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Araştırma Makalesi / Research Article

Characterization of Curves Whose Tangents Intersect a Straight Line in Euclidean 3-Space

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Abstract

In this study, we investigated the space curves in Euclidean 3-space whose tangent lines at each point intersect a given straight line passing the origin and intersect a fixed point, and we gave some characterizations in these cases.

Keywords: Frenet frame, tanget vector, space curve.

1. Introduction

The space curves whose principal normals intersecting a given straight line were first investigated by G. Pirondini, and further considered by E. Cesaro [1]. The corresponding question in affine space had been introduced by B. Su in 1929, He classified the curves and gave some remarkable results in affine 3-space by using equi-affine frame [3].

Let $\alpha : \mathrm{I} \to E^3$ be unit speed curve and $\{T(s), N(s), B(s)\}$ is the Frenet frame of $\alpha(s)$. $T(s)$, $N(s)$ and $B(s)$ are called the unit tangent, principal normal and binormal vectors respectively. Frenet formulae are given by

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$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \tag{1}$$

where $\kappa(s)$ and $\tau(s)$ are called the curvature and the torsion of the curve $\alpha(s)$. A space curve $\alpha(s)$ is determined by its curvature $\kappa(s)$ and its torsion $\tau(s)$, uniqely [2, 4].

2. The Space Curves Whose Tangents Intersect a Fixed Line

Let $\alpha : \mathrm{I} \to E^3$ be a curve with arclength parameter and l be the line passing the origin. We assume that the tangents lines intersect the fixed l directed constant and unit vector u at each point of the curve ,then we can write the following relation

$$\alpha(s) + \lambda(s)T(s) = \beta(s).u \tag{2}$$

where $\beta(s) = \varphi(s).u$ ve $\varphi(s)$ are the differentiable vector depending s so since $\beta(s)$ is a line then we quaranteed $\beta' \wedge \beta'' = 0$. By taking the first and the second derivatives of (2), we get

$$(1 + \lambda'(s))T(s) + \lambda(s)\kappa(s)N(s) = \beta'(s)u \tag{3}$$

$$\left\{ \begin{array}{l} \{\lambda''(s) - \lambda(s)\kappa^2(s)\}T(s) \\ + \{\kappa(s) + 2\lambda'(s)\kappa(s) + \lambda(s)\kappa'(s)\}N(s) \\ + \{\ \lambda(s)\kappa(s)\tau(s)\ \}B(s) \end{array} \right\} = \beta''(s)u \tag{4}$$

by using (2) and (4). If the tangents of the curve $\alpha(s)$ intersect a fixed point on l then, $\beta' = 0$ and also $\kappa(s) = 0$ and $\lambda(s) = -s + c$. In this case, β is the involute of $\alpha(s)$. Conversely, $\alpha(s)$ is involute of β , then $\alpha(s)$ is a line intersecting a fixed point of fixed line l , so following corollary is concerned.

Corallary 2.1: The tangents of the curve $\alpha(s)$ intersect a fixed point if and only if β is the involute of α and $\alpha(s)$ is a line.

If $\beta' \neq 0$ and $\beta'' = 0$ then from (4), we have

$$\lambda''(s) - \lambda(s)\kappa^2(s) = 0 \tag{5}$$

$$\kappa(s) + 2\lambda'(s)\kappa(s) + \lambda(s)\kappa'(s) = 0 \tag{6}$$

$$\lambda(s)\kappa(s)\tau(s) = 0 \tag{7}$$

Thus, we can say that there is no solution in the case $\beta''=0$ for $\kappa(s)\neq 0$ by considering (6), so there is no curve whose tangent lines intersect a fixed line.

Let $\beta''\neq 0$ then from (3) and (4), we have

$$(\lambda''(s)-\lambda(s)\kappa^2(s))\beta'(s)-(1+\lambda'(s))\beta''(s)=0 \tag{8}$$

$$(\kappa(s)+2\lambda'(s)\kappa(s)+\lambda(s)\kappa'(s))\beta'(s)-\lambda(s)\kappa(s)\beta''(s)=0 \tag{9}$$

$$\lambda(s)\kappa(s)\tau(s)=0 \tag{10}$$

It is clear from (10) that $\alpha(s)$ has to be planar, from (8), we get the solution

$$\beta(s)=c_1+c_2\int e^{\int\frac{\lambda''(s)-\lambda(s)\kappa^2(s)ds}{1+\lambda'(s)}ds}ds\,. \tag{11}$$

Rewrite (11) in (9),

$$\left\{\begin{matrix} c_2\{\kappa(s)+2\lambda'(s)\kappa(s)+\lambda(s)\kappa'(s)\}(1+\lambda'(s)) \\ -\lambda(s)\kappa(s)\{\lambda''(s)-\lambda(s)\kappa^2(s)\} \end{matrix}\right\}=0 \tag{12}$$

and the solution of (12) is,

$$\lambda(s)=\frac{-c_2\int e^{\int i\kappa(s)ds}ds-\int e^{-\int i\kappa(s)ds}ds-c_1}{c_2e^{\int i\kappa(s)ds}+e^{-\int i\kappa(s)ds}} \tag{13}$$

Here, $\lambda(s)$ is the real solution iff $c_2=1$,so the real solution of (12) is

$$\lambda(s)=-\frac{2\int\cos(\theta)ds+c_1}{2\cos(\theta)} \tag{14}$$

and from (11), $\beta(s)$ is

$$\beta(s)=c_1+\int e^{\int\phi ds}ds \tag{15}$$

where $\theta=\int\kappa(s)ds$ and

$$\phi=\frac{\{4\kappa^2(s)\sin(\theta)+2\kappa'(s)\cos(\theta)\}\int\cos(\theta)ds+2c_1\kappa^2(s)\sin(\theta)+2\kappa(s)\cos^2(\theta)+c_1\kappa^2(s)\cos(\theta)}{\kappa(s)\cos(\theta)(2\int\cos(\theta)ds+c_1)} \tag{16}$$

and c_1 is an arbitrary constant. For any c_2 and nonzero constant $\kappa(s)$ in (13), $\lambda(s)$ is

$$\lambda(s) = -\frac{c_2 \sin(\kappa s) + \cos(\kappa s) + c_1}{\kappa(c_2 \cos(\kappa s) - \sin(\kappa s))} \quad . \tag{17}$$

Hence following corallary is concerned.

Teorem 2.1: Let $\alpha(s)$ be a planar curve with non-constant curvature and the tangent lines at each point of $\alpha(s)$, intersect fixed line l then

$$\lambda(s) = -\frac{2\int \cos(\theta) ds + c_1}{2\cos(\theta)}$$

and

$$\beta(s) = c_1 + \int e^{\int \phi ds} ds$$

where $\theta = \int \kappa(s) ds$ and

$$\phi = \frac{\{4\kappa^2(s)\sin(\theta) + 2\kappa'(s)\cos(\theta)\}\int \cos(\theta) ds + 2c_1\kappa^2(s)\sin(\theta) + 2\kappa(s)\cos^2(\theta) + c_1\kappa^2(s)\cos(\theta)}{\kappa(s)\cos(\theta)(2\int \cos(\theta) ds + c_1)}$$

Corallary 2.2: If $\alpha(s)$ is a planar curve with constant nonzero curveture and the tangent lines at each points of $\alpha(s)$ intersect fixed line l ,then

$$\lambda(s) = -\frac{c_2 \sin(\kappa s) + \cos(\kappa s) + c_1}{\kappa(c_2 \cos(\kappa s) - \sin(\kappa s))}$$

and

$$\beta(s) = c_1 + c_2 \int e^{\int \frac{\lambda''(s) - \lambda(s)\kappa^2(s) ds}{1 + \lambda'(s)} ds} ds .$$

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