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Research Article

A new double-step method for solving complex Helmholtz equation

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Abstract

We present a new double-step iteration method for solving the systems of linear equations that arise from finite difference discretizations of the complex Helmholtz equations. Convergence analysis of the method is discussed. An upper bound on the spectral radius of the iteration matrix of the method is presented and the parameter which minimizes this upper bound is computed. The proposed method is compared theoretically and numerically with some existing methods.

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1. Introduction

The Helmholtz equations arise in many physical applications, e.g. scattering problems, electromagnetics, and acoustics (see [1,10]). Discretization of the 2-D and 3-D Helmholtz equations by means of finite difference methods lead to complex symmetric systems of linear equations.

The complex Helmholtz equation in 2-D and 3-D can be written as

$$\begin{cases} -\triangle u + \sigma_1 u + i\sigma_2 u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\triangle \equiv \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}, \quad d = 2, 3,$$

$\sigma_1 \in \mathbb{R}$, $\sigma_2 \geq 0$, $\Omega = [0,1]^d$, and $i = \sqrt{-1}$. The discretization of the equation above in 2-D ($d = 2$), using the second order central difference scheme on an $(m+2) \times (m+2)$ grid of Ω with mesh-size $h = 1/(m+1)$ leads to a system of linear equations with the coefficient matrix $A = W + iT \in \mathbb{C}^{n\times n}$, such that $n = m^2$, and

$$W = K + \sigma_1 h^2 (I_m \otimes I_m) \quad \text{and} \quad T = \sigma_2 h^2 (I_m \otimes I_m),$$

with $K = I_m \otimes V_m + V_m \otimes I_m$ and $V_m = \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Here, $\otimes$ and I_m denote the Kronecker product and the identity matrix of order m, respectively. The

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matrix K is symmetric positive definite, since V_m is symmetric positive definite and the Kronecker products with I_m keep the eigenvalues the same, but raise the multiplicities of the eigenvalues m times. For $\sigma_1 \geq 0$, the sum of a symmetric positive definite matrix with a symmetric positive semidefinite one is also symmetric positive definite. Hence, W is symmetric positive definite. If $\sigma_1 < 0$, then the matrix W is not necessarily symmetric positive definite. However, when the mesh-size h is small enough, then the matrix W is symmetric positive definite. On the other hand, the matrix T is symmetric positive semidefinite.

Similar to the 2-D case, discretization of the 3-D complex Helmholtz equation using the second order central difference scheme on an $(m+2) \times (m+2) \times (m+2)$ grid of Ω with mesh-size $h = 1/(m+1)$ results in a system of linear equations with the coefficient matrix $A = W + iT \in \mathbb{C}^{n\times n}$, where $n = m^3$,

$$W = K + \sigma_1 h^2 (I_m \otimes I_m \otimes I_m) \quad \text{and} \quad T = \sigma_2 h^2 (I_m \otimes I_m \otimes I_m),$$

with $K = I_m \otimes I_m \otimes V_m + V_m \otimes I_m \otimes I_m + I_m \otimes V_m \otimes I_m$ and $V_m = \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$. Similar to the 2-D case, the matrix T is symmetric positive semidefinite and, for small values of h, the matrix W is symmetric positive definite.

To summarize, in both 2-D and 3-D, a complex system of linear equations of the form

$$Au = b, \tag{1.2}$$

is obtained where $A = W + iT$ is symmetric, i.e., W and T are symmetric. Moreover, the matrix T is symmetric positive semidefinite and, for small values of h, the matrix W is symmetric positive definite.

In recent years, many efficient iterative methods have been proposed for the numerical solutions of complex symmetric linear systems. Among these, some notable ones are the Hermitian/skew-Hermitian splitting (HSS) method [6] due to Bai et al., its modified version (MHSS) [3], preconditioned version (PMHSS) [4], the combination method of real and imaginary parts (CRI) [19] due to Wang et al.

Additionally, the second author and his colleagues introduced the scale-splitting (SCSP) method [13], more recently the second author generalized this to the two-step scale-splitting (TSCSP) method [17], which we have further generalized into the two parameter two-step scale-splitting (TTSCSP) method [18] given below formally.

The TTSCSP iteration method [18]: Let $u^{(0)} \in \mathbb{C}^n$ be given. Generate a sequence $\{u^{(k)}\}$ such that

$$(\alpha W + T)u^{(k+\frac{1}{2})} = i(W - \alpha T)u^{(k)} + (\alpha - i)b, \tag{1.3}$$

$$(W + \beta T)u^{(k+1)} = i(\beta W - T)u^{(k+\frac{1}{2})} + (1 - \beta i)b, \tag{1.4}$$

for given positive real numbers α and β .

If $\alpha = \beta$, then the TTSCSP method becomes TSCSP method. Salkuyeh in [17] showed that if both of the matrices W and T are symmetric positive definite, then for any positive α the TSCSP method is convergent. Letting $G = W^{-\frac{1}{2}} T W^{-\frac{1}{2}}$ and $\sigma(G)$ be the set of the eigenvalues of G, in [18], it was proved that

$$\sigma(\alpha, \beta) = \max_{\mu\in\sigma(G)} \left\{ \left| \frac{\mu - \beta}{1 + \beta\mu} \right| \right\} \max_{\mu\in\sigma(G)} \left\{ \left| \frac{1 - \alpha\mu}{\alpha + \mu} \right| \right\}, \tag{1.5}$$

is an upper bound for the spectral radius of the the TTSCSP iteration matrix and the conditions on α and β that ensure the convergence of the method were presented.

Axelsson and Kucherov in [2] showed that it is possible to rewrite the system $(W + iT)u = b = p + iq$ in the form

$$\mathcal{A}\mathrm{u} = \begin{bmatrix} W & -T \\ T & W \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{1.6}$$

which involves only real arithmetic.

Recently, using the idea of [8], Salkuyeh et al. in [16] solved the system (1.6) by the generalized successive overrelaxation (GSOR) iteration method. A preconditioned version of the GSOR method was presented by Hezari et al. in [12].

In this paper we present a new double-step method (hereafter, it is referred using the short-hand DSM) for solving the system (1.2) and investigate its convergence properties. From theoretical point of view, we compare the proposed method with the TSCSP method and prove that under some conditions an upper bound on the spectral radius of DSM is smaller than that of the TSCSP method. Our numerical results show that DSM is superior to several existing methods

This paper is organized as follows. In Section 2, we present DSM and discuss its convergence properties. Extreme eigenvalues of the matrix $S = W^{-1}T$, which are used for computing the convergence intervals of DSM, are given in Section 3. Inexact version of DSM is studied in Section 4. Finally, Section 5 is devoted to numerical experiments to show the effectiveness of the new method. Concluding remarks are given in Section 6.

2. The proposed method, DSM

Inspired by the ideas from the existing methods such as the TSCSP, TTSCSP, and CRI methods, we introduce DSM. Each iteration of the TSCSP, TTSCSP, and CRI methods consist of two half steps. In one step, the matrix W plays the role of the shift matrix and the other step the matrix T. In the finite difference discretization of the Helmholtz equation the matrix W dominates the matrix T. Hence, in both steps of DSM, we consider the matrix T as the shift matrix. Letting $\alpha > 0$, by adding αT to both sides of the complex system (1.2), we obtain the following equivalent system

$$(\alpha T + W)u = (\alpha - i)Tu + b. \tag{2.1}$$

Next, we multiply both sides of Eq. (1.2) by $(1 - \alpha i)$ to obtain

$$(1 - \alpha i)Au = (1 - \alpha i)b. \tag{2.2}$$

The last equation can be rewritten as the following fixed-point equation:

$$(\alpha T + W)u = i(\alpha W - T)u + (1 - \alpha i)b. \tag{2.3}$$

Now, by alternately iterating between the two systems of fixed-point equations (2.1) and (2.3), we obtain DSM for solving the complex symmetric linear system (1.2), which is formally stated next.

DSM: Let $u^{(0)} \in \mathbb{C}^n$ be an initial guess. For $k = 0, 1, 2, \ldots$, repeat until $\{u^{(k)}\}$ converges, compute $u^{(k+1)}$ based on the update rule

$$\begin{cases} (\alpha T + W)u^{(k+\frac{1}{2})} = (\alpha - i)Tu^{(k)} + b, \\ (\alpha T + W)u^{(k+1)} = i(\alpha W - T)u^{(k+\frac{1}{2})} + (1 - \alpha i)b, \end{cases} \tag{2.4}$$

where α is a positive real number.

The two subsystems at each iteration of this method require to solve the systems with the coefficient matrix $\alpha T + W$. If W and T are symmetric positive semidefinite, with at least one of them being positive definite, then the matrix $\alpha T + W$ would be symmetric positive definite. Therefore, the two subsystems of the iteration method can be exactly solved by the Cholesky factorization or inexactly by the conjugate gradient (CG) method. Unlike the other methods such as TTSCSP, TSCSCP, MHSS, and CRI we observe that the coefficient matrices of the two half-steps of the method are the same. Hence, when these systems are solved exactly, only one Cholesky factorization of $\alpha T + W$ is needed.

It is easy to see that DSM can be reformulated as

$$u^{(k+1)} = \mathcal{G}_\alpha u^{(k)} + \mathcal{C}_\alpha b, \tag{2.5}$$

where

$$\mathcal{G}_\alpha = (1+\alpha i)(\alpha T + W)^{-1}(\alpha W - T)(\alpha T + W)^{-1}T, \tag{2.6}$$

and

$$\mathcal{C}_\alpha = (\alpha T + W)^{-1}\left(i(\alpha W - T)(\alpha T + W)^{-1} + (1-\alpha i)I\right).$$

Lemma 2.1. *Let W and $T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semidefinite, respectively. Then, the eigenvalues of the matrix $S = W^{-1}T$ are all real and nonnegative.*

Proof. It follows from $S = W^{-\frac{1}{2}}(W^{-\frac{1}{2}}TW^{-\frac{1}{2}})W^{\frac{1}{2}}$ that the matrices $G = W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$ and S are similar. Now the result follows from the fact that the matrix G is symmetric positive semidefinite and its eigenvalues are nonnegative. □

Theorem 2.2. *Let the matrices $W \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ be symmetric positive definite and symmetric positive semidefinite, respectively. Let also $\mu_{\min}$ and $\mu_{\max}$ be the smallest and largest eigenvalues of the matrix $S = W^{-1}T$, respectively. Set $\xi = \sqrt{\frac{1+\sqrt{5}}{2}} \approx 1.272$ and $\eta = 2\sqrt{2+\sqrt{5}} \approx 4.116$.*

(i) *When $0 < \alpha < \xi$, DSM for solving the (1.2) is convergent, if*

$$\mu_{\max} < r(\alpha) = \frac{2}{-\alpha(\sqrt{1+\alpha^2}+2)+\sqrt{\Delta}},$$

where $\Delta = (1+\alpha^2)(\alpha^2 + 4\sqrt{1+\alpha^2})$.

(ii) *When $\xi \le \alpha < \eta$, DSM is unconditionally convergent.*

(iii) *When $\alpha \ge \eta$, DSM for solving (1.2) is convergent, if*

$$\mu_{\max} < s(\alpha) = \frac{-2\alpha + \alpha\sqrt{1+\alpha^2} - \sqrt{\hat{\Delta}}}{2(\alpha^2+\sqrt{1+\alpha^2})},$$

or

$$\mu_{\min} > q(\alpha) = \frac{-2\alpha + \alpha\sqrt{1+\alpha^2} + \sqrt{\hat{\Delta}}}{2(\alpha^2+\sqrt{1+\alpha^2})},$$

where $\hat{\Delta} = (1+\alpha^2)(\alpha^2 - 4\sqrt{1+\alpha^2})$.

Proof. We can rewrite the matrix $\mathcal{G}_\alpha$ as

$$\mathcal{G}_\alpha = (1+\alpha i)(\alpha S + I)^{-1}(\alpha I - S)(\alpha S + I)^{-1}S,$$

where $S = W^{-1}T$. The eigenvalues of the matrix $\mathcal{G}_\alpha$ are of the form

$$\lambda = \frac{(1+\alpha i)(\alpha-\mu)\mu}{(\alpha\mu+1)^2}, \tag{2.7}$$

where μ is the eigenvalue of the matrix S. Hence, we obtain

$$|\lambda| = \frac{\sqrt{1+\alpha^2}\,|\alpha-\mu|\mu}{(\alpha\mu+1)^2}.$$

Having $|\lambda| < 1$, it is equivalent to

$$\frac{\sqrt{1+\alpha^2}\,|\alpha-\mu|\,\mu}{(\alpha\mu+1)^2} < 1.$$

The latter inequality holds if and only if

$$-(\alpha\mu+1)^2 < \sqrt{1+\alpha^2}(\alpha-\mu)\mu < (\alpha\mu+1)^2. \tag{2.8}$$

We determine the values of α in terms of the eigenvalues μ so that both of the inequalities in (2.8) are satisfied. By direct computation, the left-hand side of the above inequality is equivalent to

$$(-\sqrt{1+\alpha^2}+\alpha^2)\mu^2+\alpha(\sqrt{(1+\alpha^2)}+2)\mu+1>0. \tag{2.9}$$

From Lemma 2.1, μ is nonnegative. If $\mu=0$, then inequality (2.9) holds trivially, so let us assume $\mu>0$. Letting $x=1/\mu$, equation (2.9) can be rewritten as

$$h(x)=x^2+\alpha(\sqrt{1+\alpha^2}+2)x+(\alpha^2-\sqrt{1+\alpha^2})>0. \tag{2.10}$$

The discriminant of the quadratic equation $h(x)=0$ is

$$\Delta=(1+\alpha^2)(\alpha^2+4\sqrt{1+\alpha^2}).$$

Obviously, $\Delta>0$. Hence the quadratic equation $h(x)=0$ has two solutions

$$x_1=\frac{-\alpha(\sqrt{1+\alpha^2}+2)-\sqrt{\Delta}}{2} \quad \text{and} \quad x_2=\frac{-\alpha(\sqrt{1+\alpha^2}+2)+\sqrt{\Delta}}{2}.$$

Clearly, $x_1<0$. Additionally, as $x_1x_2=\alpha^2-\sqrt{1+\alpha^2}$ the signs of x_2 and $\sqrt{1+\alpha^2}-\alpha^2$ must be the same. It follows that $x_2\le 0$ if and only if $\sqrt{1+\alpha^2}-\alpha^2\le 0$ which happens when $\alpha\ge\xi:=\sqrt{\frac{1+\sqrt{5}}{2}}$. Recalling also $\alpha>0$ and $\mu>0$, there are only the following two possibilities:

- If $\alpha\ge\xi$, then $h(x)>0$.
- If $0<\alpha<\xi$, then $x_2>0$. Therefore, for $x>x_2$ the convergence condition (2.10) holds true. But the condition $x>x_2$ can equivalently be expressed as $\mu<r(\alpha)$ for all $\mu\in\sigma(S)$, where $\sigma(S)$ is the spectrum of the matrix S. Since $\mu_{\min}\le\mu\le\mu_{\max}$, the inequality $\mu<r(\alpha)$ holds for all $\mu\in\sigma(S)$ if $\mu_{\max}<r(\alpha)$.

The right-hand side of (2.8) holds if and only if

$$K(\mu)=(\sqrt{1+\alpha^2}+\alpha^2)\mu^2+\alpha(2-\sqrt{1+\alpha^2})\mu+1>0. \tag{2.11}$$

The discriminant of the quadratic equation $K(\mu)=0$ is

$$\hat{\Delta}=(1+\alpha^2)(\alpha^2-4\sqrt{1+\alpha^2}).$$

Therefore, according to the sign of $\hat{\Delta}$, we have two cases

- If $0<\alpha<\eta$, then $\alpha^2-4\sqrt{1+\alpha^2}<0$. Therefore $\hat{\Delta}<0$ and inequality (2.11) holds true, where $\eta=\sqrt{8+4\sqrt{5}}$.

- If $\alpha\ge\eta$, then $\hat{\Delta}\ge 0$ and equation $K(x)=0$ has the following two nonnegative solutions

  $$s(\alpha)=\frac{\alpha(\sqrt{1+\alpha^2}-2)-\sqrt{\hat{\Delta}}}{2(\alpha^2+\sqrt{1+\alpha^2})}, \qquad q(\alpha)=\frac{\alpha(\sqrt{1+\alpha^2}-2)+\sqrt{\hat{\Delta}}}{2(\alpha^2+\sqrt{1+\alpha^2})}.$$

  Hence, the convergence condition (2.11) holds if and only if $\mu<s(\alpha)$ or $\mu>q(\alpha)$. Therefore it is sufficient to have $\mu_{\max}<s(\alpha)$ or $\mu_{\min}>q(\alpha)$.

□

In general it is difficult to find the optimal value of the parameter α and it is problem-based. In the sequel, we consider an upper bound for the spectral radius of $\mathcal{G}_\alpha$ and compute its minimizer. To do so, from Eq. (2.7) we have

$$\begin{aligned}|\lambda| &\le \sqrt{1+\alpha^2}\max_{\mu\in\sigma(S)}\left\{\frac{|\alpha-\mu|\mu}{(1+\alpha\mu)^2}\right\}\\ &\le \sqrt{1+\alpha^2}\max_{\mu\in\sigma(S)}\left\{\frac{|\alpha-\mu|}{1+\alpha\mu}\right\}\max_{\mu\in\sigma(S)}\left\{\frac{\mu}{1+\alpha\mu}\right\}=:\gamma(\alpha).\end{aligned} \tag{2.12}$$

In the next theorem, we find the value of α that minimizes the upper bound $\gamma(\alpha)$.

Theorem 2.3. *Let all the assumptions of Theorem 2.2 hold and*

$$\alpha^* = \arg\min_{\alpha>0} \gamma(\alpha).$$

Then

$$\alpha^* = \frac{\mu_{\min}\mu_{\max} - 1 + \sqrt{(1+\mu_{\min}{}^2)(1+\mu_{\max}{}^2)}}{\mu_{\min} + \mu_{\max}}.$$

Proof. Let

$$f(\alpha,\mu) = \frac{\alpha-\mu}{1+\alpha\mu} \quad \text{and} \quad g(\alpha,\mu) = \frac{\mu}{1+\alpha\mu}.$$

Therefore, we have

$$\alpha^* = \arg\min_{\alpha>0} \left\{ \sqrt{1+\alpha^2} \max_{\mu\in\sigma(S)} \{|f(\alpha,\mu)|\} \max_{\mu\in\sigma(S)} \{g(\alpha,\mu)\} \right\}.$$

We first study some properties of the function $g(\alpha,\mu)$. The partial derivative of $g(\alpha,\mu)$ with respect to μ is positive, because

$$\frac{\partial g(\alpha,\mu)}{\partial \mu} = \frac{1}{(1+\alpha\mu)^2}.$$

This shows that the function $g(\alpha,\mu)$ is increasing with respect to the variable μ, and hence we get

$$\max_{\mu\in\sigma(S)} \{g(\alpha,\mu)\} = \frac{\mu_{\max}}{1+\alpha\mu_{\max}}.$$

It follows from Theorem 2 of [18] that

$$\max_{\mu\in\sigma(S)} \{|f(\alpha,\mu)|\} = \max\left\{ \frac{\alpha-\mu_{\min}}{1+\alpha\mu_{\min}}, \frac{\mu_{\max}-\alpha}{1+\alpha\mu_{\max}} \right\} = \begin{cases} \dfrac{\mu_{\max}-\alpha}{1+\alpha\mu_{\max}}, & \alpha \le \alpha^*, \\ \dfrac{\alpha-\mu_{\min}}{1+\alpha\mu_{\min}}, & \alpha \ge \alpha^*, \end{cases}$$

where α^* satisfies the relation

$$\frac{\alpha^*-\mu_{\min}}{1+\alpha^*\mu_{\min}} = \frac{\mu_{\max}-\alpha^*}{1+\alpha^*\mu_{\max}},$$

which gives the following positive value for α^*

$$\alpha^* = \frac{\mu_{\min}\mu_{\max} - 1 + \sqrt{(1+\mu_{\min}{}^2)(1+\mu_{\max}{}^2)}}{\mu_{\min} + \mu_{\max}}.$$

Therefore, from Eq. (2.12) we have

$$\gamma(\alpha) = \begin{cases} p_1(\alpha) = \dfrac{\sqrt{1+\alpha^2}\mu_{\max}(\mu_{\max}-\alpha)}{(1+\alpha\mu_{\max})^2}, & \alpha \le \alpha^*, \\ p_2(\alpha) = \dfrac{\sqrt{1+\alpha^2}\mu_{\max}(\alpha-\mu_{\min})}{(1+\alpha\mu_{\min})(1+\alpha\mu_{\max})}, & \alpha \ge \alpha^*. \end{cases}$$

It is straightforward to verify that

$$p_1'(\alpha) = -\mu_{\max}\frac{1+\alpha^2+\mu_{\max}^2+\alpha^2\mu_{\max}^2+(\alpha-\mu_{\max})^2}{\sqrt{1+\alpha^2}(1+\alpha\mu_{\max})^3} < 0,$$

$$p_2'(\alpha) = \mu_{\max}\frac{\alpha^2\mu_{\min}+2\mu_{\min}+\alpha^3+\alpha^3\mu_{\min}\mu_{\max}+\alpha\mu_{\min}^2+\mu_{\min}^2\mu_{\max}+\alpha^3\mu_{\min}^2}{\sqrt{1+\alpha^2}(\alpha+\mu_{\max})^2(1+\alpha\mu_{\min})^2} > 0,$$

for all $\alpha > 0$, which show that the functions p_1 and p_2 are decreasing and increasing, respectively. Therefore $\alpha^* = \arg\min_{\alpha>0} \gamma(\alpha)$. □

In the sequel, we compare DSM with the TTSCSP iteration method. DSM is a single-parameter method, hence we compare DSM with the TTSCSP method when $\alpha = \beta$, that is, with the TSCSP iteration method. In general it is difficult to compute the spectral radius of the TSCSP method and DSM, because they are problem-based. Hence, we compare the upper bounds on the spectral radii of their iteration matrices. In the next theorem we prove that under some mild conditions the upper bound on the spectral radius of DSM is less than or equal to that of the TSCSP method.

Theorem 2.4. *Let the conditions of Theorem 2.2 be satisfied,* $\mu_{\min} < 1$, $\mu_{\max} \geq 1$, *and*

$$\tilde{\alpha} = \frac{1 - \mu_{\max} - 2\mu_{\min}\mu_{\max} + \sqrt{\Delta}}{2(\mu_{\max} + \mu_{\min})},$$

where $\Delta = (\mu_{\max} - 1)^2 + 4\mu_{\max}(\mu_{\max} + \mu_{\min}^2(\mu_{\max} - 1)) > 0$. *Then,*

$$\gamma(\alpha) \leq \sigma(\alpha, \alpha), \quad \forall \alpha \in (0, \min\{\hat{\alpha}, \tilde{\alpha}\}),$$

where $\sigma(\alpha, \beta)$ *is defined in Eq.* (1.5), *and*

$$\hat{\alpha} = \frac{1 - \mu_{\min}\mu_{\max} + \sqrt{(1 + {\mu_{\min}}^2)(1 + {\mu_{\max}}^2)}}{\mu_{\min} + \mu_{\max}}.$$

Proof. From Eq. (1.5) an upper bound for the spectral radius of the TSCSP iteration method is given by

$$\sigma(\alpha, \alpha) = \max_{\mu \in \sigma(S)} \left\{ \left| \frac{\mu - \alpha}{1 + \alpha\mu} \right| \right\} \max_{\mu \in \sigma(S)} \left\{ \left| \frac{1 - \alpha\mu}{\alpha + \mu} \right| \right\}.$$

Similar to Theorem 2.3, one can see that

$$\sigma(\alpha, \alpha) = \max\left\{ \frac{1 - \alpha\mu_{\min}}{\alpha + \mu_{\min}}, \frac{\alpha\mu_{\max} - 1}{\alpha + \mu_{\max}} \right\} \max\left\{ \frac{\alpha - \mu_{\min}}{1 + \alpha\mu_{\min}}, \frac{\mu_{\max} - \alpha}{1 + \alpha\mu_{\max}} \right\}.$$

The inequality $\gamma(\alpha) \leq \sigma(\alpha, \alpha)$ is equivalent to

$$\sqrt{1 + \alpha^2} \frac{\mu_{\max}}{\alpha + \mu_{\max}} \leq \max\left\{ \frac{1 - \alpha\mu_{\min}}{\alpha + \mu_{\min}}, \frac{\alpha\mu_{\max} - 1}{\alpha + \mu_{\max}} \right\} = \eta(\alpha). \tag{2.13}$$

Hence, let us consider $\eta(\alpha)$, in particular define

$$g(\alpha) = \frac{1 - \alpha\mu_{\min}}{\alpha + \mu_{\min}} \quad \text{and} \quad h(\alpha) = \frac{\alpha\mu_{\max} - 1}{\alpha + \mu_{\max}}.$$

The strictly decreasing function g passes through the points $(0, \frac{1}{\mu_{\min}})$ and $(\frac{1}{\mu_{\min}}, 0)$, and $y = -\mu_{\min}$ is a horizontal asymptote for the function g. Also the function h is a strictly increasing that passes through the points $(0, \frac{-1}{\mu_{\max}})$ and $(\frac{1}{\mu_{\max}}, 0)$, and $y = \mu_{\max}$ is a horizontal asymptote for the function h (see Figure 1). The intersection point of the functions h and g satisfies

$$\frac{1 - \hat{\alpha}\mu_{\min}}{\hat{\alpha} + \mu_{\min}} = \frac{\hat{\alpha}\mu_{\max} - 1}{\hat{\alpha} + \mu_{\max}}.$$

By direct computation we can see that the intersection point, $\hat{\alpha}$, is equal to

$$\hat{\alpha} = \frac{1 - \mu_{\min}\mu_{\max} + \sqrt{(1 + {\mu_{\min}}^2)(1 + {\mu_{\max}}^2)}}{\mu_{\min} + \mu_{\max}}.$$

Hence, we have

$$\eta(\alpha) = \begin{cases} \dfrac{1 - \alpha\mu_{\min}}{\alpha + \mu_{\min}}, & \alpha < \hat{\alpha}, \\ \dfrac{\alpha\mu_{\max} - 1}{\alpha + \mu_{\max}}, & \alpha \geq \hat{\alpha}. \end{cases}$$

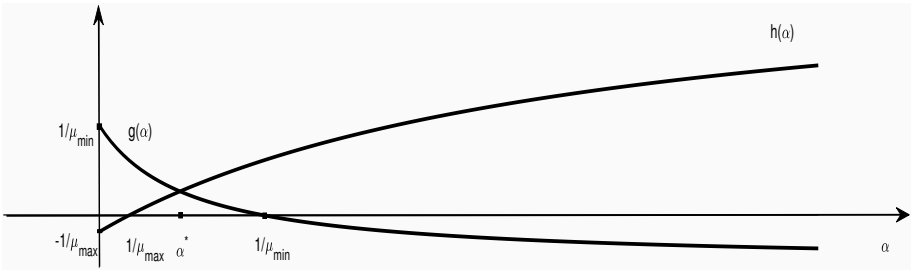

Figure 1. Graph of $\eta(\alpha)$.

For $\alpha < \hat{\alpha}$, the inequality (2.13) holds if and only if

$$\sqrt{1+\alpha^2}\frac{\mu_{\max}}{\alpha+\mu_{\max}} \leq \frac{1-\alpha\mu_{\min}}{\alpha+\mu_{\min}}.$$

From the fact that $\sqrt{1+\alpha^2} \leq 1+\alpha$, we deduce that the latter inequality holds if

$$(1+\alpha)\frac{\mu_{\max}}{\alpha+\mu_{\max}} \leq \frac{1-\alpha\mu_{\min}}{\alpha+\mu_{\min}}.$$

This inequality can be simplified as

$$q(\alpha) := (\mu_{\max}+\mu_{\min})\alpha^2 + (\mu_{\max}+2\mu_{\min}\mu_{\max}-1)\alpha + \mu_{\max}(\mu_{\min}-1) \leq 0.$$

The roots of $q(\alpha) = 0$ are

$$\alpha_{\pm} = \frac{1-\mu_{\max}-2\mu_{\min}\mu_{\max} \pm \sqrt{\Delta}}{2(\mu_{\max}+\mu_{\min})},$$

where $\Delta = (\mu_{\max}-1)^2 + 4\mu_{\max}(\mu_{\max}+\mu_{\min}^2\mu_{\max}-\mu_{\min}^2) \geq 0$. It is straightforward to verify that if $\mu_{\min} < 1$ and $\mu_{\max} \geq 1$, then $\alpha_- < 0$ and $\alpha_+ > 0$. This completes the proof. □

3. Extreme eigenvalues of S

To employ Theorem 2.2 for the convergence of DSM, we need to compute the extreme eigenvalues of S. In this section, we obtain the smallest and largest eigenvalues of S for both the two- or three-dimensional Helmholtz equation. We can see that the eigenvalues of the matrix

$$S = W^{-1}T = h^2\sigma_2\,(h^2\sigma_1 I_{m^2} + K)^{-1}$$

are as

$$\mu_{i,j} = \frac{h^2\sigma_2}{h^2\sigma_1+\eta_{i,j}},$$

where $\eta_{i,j}$'s are the eigenvalues of K. In 2-D, the eigenvalues of the matrix K are given by (see [14, 15])

$$\eta_{i,j} = 4\left(\sin^2\left(\frac{i\pi}{2(m+1)}\right) + \sin^2\left(\frac{j\pi}{2(m+1)}\right)\right), \quad i,j = 1,2,\ldots,m.$$

Then we get

$$\begin{aligned}
\eta_{\min} &= \min_{i,j=1,\ldots,m}\eta_{i,j} = 8\sin^2\frac{\pi}{2(m+1)},\\
\eta_{\max} &= \max_{i,j=1,\ldots,m}\eta_{i,j} = 8\sin^2\frac{m\pi}{2(m+1)} = 8\cos^2\frac{\pi}{2(m+1)}.
\end{aligned}$$

Therefore, we obtain

$$\mu_{\min} = \min_{i,j=1,\ldots,m} \mu_{i,j} = \frac{h^2\sigma_2}{h^2\sigma_1 + \eta_{\max}} = \frac{h^2\sigma_2}{h^2\sigma_1 + 8\cos^2\frac{\pi}{2(m+1)}},$$

$$\mu_{\max} = \max_{i,j=1,\ldots,m} \mu_{i,j} = \frac{h^2\sigma_2}{h^2\sigma_1 + \eta_{\min}} = \frac{h^2\sigma_2}{h^2\sigma_1 + 8\sin^2\frac{\pi}{2(m+1)}}.$$

For 3-D, the eigenvalues of K are given by (see [14, 15])

$$\eta_{i,j,k} = 4\left(\sin^2\left(\frac{i\pi}{2(m+1)}\right) + \sin^2\left(\frac{j\pi}{2(m+1)}\right) + \sin^2\left(\frac{k\pi}{2(m+1)}\right)\right),$$

for $i, j, k = 1, 2, \ldots, m$. Similar to the 2-D case the eigenvalues of $S = W^{-1}T = h^2\sigma_2\,(h^2\sigma_1 + K)^{-1}$ are given by

$$\mu_{i,j,k} = \frac{h^2\sigma_2}{h^2\sigma_1 + \eta_{i,j,k}},$$

where $\eta_{i,j,k}$'s are the eigenvalues of K. Hence, following the arguments for the 2-D case, we deduce that

$$\mu_{\min} = \min_{i,j,k=1,\ldots,m} \mu_{i,j,k} = \frac{h^2\sigma_2}{h^2\sigma_1 + 12\cos^2\frac{\pi}{2(m+1)}}, \tag{3.1}$$

$$\mu_{\max} = \max_{i,j,k=1,\ldots,m} \mu_{i,j,k} = \frac{h^2\sigma_2}{h^2\sigma_1 + 12\sin^2\frac{\pi}{2(m+1)}}. \tag{3.2}$$

4. Inexact DSM

The two half steps at each DSM iteration require the solutions of two systems of linear equations with the coefficient matrix $\alpha T + W$. This may be costly and impractical in practical implementations. To overcome this disadvantage we can solve the two subsystems iteratively. Since W and T are symmetric positive definite and symmetric positive semidefinite matrices, we conclude that the matrix $\alpha T + W$ is symmetric positive definite and we can employ CG to solve the two subsystems of DSM.

The subsystems involving DSM are solved by the CG method such that the relative residual norms are less than $\epsilon_k > 0$ and $\eta_k > 0$ for the inexact solutions of the first and the second linear systems, respectively. To do so, letting

$$u^{(k+\frac{1}{2})} = u^{(k)} + \delta^{(k)},$$

and substituting it in the first subsystem (2.4), gives

$$(\alpha T + W)\delta^{(k)} = b - (W + iT)u^{(k)} = b - Au^{(k)} = r^{(k)}.$$

In the same way, by setting

$$u^{(k+1)} = u^{(k+\frac{1}{2})} + \delta^{(k+\frac{1}{2})},$$

the second subsystem of (2.4) can be written as

$$(\alpha T + W)\delta^{(k+\frac{1}{2})} = (1 - \alpha i)r^{(k+\frac{1}{2})},$$

where $r^{(k+\frac{1}{2})} = b - Au^{(k+\frac{1}{2})}$. Putting the remarks above into use gives the following inexact version of the DSM algorithm.

The Inexact DSM (IDSM)

(1) `Choose an initial guess` $u^{(0)}$ `and compute` $r^{(0)} = b - Au^{(0)}$
(2) `For` $k = 0, 1, 2, \ldots$ `until convergence, Do`
(3) `Compute` $r^{(k)} = b - Au^{(k)}$.

(4) Solve $(\alpha T+W)\delta^{(k)} = r^{(k)}$ by the CG method to compute the approximate solution $\delta^{(k)}$ satisfying $\|r^{(k)} - (\alpha T + W)\delta^{(k)}\|_2 \leq \epsilon_k \|r^{(k)}\|_2$

(5) $u^{(k+\frac{1}{2})} := u^{(k)} + \delta^{(k)}$

(6) Compute $r^{(k+\frac{1}{2})} = b - Au^{(k+\frac{1}{2})}$ and set $r^{(k+\frac{1}{2})} = (1 - \alpha i)r^{(k+\frac{1}{2})}$

(7) Solve $(\alpha T + W)\delta^{(k+\frac{1}{2})} = r^{(k+\frac{1}{2})}$ by the CG method to compute the approximate solution $\delta^{(k+\frac{1}{2})}$ satisfying $\|r^{(k+\frac{1}{2})} - (\alpha T + W)\delta^{(k+\frac{1}{2})}\|_2 \leq \eta_k \|r^{(k+\frac{1}{2})}\|_2$

(8) $u^{(k+1)} := u^{(k+\frac{1}{2})} + \delta^{(k+\frac{1}{2})}$

(9) EndDo

Convergence of IDSM can be proved similar to that of the TTSCSP method (see [18]).

5. Numerical experiments

In this section, we apply DSM and its inexact version for solving the Helmholtz equations (1.2), for different values of σ_1 and σ_2. The right-hand side vector b is set to be $b = (1+i)A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. It is shown that DSM is more efficient than the PMHSS, CRI, and TTSCSP methods. Numerical comparisons of the inexact version of these algorithms are also performed.

Numerical results are compared in terms of both the number of iterations and the CPU time which are, respectively, denoted by "Iter" and "CPU" in the tables. In all the tests, we use a zero vector as an initial guess and the stopping criterion

$$\frac{\|b - Au^{(k)}\|_2}{\|b\|_2} < 10^{-6},$$

is always used. In the implementation of DSM, the systems with the coefficient matrix $\alpha T+W$ is solved using the Cholesky factorization of the matrix. For all the exact methods, we apply the sparse Cholesky factorization together with the symmetric approximate minimum degree reordering [15] for solving the subsystems. To do so, we have used the `symamd.m` command of MATLAB. In all the inexact methods, we apply the CG method (or its preconditioned version, PCG) for solving the two subsystems with tolerance 10^{-2}. In all of the inexact 2-D versions of the algorithms we apply the preconditioned CG (PCG) iteration method in conjunction with the modified incomplete Cholesky factorization as the preconditioner for solving the subsystems. In the MATLAB notation the preconditioner can be computed using the following command

```
LC=ichol(C,struct('michol','on','type','ict','droptol',1e-2));
```

where C is a given symmetric positive definite matrix.

All runs are implemented in MATLAB R2014b with a Laptop with 2.40 GHz central processing unit (Intel(R) Core(TM) i7-5500), 8 GB memory and Windows 10 operating system.

5.1. Existing methods used for comparison

In our numerical experiments, we compare the proposed DSM with

1. the two parameter two-step scale-splitting (TTSCSP) method defined in the introduction, as well as
2. the preconditioned Hermitian/skew-Hermitian splitting (PMHSS) method [4], and
3. the combination method of real and imaginary parts (CRI) [19].

The formal definitions of the latter two methods are given below. Note that, in the definition of the PMHSS method, the matrix V corresponds to a preconditioner and is supposed to be symmetric positive definite. In our experiments, we always choose $V = W$, which is a standard practice followed in the literature.

Table 1. Numerical results for the 2-D case with $\sigma_1 = -10$ and $\sigma_2 = 10$.

Method	m	32	64	128	256	512
DSM	α_{opt}	0.74	0.74	0.74	0.74	0.74
	Iter	5	5	4	4	4
	CPU	0.02	0.03	0.11	0.58	3.74
DSM	α^*	0.42	0.42	0.42	0.42	0.42
	Iter	10	9	8	7	6
	CPU	0.04	0.06	0.16	0.62	4.89
TTSCSP	α_{opt}	0.75	0.71	0.80	0.72	0.68
	β_{opt}	0.04	0.03	0.02	0.04	0.01
	Iter	5	5	4	4	4
	CPU	0.02	0.03	0.15	0.88	4.81
PMHSS	α_{opt}	0.78	0.86	0.89	0.91	0.91
	Iter	40	40	40	40	40
	CPU	0.02	0.07	0.42	2.24	16.85
CRI	α_{opt}	0.85	0.71	0.87	0.68	0.56
	Iter	15	14	12	11	10
	CPU	0.02	0.05	0.27	1.26	8.71
$u = A\backslash b$	CPU	0.005	0.04	0.105	0.45	2.44

The PMHSS iteration method [4]: `Given` $u^{(0)} \in \mathbb{C}^n$ `, generate a sequence` $\{u^{(k)}\}$ `such that`

$$\begin{cases} (\alpha V + W)u^{(k+\frac{1}{2})} = (\alpha V - iT)u^{(k)} + b, \\ (\alpha V + T)u^{(k+1)} = (\alpha V + iW)u^{(k+\frac{1}{2})} - ib, \end{cases}$$

`where` α `is a given positive real number.`

The CRI iteration method [19]: `Given` $u^{(0)} \in \mathbb{C}^n$ `, generate a sequence` $\{u^{(k)}\}$ `such that`

$$\begin{cases} (\alpha T + W)u^{(k+\frac{1}{2})} = (\alpha - i)Tu^{(k)} + b, \\ (\alpha W + T)u^{(k+1)} = (\alpha + i)Wu^{(k+\frac{1}{2})} - ib, \end{cases}$$

`where` α `is a given positive real number.`

5.2. Numerical comparison

In Tables 1 and 2, we list the numerical results on the 2-D complex Helmholtz equation (1.1) for the exact methods with $(\sigma_1, \sigma_2) = (-10, 10)$ and $(\sigma_1, \sigma_2) = (100, 10)$, respectively. For all of the methods, the optimal value of α (α_{opt}) were found experimentally and are the ones resulting in the least numbers of iterations. In these tables, we see that for all the examples the values of α_{opt} are constant. We also present the numerical results for the DSM method with parameter α^* provided by Theorem 2.3. To show the advantages of the iteration methods over the direct sparse solvers, we present the CPU times of computing the solution of the system by the backslash command ("\") of MATLAB.

The inexact version of the DSM, TTSCSP, PMHSS, and CRI methods are denoted by IDSM, ITTSCSP, IPMHSS, and ICRI, respectively. Tables 3 and 4 present the numerical results of the inexact versions of the methods. In Table 3, for the ITTSCSP and IPMHSS iteration method for $m = 64$, the `ict` function of MATLAB encounters a nonpositive pivot during the computation of the inexact Cholesky factorization. Therefore, we have used a smaller value of the dropping tolerance (droptol) which have been presented in the table. Numerical results corresponding to the exact versions of the methods for the 3-D cases have been reported in Tables 5 and 6. In these tables, recall that $n = 8^3, 16^3, 32^3, 64^3$. As

Table 2. Numerical results for the 2-D case with $\sigma_1 = 100$ and $\sigma_2 = 10$.

Method	m	32	64	128	256	512
DSM	α_{opt}	0.06	0.06	0.06	0.06	0.06
	Iter	2	2	2	2	2
	CPU	0.01	0.02	0.08	0.38	2.32
DSM	α^*	0.04	0.04	0.04	0.04	0.04
	Iter	3	3	2	2	2
	CPU	0.02	0.03	0.08	0.39	2.30
TTSCSP	α_{opt}	1.40	0.93	0.75	0.65	0.50
	β_{opt}	0.01	0.03	0.02	0.01	0.01
	Iter	4	4	4	4	4
	CPU	0.02	0.03	0.15	0.79	4.85
PMHSS	α_{opt}	0.76	0.85	0.89	0.76	0.76
	Iter	40	40	40	41	41
	CPU	0.02	0.07	0.42	2.28	17.20
CRI	α_{opt}	0.51	0.73	0.49	0.36	0.43
	Iter	7	6	6	6	5
	CPU	0.02	0.04	0.17	0.87	5.21
$u = A\backslash b$	CPU	0.005	0.02	0.10	0.46	2.63

Table 3. Numerical results for the inexact version of the methods for the 2-D case with $\sigma_1 = -10$ and $\sigma_2 = 10$.

Method	m	32	64	128	256	512
IDSM	α_{opt}	0.74	0.74	0.74	0.74	0.74
	Iter	5	5	4	4	4
	CPU	0.10	0.17	0.32	1.37	14.89
ITTSCSP	α_{opt}	0.74	0.70	0.65	0.72	0.68
	β_{opt}	0.04	0.03	0.02	0.04	0.01
	Iter	5	6	5	5	5
	CPU	0.11	0.16	0.47	1.51	22.55
	droptol	1e-2	1e-3	1e-3	1e-3	1e-4
IPMHSS	α_{opt}	0.78	0.86	0.90	0.91	0.76
	Iter	40	40	40	40	41
	CPU	0.18	0.35	1.58	8.26	131.01
	droptol	1e-2	1e-3	1e-3	1e-4	1e-4
ICRI	α_{opt}	0.85	0.71	0.87	0.74	0.63
	Iter	15	14	12	11	10
	CPU	0.16	0.30	0.77	3.78	41.63

seen DSM outperforms the other methods in terms of both the number of iterations and the CPU time, especially for large systems. Tables 7 and 8 present numerical results in 3-D for the IDSM, ITTSCSP, IPMHSS, and ICRI methods.

It is apparent from the tables that DSM is very effective in reducing the number of iterations, as well as the CPU time; this remark applies both the exact and the inexact versions. Also, the tables show that the optimal value of the parameter α remains almost constant with respect to the problem size for both of DSM and IDSM. On the other hand, we observe that Theorem 2.3 provides good estimates for the optimal values of the parameter α. Another observation is that in most cases (except in Table 1) the CPU time

Table 4. Numerical results for the inexact version of the methods for the 2-D case with $\sigma_1 = 100$ and $\sigma_2 = 10$.

Method	m	32	64	128	256	512
IDSM	α_{opt}	0.06	0.06	0.06	0.06	0.06
	Iter	3	2	2	2	2
	CPU	0.06	0.07	0.13	0.36	3.00
ITTSCSP	α_{opt}	1.02	0.94	0.90	0.85	0.80
	β_{opt}	0.03	0.02	0.01	0.01	0.01
	Iter	4	5	4	4	4
	CPU	0.09	0.13	0.20	0.60	4.99
IPMHSS	α_{opt}	0.76	0.85	0.89	0.76	0.76
	Iter	40	40	40	41	41
	CPU	0.14	0.31	0.64	2.38	22.40
ICRI	α_{opt}	0.51	0.73	0.49	0.38	0.32
	Iter	7	6	6	6	6
	CPU	0.10	0.15	0.25	0.99	9.61

Table 5. Numerical results for the 3-D case with $\sigma_1 = -10$ and $\sigma_2 = 10$.

Method	m	8	16	32	64
DSM	α_{opt}	0.07	0.07	0.07	0.07
	Iter	2	2	2	2
	CPU	0.01	0.06	1.01	69.75
DSM	α^*	0.05	0.04	0.04	0.04
	Iter	3	3	3	2
	CPU	0.02	0.07	1.02	69.06
TTSCSP	α_{opt}	0.89	0.89	0.89	0.89
	β_{opt}	0.05	0.04	0.03	0.02
	Iter	4	4	4	4
	CPU	0.02	0.11	2.84	2165.19
PMHSS	α_{opt}	0.70	0.70	0.70	0.70
	Iter	36	39	41	42
	CPU	0.03	0.55	17.82	563.42
CRI	α_{opt}	1.0	1.0	1.0	1.0
	Iter	7	7	7	7
	CPU	0.02	0.14	3.95	2176.46
$u = A\backslash b$	CPU	0.004	0.092	3.15	273.09

to compute the solution by the backslash command of MATLAB is greater than that of DSM.

According to the numerical results presented in Tables 1-4 we observe that in all of the 2-D cases, IDSM is slower than DSM. The main reason for obtaining such results is that the elapsed CPU time for solving the subsystems by the PCG method is greater then that of solving these systems using the Cholesky factorizations. This is not the case in 3-D. Indeed, in the 3-D case the Cholesky factor of the matrix $\alpha T + W$ is of high-density and solving the systems with the Cholesky factor of $\alpha T + W$ (forward and backward substitutions) increases the CPU time of DSM.

Finally, we consider the Helmholtz equation in both 2-D and 3-D for $m = 32$ and several values of σ_1 and σ_2. In Table 9, we list the optimal values of α along with the values of

Table 6. Numerical results for the 3-D case with $\sigma_1 = 10$ and $\sigma_2 = 10$.

Method	m	8	16	32	64
DSM	α_{opt}	0.07	0.07	0.07	0.07
	Iter	2	2	2	2
	CPU	0.01	0.06	1.37	76.36
DSM	α^*	0.05	0.04	0.04	0.04
	Iter	3	3	3	2
	CPU	0.02	0.07	1.52	76.21
TTSCSP	α_{opt}	0.89	0.89	0.89	0.89
	β_{opt}	0.05	0.04	0.03	0.02
	Iter	4	4	4	4
	CPU	0.02	0.11	2.76	1765.45
PMHSS	α_{opt}	0.70	0.84	0.77	0.85
	Iter	36	38	40	40
	CPU	0.02	0.28	8.64	431.11
CRI	α_{opt}	1.0	1.0	1.0	1.0
	Iter	7	7	7	6
	CPU	0.02	0.14	4.01	2084.56
$u = A\backslash b$	CPU	0.01	0.09	3.01	323.57

Table 7. Numerical results for the inexact version of the methods for the 3-D case with $\sigma_1 = -10$ and $\sigma_2 = 10$.

Method	m	8	16	32	64
IDSM	α_{opt}	0.07	0.07	0.07	0.07
	Iter	3	2	2	2
	CPU	0.04	0.06	0.32	8.79
ITTSCSP	α_{opt}	0.89	0.89	0.89	0.89
	β_{opt}	0.05	0.03	0.03	0.02
	Iter	4	5	5	4
	CPU	0.03	0.08	0.55	15.79
IPMHSS	α_{opt}	0.70	0.70	0.70	0.70
	Iter	36	39	41	42
	CPU	0.05	0.22	1.88	61.81
ICRI	α_{opt}	1.0	1.0	1.0	1.0
	Iter	7	7	7	7
	CPU	0.04	0.10	0.61	20.93

$\mu_{\max}$ and $r(\alpha)$. In all the cases, we observe $\alpha_{opt} < \xi$ and $\mu_{\max} < r(\alpha)$. This confirms the convergence of DSM by the first part of Theorem 2.2.

Table 8. Numerical results for the inexact version of the methods for the 3-D case with $\sigma_1 = 10$ and $\sigma_2 = 10$.

Method	m	8	16	32	64
IDSM	α_{opt}	0.07	0.07	0.07	0.07
	Iter	3	2	2	2
	CPU	0.03	0.05	0.25	5.82
ITTSCSP	α_{opt}	0.81	0.60	0.50	0.60
	β_{opt}	0.05	0.03	0.03	0.03
	Iter	4	4	5	5
	CPU	0.03	0.08	0.47	12.17
IPMHSS	α_{opt}	0.70	0.84	0.77	0.85
	Iter	36	38	40	40
	CPU	0.05	0.18	1.64	41.63
CRI	α_{opt}	1.0	1.0	1.0	1.0
	Iter	7	7	7	6
	CPU	0.04	0.09	0.60	14.12

Table 9. Convergence study of DSM for $m = 32$ and some values of σ_1 and σ_2.

Method	σ_1	σ_2	α	$\mu_{\max}$	$r(\alpha)$
2-D case	-10	10	$0.74 < \xi$	1.028	3.823
2-D case	100	10	$0.06 < \xi$	0.084	1.095
3-D case	-10	10	$0.07 < \xi$	0.511	1.112
3-D case	10	10	$0.07 < \xi$	0.252	1.112

6. Conclusions

We have presented a new double-step method (DSM) for solving the system of linear equations that arise from finite difference discretization of the 2-D and 3-D complex Helmholtz equations. We have provided conditions under which our method is convergent. Numerical results with several parameter values indicate DSM outperforms the CRI, PMHSS, and TTSCSP methods on the selected set of examples.

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