$$(1-\lambda)\frac{z(\varphi * f_{2k})'(z)}{(\varphi * f_{2k})(z)} + \lambda\left(\frac{(z(\varphi * f_{2k})')'(z)}{(\varphi * f_{2k})'(z)}\right) = \frac{1}{2k}\sum_{\upsilon=0}^{k-1}[k(\varepsilon^\upsilon z) + \overline{k(\varepsilon^\upsilon \overline{z})}] = h(z). \tag{2.17}$$

Let us define F as

$$(1-\lambda)\frac{z(\varphi * f_{2k})'(z)}{(\varphi * f_{2k})(z)} + \lambda\left(\frac{(z(\varphi * f_{2k})')'(z)}{(\varphi * f_{2k})'(z)}\right) = \frac{zF'(z)}{F(z)},$$

then

$$(\varphi * f_{2k})(z) = \left(\frac{1}{\lambda}\int_0^z \frac{(F(t))^{\frac{1}{\lambda}}}{t}dt\right)^{\lambda}, \tag{2.18}$$

and the function F is analytic with $F(0) = 0$ and from (2.18) we can write

$$\frac{zF'(z)}{F(z)} = h(z).$$

Now by solving the last equation and inserting the solution in the equality (2.16) we get the desired result. □

Theorem 2.8. *Let* $0 < \lambda \leq 1$ *and* $f \in \mathcal{M}^k_{\lambda,\mu}(\varphi, h)$. *Then there exists a function* $k \in \mathcal{K}_\mu(h)$ *such that*

$$zf'(z) = \frac{1}{\lambda^\lambda}\frac{\int_0^z t^{\frac{1-\lambda}{\lambda}}\exp(\frac{1}{\lambda}\int_0^t \frac{h(v)-1}{v}dv)k(t)dt}{(\int_0^1 u^{\frac{1-\lambda}{\lambda}}\exp(\frac{1}{\lambda}\int_0^u \frac{h(t)-1}{t}dt)du)^{1-\lambda}} * \Psi, \tag{2.19}$$

where $\Psi * \varphi = \frac{z}{1-z}$ *and* h *is given by* (2.15).

Proof. Suppose that $f \in \mathcal{M}^k_{\lambda,\mu}(\varphi, h)$, we can get

$$(1-\lambda)\frac{z(\varphi * f)'(z)}{(\varphi * f_{2k})(z)} + \lambda\left(\frac{(z(\varphi * f)')'(z)}{(\varphi * f_{2k})'(z)}\right) \in \mathcal{K}_\mu(h),$$

so there exists a function $k \in \mathcal{K}_\mu(h)$ such that

$$(1-\lambda)\frac{z(\varphi * f)'(z)}{(\varphi * f_{2k})(z)} + \lambda\left(\frac{(z(\varphi * f)')'(z)}{(\varphi * f_{2k})'(z)}\right) = k(z).$$

Taking $F(z) = z(\varphi * f)'(z)$ and $G(z) = (\varphi * f_{2k})(z)$ in the above equation yields

$$(1-\lambda)\frac{F(z)}{G(z)} + \lambda\frac{F'(z)}{G'(z)} = k(z),$$

or

$$F'(z) + \frac{1-\lambda}{\lambda}\frac{G'(z)}{G(z)}F(z) = \frac{k(z)G'(z)}{\lambda}. \tag{2.20}$$

Now solving the Cauchy problem (2.20) and considering (2.14) we get our result and the proof is complete. □

Let $L(r, f)$ denote the length of the image of the circle $|z| = r$ under f. We prove the following.

Theorem 2.9. *Let* $h_1'(0) \neq 0, \mu_2 = 1, h_2(z) = \frac{1+z}{1-z}$, *and* $f \in \mathcal{CW}^k_\mu(\varphi, \boldsymbol{h})$. *Then, for* $0 < r < 1$,

$$L(r, \varphi * f) \leq 2\pi(2\mu_1 - 1)|h_1'(0)|\frac{1}{(1-r)^{\frac{k+2}{k}}}. \tag{2.21}$$

Proof. Using Theorem 2.1 and in the view of the definition of class $\mathcal{CW}^k_\mu(\varphi, \mathbf{h})$ there exists a function $g \in \mathcal{W}^k_1(\varphi, \frac{1+z}{1-z})$ such that

$$z(\varphi * f)'(z) = \psi(z)p(z), \qquad \psi = \varphi * g_{2k} \in S^*, \quad p \in \mathcal{K}_1(h_1). \tag{2.22}$$

Now for $z = re^{i\theta}$, we have

$$\begin{aligned} L(r, \varphi * f) &= \int_0^{2\pi} |z(\varphi * f)'(z)|d\theta \\ &= \int_0^{2\pi} |\psi(z)p(z)|d\theta. \end{aligned}$$

Hence, using the Hölder's inequality, we obtain

$$L(r,\varphi*f)\le 2\pi\left(\frac{1}{2\pi}\int_0^{2\pi}|\psi(z)|^2d\theta\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2d\theta\right)^{\frac{1}{2}}. \tag{2.23}$$

For $p\in\mathcal{K}_{\mu_1}(h_1)$, from (1.4) we have

$$\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2d\theta\le\frac{1+[(2\mu_1-1)^2|h_1'(0)|^2-1]r^2}{1-r^2}. \tag{2.24}$$

Also for k-fold symmetric function ψ it is known that [10]

$$|\psi(z)|\le\frac{|z|}{(1-|z|^k)^{\frac{2}{k}}}. \tag{2.25}$$

Using (2.24) and (2.25) in (2.23), it follows that

$$\begin{aligned}L(r,\varphi*f)&\le 2\pi\left(\frac{1+[(2\mu_1-1)^2|h_1'(0)|^2-1]r^2}{1-r^2}\right)^{\frac{1}{2}}\left(\frac{r}{(1-r^k)^{\frac{2}{k}}}\right)\\&\le 2\pi(2\mu_1-1)|h_1'(0)|\frac{1}{(1-r)^{1+\frac{2}{k}}}.\end{aligned}$$

This completes the proof. □

Theorem 2.10. *Let $h_1'(0)\neq 0$ and $f\in\mathcal{CW}_\mu^k(\varphi,\boldsymbol{h})$ with $\mu_2=1, h_2(z)=\frac{1+z}{1-z}$. Then, for $0<r<1$,*

$$|a_n|\le 2\pi(2\mu_1-1)n^{\frac{2}{k}}, \tag{2.26}$$

where a_n are the coefficients of $\varphi*f$.

Proof. Since, with $z=re^{i\theta}$, Cauchy Theorem gives

$$na_n=\frac{1}{2\pi r^n}\int_0^{2\pi}z(\varphi*f)'(z)e^{-in\theta}d\theta,$$

then

$$n|a_n|\le\frac{1}{2\pi r^n}\int_0^{2\pi}|z(\varphi*f)'(z)|d\theta=\frac{1}{2\pi r^n}L(r,\varphi*f).$$

Using Theorem 2.9 and putting $r=1-\frac{1}{n}, (n\longrightarrow\infty)$, we obtain the required result. □

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