Research Article GU J Sci 34 (1): 251-270 (2021) DOI: 10.35378/gujs.660845

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

A Comparison Analysis of Fuzzy and Bayesian Linear Model Parameter Estimates for Replicated Response Measures

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Highlights

• This paper focuses on comparison of fuzzy and Bayesian model parameter estimates.
• Interval arithmetic was used to compare the model parameter estimates in the study.
• Fuzzy linear modeling can be preferred to Bayesian modeling without any strict modeling assumptions.

Article Info

Received: 17/12/2019
Accepted: 05/07/2020

Keywords

Replicated response measures,
Fuzzy modeling
Bayesian modeling
Interval arithmetic

Abstract

It is possible to define functional relationship between replicated response measures and input variables by using fuzzy and Bayesian modeling approaches. The main aim of the study is to present the alternative usability of fuzzy modeling approach to Bayesian modeling approach with defining a proper alpha-cut level among the many alpha-cut levels. In this study, the uncertainty of estimated model parameters were compared by transforming the estimated parameter values to intervals. Interval valued parameter estimates were obtained through alpha-cut level presentation and credible intervals for fuzzy and Bayesian approaches, respectively. Thus, it was achieved to model the replicated response measured (RRM) data set without making any probabilistic modeling assumptions which were hard to satisfy for small sized RRM data set. To compare the interval valued model parameter estimates in the proposed study, midpoint, width, radius and Hausdorff metrics were used. And also, interval type residuals were calculated to see the performance of predicted fuzzy and Bayesian models for making clear comparison. Two data sets from the literature, which were called Roman Catapult and Printing Ink, were used and the obtained results were discussed in application part.

1. INTRODUCTION

Experimental designs are commonly used in many fields of science before composing a predicted model of a response of interest and a number of associated input variables. In some cases, experimenters need to measure the response values with the same input levels in different experimental runs, which is called replication, to increase the precision of the predicted model. Analysis of replicated response measured (RRM) data set enables to examine multiple source of variability, e.g. changing equipment settings, environmental factors, which cause experimental error. In general, the unknown relationship between the variables is approximated by a low-degree polynomial statistical model in which the error is assumed to be random and the model parameters are crisp [1]. However, the unknown model parameters should have flexible structure for modeling of the RRM data set since the qualification of the replicated measures has uncertainty. It is clear here that the replicated response measures cannot be exactly represented with a single numerical quantity. In this case, fuzzy and Bayesian modeling approaches can be used alternatively from possibilistic and probabilistic modeling perspective, respectively.

In the literature, it is possible to see the fuzzy approach applications for response modeling due to its ability about defining uncertainty of the data sets. Fuzzy set theory, firstly introduced by [2], helps to analyse the uncertainty by the use of fuzzy numbers with a membership function. In response surface studies, one of

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the common way of expressing uncertainty with fuzzy approach is using fuzzy regression at modeling stage of the data set. The fuzzy regression model was used for an off-line quality engineering problem for the first time by [3]. Fuzzy approach was applied for modeling of dual responses in the study of [4]. A new response surface method was proposed by using fuzzy logic models by [5]. In [6], the application of fuzzy logic and gave comparative results with classical modeling approach for the development of sugarcane juice was presented. Fuzzy least squares (FLS) approach was applied to response surface problems by using triangular type-1 fuzzy numbers (TT1FNs) in the study of [7]. And also, application of fuzzy modeling approach to RRM data set through TT1FNs can be seen in the studies of [8-12].

There have been several studies about Bayesian approach for response modeling. A Bayesian approach for empirical regression modeling was presented in [13]. Bayesian hierarchical approach was used to model dual response system by [14-16]. Bayesian predictive approach for multiresponse problem was applied by [17]. The application of Bayesian approach to the robust parameter design problem of multiresponse was presented in the study of [18]. Bayesian methods in response surface methodology in the context of “off-line quality” improvement was used by [19]. Even there have been some studies about Bayesian approach for response surface modeling in the literature, there is only a work for application of Bayesian approach to RRM data set, studied by [11].

The main aim of this study is to present the alternative usability of the fuzzy approach to the Bayesian approach for modeling of RRM data set. The model parameters were assumed as random variables and TT1FNs for Bayesian and fuzzy modeling approaches, respectively. In this study, interval valued parameter estimates were obtained to make comparison for the uncertainty of estimated model parameters on the same domain. There have been several studies about application of interval type uncertainty. A new reliability estimation model based on the level cut strategy and volume ratio theory was proposed by [20] in the case that interval and fuzzy uncertainties exist simultaneously. [21] presented a non-probabilistic multidisciplinary uncertainty analysis method to obtain the bounds of system output variables when the uncertain variables were described as interval variables. A new analytic method of time-dependent reliability based on theory of non-probabilistic interval process for cracked structures with limited uncertainty information was proposed by [22].

In this study, interval indication of model parameter estimates was achieved by using alpha-cut level set presentation and highest posterior density (HPD) credible interval related to fuzzy approach and Bayesian approach, respectively. The purpose of the study was proposing an alternative flexible model, based on interval type fuzzy regression, instead of using Bayesian modeling for several situations, e.g. small sized RRM data sets, unsatistified probabilistic modeling assumptions. This was achieved by choosing a proper alpha-cut level among some alpha-cut levels in terms of similarity of interval valued Bayesian model parameter estimates. Several comparison metrics, e.g. midpoint, width, radius and Hausdorff, were used to make decision. Among them, the Hausdorff-metric is a well-known distance metric to compare the similarity of two closed intervals through midpoints and radius metrics. Detailed information about Hausdorff-metric and interval data can be seen in the study of [23, 24].

The paper was organized as follows. The fuzzy and Bayesian linear modeling approaches for RRM experiments were presented in section 2. In section 3, interval valued parameter estimates and comparison metrics were defined in detail. The applicability of preferable approach for fuzzy modeling was illustrated with the data sets from the literature in section 4. Conclusions with comparison results were given in section 5.

2. FUZZY AND BAYESIAN LINEAR MODELING APPROACHES FOR REPLICATED RESPONSE MEASURES

Composing a proper experimental design should be considered as a basic necessary step to get most valuable information about a modeling problem. In some cases, the researcher needs replicated measures of response variables for each experimental unit (observation). So, the experimental design is composed with replicated response measures as given in Table 1.

Table 1. *Experimental design with replicated response measures*

No	Input levels				Response			
	$\mathbf{X}_1$	$\mathbf{X}_2$	$\cdots$	$\mathbf{X}_p$	**Y**			
1	x_{11}	x_{12}	$\ldots$	x_{1p}	y_{11}	y_{12}	$\ldots$	y_{1r}
2	x_{21}	x_{22}	$\ldots$	x_{2p}	y_{21}	y_{22}	$\ldots$	y_{2r}
.	.	.		.	.	.		
.	.	.	$\ldots$	.	.	.	$\ldots$	
.	.	.		.	.	.		
n	x_{n1}	x_{n2}	$\ldots$	x_{np}	y_{n1}	y_{n2}	$\ldots$	y_{np}

In Table 1, *n* denotes the number of experimental units and *r* is the number of replications for each response. It should be noted that replications are measured for each setting of a group of *p* input variables. The main purpose of creating an experimental design as given in Table 1 is obtaining much more accurate estimates with measuring the quantity of uncertainty for the response.

A modeling methodology of the RRM data set, given in Table 1, can be considered as linear regression model which is well-known and commonly used one. General form of the linear regression model, used in this study, can be given in a matrix form as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{1}$$

in which $\mathbf{Y} = \left[Y_1\ Y_2 \ldots Y_n\right]'$ is a response vector with *n* observations $(n \times 1)$, $\boldsymbol{\beta} = \left[\beta_0\ \beta_1 \ldots \beta_p\right]'$ is a vector of unknown model parameters $\left((p+1) \times 1\right)$, $\boldsymbol{\varepsilon} = \left[\varepsilon_1\ \varepsilon_2 \ldots \varepsilon_n\right]'$ is an error vector with independent and identically distributed (i.i.d.) random vectors $(n \times 1)$, and $\mathbf{X}$ is a matrix of settings of the input variables $\left(n \times (p+1)\right)$. It should be noted here that the assumptions on $\boldsymbol{\varepsilon}$ are zero mean and common variance.

Although the model given in Equation (1) is well-used one, the application of the model to RRM data set will need some adaptations. It is clear that representing of replicated measures with a single quantity will cause loss of information for each unit. Therefore, it is necessary to use modeling approaches with conserving natural data structure of replications which may contain uncertainty different than randomness.

In this study, two different types of modeling approaches, fuzzy and Bayesian, are used to estimate the unknown model parameters, $\boldsymbol{\beta}$, for RRM data set.

2.1. Fuzzy Linear Modeling Approach

Fuzzy modeling is considered to be one of the proper modeling approach for RRM data set due to its ability of dealing with the uncertainties of the nature of the replicated measures. In order to apply fuzzy linear modeling approach to unknown response, the linear model given in Equation (1) can be written as

$$\tilde{\mathbf{Y}} = \mathbf{X}\tilde{\boldsymbol{\beta}} + \tilde{\boldsymbol{\varepsilon}} \tag{2}$$

where the observed response, model parameters and error vector are assumed as fuzzy numbers denoted as $\tilde{\mathbf{Y}} = \left[\tilde{Y}_1\ \tilde{Y}_2 \ldots \tilde{Y}_n\right]'$, $\tilde{\boldsymbol{\beta}} = \left[\tilde{\beta}_0\ \tilde{\beta}_1 \ldots \tilde{\beta}_p\right]'$ and $\boldsymbol{\varepsilon} = \left[\tilde{\varepsilon}_1\ \tilde{\varepsilon}_2 \ldots \tilde{\varepsilon}_n\right]'$, respectively. Throughout the paper TT1FNs are preferred to use for the sake of simplicity. Here, the input variables are considered to be crisp.

To present the observed replicated response measures as TT1FNs, some of the descriptive statistics of the replicated response measures are calculated. Therefore, fuzzification of replicates is achieved by taking into account the data structure of replications in statistical framework.
Let the matrix of observed replicated response measures be

$$\mathbf{y}=\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \cdot\\ \cdot\\ \cdot\\ \mathbf{y}_n\end{bmatrix}=\begin{bmatrix}y_{11} & y_{12} & \cdots & y_{1r}\\ y_{21} & y_{22} & \cdots & y_{2r}\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & & \cdot\\ y_{n1} & y_{n2} & \cdots & y_{nr}\end{bmatrix}. \tag{3}$$

The TT1FN representations for each unit is given as

$$\mathbf{y}=\begin{bmatrix}\tilde{\mathbf{y}}_1\\ \tilde{\mathbf{y}}_2\\ \cdot\\ \cdot\\ \cdot\\ \tilde{\mathbf{y}}_n\end{bmatrix}=\begin{bmatrix}\left(y_1^l,y_1^c,y_1^u\right)\\ \left(y_2^l,y_2^c,y_2^u\right)\\ \cdot\\ \cdot\\ \cdot\\ \left(y_n^l,y_n^c,y_n^u\right)\end{bmatrix}. \tag{4}$$

Here, each fuzzy response is denoted as $\tilde{\mathbf{y}}_i=\left(y_i^l,y_i^c,y_i^u\right),\ i=1,2,...,n$. In the studies of [10, 11], replicated response measures were transformed to TT1FNs by using mean and standard deviation statistics of replicated measures. However, these statistics may not be sufficient to present the data structure of replications during fuzzification. Different type of fuzzification rules for presenting the replicated response measures as TT1FNs were given in [12]. It was seen that the fuzzification with descriptive statistics and golden ratio was the most proper one. Therefore, in this study, replicated response measures are fuzzified by using the rule in the study of [12] as following

$$\begin{aligned} y_i^l &= y_{i(1)}\\ y_i^c &= y_{i(1)}+r_i\times\rho\ ,\quad i=1,2,\dots,n\ ,\quad j=1,2,\dots,r\\ y_i^u &= y_{i(r)} \end{aligned} \tag{5}$$

where $y_{i(1)}$ and $y_{i(r)}$ are the smallest and the largest order statistics for each unit, respectively. r_i is called as the range and obtained by $r_i=y_{i(r)}-y_{i(1)}$, $i = 1,2,\dots,n$. ρ is the golden ratio and taken equal to 0.618.

After fuzzification of replicated response measures, it is aimed to obtain predicted fuzzy linear response model. In this case, the predicted fuzzy linear regression model is written as

$$\hat{\tilde{\mathbf{Y}}}=\mathbf{X}\hat{\tilde{\boldsymbol{\beta}}}+\hat{\tilde{\mathbf{e}}} \tag{6}$$

in which $\hat{\tilde{\mathbf{Y}}}$, $\hat{\tilde{\boldsymbol{\beta}}}$ and $\hat{\tilde{\mathbf{e}}}$ are predicted fuzzy response values, estimated fuzzy model parameters and predicted fuzzy errors, denoted as $\hat{\tilde{\mathbf{Y}}}=\left(\hat{\mathbf{Y}}^l,\hat{\mathbf{Y}}^c,\hat{\mathbf{Y}}^u\right)$, $\hat{\tilde{\boldsymbol{\beta}}}=\left(\hat{\boldsymbol{\beta}}^l,\hat{\boldsymbol{\beta}}^c,\hat{\boldsymbol{\beta}}^u\right)$ and $\hat{\tilde{\mathbf{e}}}=\left(\hat{\mathbf{e}}^l,\hat{\mathbf{e}}^c,\hat{\mathbf{e}}^u\right)$, respectively. In order to obtain the estimates of triangular type fuzzy model parameters FLS approach is used. The FLS is based on minimizing the sum of the squared distance between observed and predicted fuzzy response values, formalized as

$$\min_{\tilde{\boldsymbol{\beta}}} \phi\left(\tilde{\boldsymbol{\beta}}\right) = \min_{\tilde{\boldsymbol{\beta}}} \left(d\left(\tilde{\mathbf{Y}}, \hat{\tilde{\mathbf{Y}}}\right)\right) \tag{7}$$

where d denotes the Diamond's distance metric, called vertex method, given in [25] as following

$$d\left(\tilde{\mathbf{Y}}, \hat{\tilde{\mathbf{Y}}}\right) = \sqrt{\frac{1}{3}\left(\left(\mathbf{Y}^l - \hat{\mathbf{Y}}^l\right)^2 + \left(\mathbf{Y}^c - \hat{\mathbf{Y}}^c\right)^2 + \left(\mathbf{Y}^u - \hat{\mathbf{Y}}^u\right)^2\right)}. \tag{8}$$

The estimates of fuzzy model parameters are obtained by minimizing the Equation (7) and calculated as given below

$$\hat{\tilde{\boldsymbol{\beta}}} = W\tilde{\mathbf{Y}} \tag{9}$$

in which W is equal to $\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'$. Assuming that the $\left(\mathbf{X}'\mathbf{X}\right)^{-1}$ is nonsingular, matrix W may have negative values. In this case, scalar multiplication should be done with considering the elementary operations rules of TT1FNs. Let matrix W be

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & & \cdot \\ w_{p+1,1} & w_{p+1,2} & \cdots & w_{p+1,n} \end{bmatrix}_{(p+1)\times n} \tag{10}$$

and the multiplication is

$$W\tilde{\mathbf{Y}} = \begin{cases} \left(w_{ij}\mathbf{Y}^l, w_{ij}\mathbf{Y}^c, w_{ij}\mathbf{Y}^u\right) & , \quad w_{ij} > 0 \\ \left(-w_{ij}\mathbf{Y}^u, -w_{ij}\mathbf{Y}^c, -w_{ij}\mathbf{Y}^l\right) & , \quad w_{ij} < 0 \end{cases} \quad , \quad i = 1,2,\ldots,p+1 \ , \ j=1,2,\ldots,n \tag{11}$$

where w_{ij} is the element positioned in the i th row and j th column of matrix W. Therefore, according to the matrix operators given in Equation (11), the triangular fuzzy model parameter estimates, $\hat{\tilde{\boldsymbol{\beta}}}$, are obtained.

2.2. Bayesian Linear Modeling Approach

Bayesian can be considered another proper modeling approach for RRM data set since the ability of representing uncertainty of replicated response measures in the predicted model with assuming model parameters as random variables instead of crisp values. And also, Bayesian approach allows learning from the data with combining prior information of model parameters. Posterior distribution of unknown model parameters is called as Bayesian estimates of model parameters. Linear regression methods can be thought of as Bayesian posterior inference based on a non-informative prior distribution for the parameters of the normal linear model.

Consider the linear regression model given in matrix form as in Equation (1) and recall the assumptions concerning the elements of $\boldsymbol{\varepsilon}$ with Normally distributed, $\boldsymbol{\varepsilon} \sim N_n\left(0, \sigma^2 I_n\right)$. It is clear that the model which relates parameters and observations is written as

$$\mathbf{Y}|\boldsymbol{\beta},\sigma^2,\mathbf{X} \sim N_n\left(\mathbf{X}\boldsymbol{\beta},\sigma^2 I_n\right). \tag{12}$$

After defining the model, Bayesian analysis focuses to discover the posterior distribution for the parameters. The analysis begins with a prior distribution for the parameters. A non-informative prior distribution, commonly used for linear regression, can be defined as

$$h\left(\boldsymbol{\beta},\sigma^2\right) \propto \left(\sigma^2\right)^{-1}, \quad \boldsymbol{\beta} \in R^p \ , \quad \sigma^2 > 0. \tag{13}$$

The expression given in Equation (13) means that the joint probability distribution of $\boldsymbol{\beta}$ and σ^2 is a flat surface with a constant level proportional to $\frac{1}{\sigma^2}$. Applying Bayes Theorem, the posterior density of the $\boldsymbol{\beta}$ and σ^2, conditional on the data, is given by a proportion below [11]

$$h\left(\boldsymbol{\beta},\sigma^2|\mathbf{Y}\right) \propto \left(\sigma^2\right)^{-\left(\frac{k}{2}+1\right)}.e^{-\frac{kS^2}{2\sigma^2}} \times \left(\sigma^2\right)^{-\frac{p}{2}}.e^{-\frac{1}{2\sigma^2}\left[\left(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}}\right)'\mathbf{X}'\mathbf{X}\left(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}}\right)\right]}. \tag{14}$$

While the posterior distribution of $\boldsymbol{\beta}$ given σ^2 and $\mathbf{Y}$ is

$$\boldsymbol{\beta}|\sigma^2,\mathbf{Y} \sim N_{p+1}\left(\hat{\boldsymbol{\beta}},\sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}\right) \tag{15}$$

and the marginal posterior distribution of $\boldsymbol{\beta}$ is multivariate Student-t distribution given as

$$\boldsymbol{\beta}|\mathbf{Y} \sim t_{p+1}\left(k;\hat{\boldsymbol{\beta}},S^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}\right) \tag{16}$$

where $\hat{\boldsymbol{\beta}}$ and S^2 are the maximum likelihood estimates of $\boldsymbol{\beta}$ and σ^2 parameters, respectively. Here, $kS^2 = \left(\mathbf{Y}-\mathbf{X}\hat{\boldsymbol{\beta}}\right)'\left(\mathbf{Y}-\mathbf{X}\hat{\boldsymbol{\beta}}\right)$, $k = n-(p+1)$ and $\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{Y}$ [26]. Thus, the marginal posterior distribution of each regression model parameter β_j, $j = 0,1,2,...,p$ is obtained as

$$\beta_j|Y \sim t_1\left(k;\hat{\beta}_j,S^2\left(\mathbf{X}'\mathbf{X}\right)_{jj}^{-1}\right). \tag{17}$$

3. INTERVAL VALUED PARAMETER ESTIMATES

Determining the uncertainty of model parameters through fuzzy and Bayesian approaches are distinct from each other. For this purpose, it is necessary to define a common uncertainty presentation for the model parameters. In this study, interval valued parameter estimates are obtained to present the fuzzy and Bayesian linear model parameters since transforming the model parameter estimates to intervals provides convenience for comparison of the parameters uncertainty on the same domain.

3.1. Fuzzy alpha-cut level interval

To define the fuzzy parameter estimates as crisp intervals, a well-known approach is representing the $\hat{\tilde{\boldsymbol{\beta}}} = \left(\hat{\boldsymbol{\beta}}^l,\hat{\boldsymbol{\beta}}^c,\hat{\boldsymbol{\beta}}^u\right)$ as a family of sets called alpha-level set. The alpha-level set of $\hat{\tilde{\boldsymbol{\beta}}}$ is defined as

$$\hat{\tilde{\boldsymbol{\beta}}} = \left\{\hat{\boldsymbol{\beta}} \in R : \mu_{\hat{\tilde{\boldsymbol{\beta}}}}\left(\hat{\boldsymbol{\beta}}\right) \geq \alpha\right\} \tag{18}$$

where $\mu_{\hat{\tilde{\boldsymbol{\beta}}}}\left(\hat{\boldsymbol{\beta}}\right)$ is identified as below

$$\mu_{\hat{\tilde{\boldsymbol{\beta}}}}\left(\hat{\boldsymbol{\beta}}\right)=\begin{cases}\dfrac{\hat{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}}^{l}}{\hat{\boldsymbol{\beta}}^{c}-\hat{\boldsymbol{\beta}}^{l}} & , \quad \hat{\boldsymbol{\beta}}^{l}\le\hat{\boldsymbol{\beta}}\le\hat{\boldsymbol{\beta}}^{c}\\ \dfrac{\hat{\boldsymbol{\beta}}^{u}-\hat{\boldsymbol{\beta}}}{\hat{\boldsymbol{\beta}}^{u}-\hat{\boldsymbol{\beta}}^{c}} & , \quad \hat{\boldsymbol{\beta}}^{c}\le\hat{\boldsymbol{\beta}}\le\hat{\boldsymbol{\beta}}^{u}\ .\\ 0 & , \qquad o.w.\end{cases} \tag{19}$$

It is possible to obtain a crisp interval of $\hat{\tilde{\boldsymbol{\beta}}}$ easily by alpha-cut operation, denoted with $\hat{\tilde{\boldsymbol{\beta}}}_{\alpha}$, as follows

$$\begin{aligned}\hat{\tilde{\boldsymbol{\beta}}}_{\alpha}&=\left[L(\alpha),U(\alpha)\right]\\&=\left[\hat{\boldsymbol{\beta}}^{l}+\left(\hat{\boldsymbol{\beta}}^{c}-\hat{\boldsymbol{\beta}}^{l}\right)\alpha,\ \hat{\boldsymbol{\beta}}^{u}-\left(\hat{\boldsymbol{\beta}}^{u}-\hat{\boldsymbol{\beta}}^{c}\right)\alpha\right]\end{aligned} \tag{20}$$

in which $\alpha\in[0,1]$. It should be noted here that the smaller the value of alpha, the higher the uncertainty.

3.2. Bayesian credible interval

In order to define the Bayesian parameter estimates as crisp intervals, Bayesian credible interval is defined as a measure of the variation of model parameters. The Bayesian credible interval is similar to frequentist confidence interval and generally called as HPD credible interval.

In this study, HPD credible interval is used for analyzing the variation of each model parameters β_j , $j=0,1,2,...,p$, defined as

$$\left[\hat{\beta}_j-\sqrt{S^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}_{jj}}F^{-1}_{t_{(k)}}\left(\frac{1-\gamma}{2}\right),\ \hat{\beta}_j+\sqrt{S^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}_{jj}}F^{-1}_{t_{(k)}}\left(\frac{1-\gamma}{2}\right)\right] \tag{21}$$

where $F^{-1}_{t_{(k)}}$ is a quantile of the Student-t posterior distribution with $(1-\gamma)\%$ confidence level.

3.3. Comparison metrics for interval valued estimates of model parameters

One of the main important aspect of the analysis of interval valued parameter estimates is the usage of convenient distance metric which is expected to be easy to calculate and interpret. In this study, midpoint, width, radius and the Hausdorff metrics are used for comparison purpose of the interval valued parameter estimates. More brief information on this topics can be seen in the study of [27, 28].

Let $[\beta_i]=\left[\beta_i^-,\beta_i^+\right]$, $i=1,2,...,p$, be an interval valued parameter estimates on R. The midpoint of $[\beta_i]$, $i=1,2,...,p$, is defined as the real number

$$mid\left([\beta_i]\right)=\frac{1}{2}\left(\beta_i^-+\beta_i^+\right),\ i=1,2,...,p \tag{22}$$

and the width and radius of $[\beta_i]$, $i=1,2,...,p$, are defined as, respectively,

$$w\left([\beta_i]\right)=\beta_i^+-\beta_i^- \ ,\ i=1,2,...,p \tag{23}$$

and

$$rad([\beta_i]) = \frac{1}{2} w([\beta_i]) = \frac{1}{2}(\beta_i^+ - \beta_i^-),\ i = 1,2,...,p. \tag{24}$$

Let $[\beta_i] = [\beta_i^-, \beta_i^+]$ and $[\eta_i] = [\eta_i^-, \eta_i^+]$, $i = 1,2,...,p$, be two intervals on R. Then, the Hausdorff-metric between $[\beta_i]$ and $[\eta_i]$ is given by

$$H_i = H([\beta_i],[\eta_i]) = \max\left\{\left|\beta_i^- - \eta_i^-\right|, \left|\beta_i^+ - \eta_i^+\right|\right\},\ i = 1,2,...,p. \tag{25}$$

The Equation (23) can also be expressed in terms of midpoints and radius:

$$H_i = H([\beta_i],[\eta_i]) = \left|mid([\beta_i]) - mid([\eta_i])\right| + \left|rad([\beta_i]) - rad([\eta_i])\right|,\ i = 1,2,...,p. \tag{26}$$

The higher the quantity of $H([\beta_i],[\eta_i])$, the lower the similarity of $[\beta_i]$ and $[\eta_i]$. In this study, Hausdorff metric is used to compare the two interval valued parameter estimates since it considers midpoint and radius metrics together as given in Equation (26).

4. APPLICATION

In this section, two data sets, called Roman Catapult data set and Printing Ink data set, are used for application purpose. Even though, the data sets are previously used in the study of [11], these data sets are preferred to use to present the alternatively usability of fuzzy approach providing that the definition of proper alpha-cut levels of fuzzy parameter estimates.

Roman Catapult data set: Roman Catapult data set is originally used in the study of [29] for an example of dual response analysis. The catapult experiment is designed with three inputs (length - X_1, stop angle - X_2, pivot height - X_3) and three replicated response measures by using second order central composite design. The RRM data set can be seen in the study of [11]. In order to apply fuzzy modeling to Roman Catapult data set, it is necessary to present the replicated response measures as TT1FNs. The fuzzification of response is achieved according to the data structure of replicated measures. In order to understand the data structure of the replicated measures, box-plots of the responses were plotted. The distribution of the replicated measures can be seen from the box-plots, given in Figure 1, for each unit. It's clear from Figure 1 that the replicated values have skewed distribution.

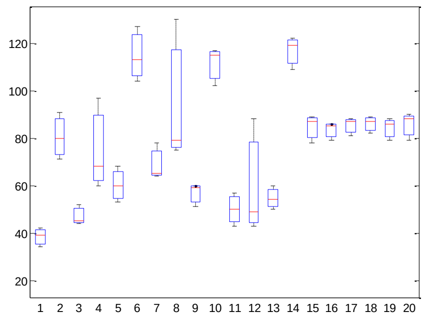

Figure 1. *Box-plots of replicated response measures for each unit of Roman Catapult data.*

By using the fuzzification rule, given in Equation (5), the data set with triangular fuzzy response values is obtained as in Table 2.

The predicted fuzzy response model is obtained by using FLS approach as below:

$$\hat{\tilde{Y}} = (75.1712, 85.2183, 92.5516) + (6.1973, 14.6454, 22.3317) X_1 + (-7.9474, 3.1401, 12.4972) X_2 + (11.1594, 19.9862, 27.1707) X_3 + (-17.8744, -8.3820, -0.7558) X_2^2 + (-15.75, -3.7625, 6.75) X_2 X_3. \quad (27)$$

Table 2. *The Roman Catapult data set with fuzzy responses*

No	**Run order**	X_1	X_2	X_3	$\tilde{\mathbf{Y}}$
1	20	-1	-1	-1	(34, 38.944, 42)
2	9	-1	-1	1	(71, 83.36, 91)
3	11	-1	1	-1	(44, 48.944, 52)
4	14	-1	1	1	(60, 82.866, 97)
5	15	1	-1	-1	(53, 62.27, 68)
6	10	1	-1	1	(104, 118.214, 127)
7	12	1	1	-1	(64, 72.652, 78)
8	1	1	1	1	(75, 108.99, 130)
9	7	-1.682	0	0	(51, 56.562, 60)
10	4	1.682	0	0	(102, 111.27, 117)
11	19	0	-1.682	0	(43, 51.652, 57)
12	18	0	1.682	0	(43, 70.81, 88)
13	12	0	0	-1.682	(50, 56.18, 60)
14	8	0	0	1.682	(109, 117.034, 122)
15	6	0	0	0	(78, 84.798, 89)
16	5	0	0	0	(79, 83.326, 86)
17	17	0	0	0	(81, 85.326, 88)
18	3	0	0	0	(82, 86.326, 89)
19	2	0	0	0	(79, 84.562, 88)
20	16	0	0	0	(79, 85.798, 90)

The predicted Bayesian model is taken into consideration as in the study of [11]. It is clear to say that the comparison of parameter estimates of fuzzy and Bayesian models is not possible unless transforming the parameter estimates to intervals. In this study, the same interval type Bayesian parameter estimates, which are obtained 95% confident with HPD credible interval, are used as in the study of [11]. The previously obtained Bayesian interval estimates of model parameters with midpoint, width and radius metrics are summarized in Table 3.

Table 3. *Interval type parameter estimates of Bayesian model with midpoint, width and radius metrics for Roman Catapult data set*

Interval type parameters $[\boldsymbol{\beta}]$	**Interval estimates of parameters** $[\underline{\boldsymbol{\beta}}, \overline{\boldsymbol{\beta}}]$	**Midpoint** $mid(\boldsymbol{\beta})$	**Width** $w(\boldsymbol{\beta})$	**Radius** $Rad(\boldsymbol{\beta})$
$[\beta_0]$	[77.3075 91.3816]	84.3445	14.0741	7.0371
$[\beta_1]$	[7.2240 20.5418]	13.8829	13.3178	6.6589
$[\beta_2]$	[-6.0375 7.2803]	0.6214	13.3178	6.6589
$[\beta_3]$	[11.9831 25.3009]	18.6420	13.3178	6.6589
$[\beta_4]$	[-17.2997 -4.4539]	-10.8768	12.8458	6.4229
$[\beta_5]$	[-14.0757 3.3257]	-5.3750	17.4014	8.7007

In order to obtain interval type fuzzy parameter estimates, alpha-cut operation is applied to fuzzy parameter estimates. The predicted fuzzy model with alpha-cut level presentation of parameter estimates is given below

$$\begin{aligned}\hat{\tilde{Y}} = & \left[75.1712+10.0471\alpha,\ 92.5516-7.3333\alpha\right]+\left[6.1973+8.4481\alpha,\ 22.3317-7.6863\alpha\right]X_1 \\ & +\left[-7.9474+11.0875\alpha,\ 12.4972-9.3571\alpha\right]X_2+\left[11.1594+8.8268\alpha,\ 27.1707-7.1845\alpha\right]X_3 \\ & +\left[-17.8744+9.4925\alpha,\ -0.7558-7.6262\alpha\right]X_2^2+\left[-15.75+11.9875\alpha,\ 6.75-10.5125\alpha\right]X_2X_3.\end{aligned} \tag{28}$$

It is possible to obtain different real valued closed interval type parameter estimates by using different alpha-cut levels, $\alpha\in[0,1]$, e.g. $\alpha\in\{0,\ 0.05,\ 0.10,\ \ldots,0.90,\ 0.95\}$. The obtained interval type fuzzy parameter estimates, midpoint, width and radius metrics are presented in Table 4 for each alpha-cut levels.

Table 4. *Interval type parameter estimates of fuzzy model with midpoint, width and radius metrics for Roman Catapult data set*

alpha-cut levels	**Interval type parameters** $\left[\boldsymbol{\beta}^{\alpha}\right]$	**Interval estimates of model parameters** $\left[\underline{\boldsymbol{\beta}}^{\alpha},\overline{\boldsymbol{\beta}}^{\alpha}\right]$	**Midpoint** $mid\left(\boldsymbol{\beta}^{\alpha}\right)$	**Width** $w\left(\boldsymbol{\beta}^{\alpha}\right)$	**Radius** $Rad\left(\boldsymbol{\beta}^{\alpha}\right)$
0	$\left[\beta_0^{\alpha}\right]$	[75.1712 92.5516]	83.8614	17.3804	8.6902
	$\left[\beta_1^{\alpha}\right]$	[6.1973 22.3317]	14.2645	16.1344	8.0672
	$\left[\beta_2^{\alpha}\right]$	[-7.9474 12.4972]	2.2749	20.4446	10.2223
	$\left[\beta_3^{\alpha}\right]$	[11.1594 27.1707]	19.165	16.0113	8.0056
	$\left[\beta_4^{\alpha}\right]$	[-17.8744 -0.7558]	-9.3151	17.1187	8.5593
	$\left[\beta_5^{\alpha}\right]$	[-15.75 6.75]	-4.5	22.5	11.25
0.05	$\left[\beta_0^{\alpha}\right]$	[75.6736 92.1849]	83.9292	16.5114	8.2557
	$\left[\beta_1^{\alpha}\right]$	[6.6197 21.94749	14.2835	15.3277	7.6639
	$\left[\beta_2^{\alpha}\right]$	[-7.3931 12.0294]	2.3181	19.4224	9.7112
	$\left[\beta_3^{\alpha}\right]$	[11.6008 26.8115]	19.2061	15.2107	7.6054
	$\left[\beta_4^{\alpha}\right]$	[-17.3998 -1.1371]	-9.2684	16.2627	8.1314
	$\left[\beta_5^{\alpha}\right]$	[-15.1506 6.2244]	-4.4631	21.3750	10.6875
0.10	$\left[\beta_0^{\alpha}\right]$	[76.1759 91.8183]	83.9971	15.6424	7.8212
	$\left[\beta_1^{\alpha}\right]$	[7.0421 21.5631]	14.3026	14.5210	7.2605
	$\left[\beta_2^{\alpha}\right]$	[-6.8387 11.5615]	2.3614	18.4002	9.2001
	$\left[\beta_3^{\alpha}\right]$	[12.0421 26.4522]	19.2472	14.4101	7.2051
	$\left[\beta_4^{\alpha}\right]$	[-16.9252 -1.5184]	-9.2218	15.4068	7.7034
	$\left[\beta_5^{\alpha}\right]$	[-14.5513 5.6988]	-4.4262	20.2500	10.1250
	$\left[\beta_0^{\alpha}\right]$	[77.1806 91.0850]	84.1328	13.9043	6.9522
	$\left[\beta_1^{\alpha}\right]$	[7.8869 20.7945]	14.3407	12.9075	6.4538

0.20	$\left[\beta_2^\alpha\right]$	[-5.7299 10.6258]	2.4479	16.3557	8.1779
	$\left[\beta_3^\alpha\right]$	[12.9248 25.7338]	19.3293	12.8090	6.4045
	$\left[\beta_4^\alpha\right]$	[-15.9759 -2.281]	-9.1285	13.6949	6.8475
	$\left[\beta_5^\alpha\right]$	[-13.3525 4.6475]	-4.3525	18.0000	9.0000
0.30	$\left[\beta_0^\alpha\right]$	[78.1853 90.3516]	84.2685	12.1663	6.0831
	$\left[\beta_1^\alpha\right]$	[8.7317 20.0258]	14.3788	11.2941	5.6470
	$\left[\beta_2^\alpha\right]$	[-4.6212 9.6901]	2.5344	14.3112	7.1556
	$\left[\beta_3^\alpha\right]$	[13.8074 25.0153]	19.4114	11.2079	5.6039
	$\left[\beta_4^\alpha\right]$	[-15.0267 -3.0436]	-9.0352	11.9831	5.9915
	$\left[\beta_5^\alpha\right]$	[-12.1538 3.5963]	-4.2788	15.75	7.875
0.40	$\left[\beta_0^\alpha\right]$	[79.1901 89.6183]	84.4042	10.4282	5.2141
	$\left[\beta_1^\alpha\right]$	[9.5765 19.2572]	14.4169	9.6807	4.8403
	$\left[\beta_2^\alpha\right]$	[-3.5124 8.7544]	2.621	12.2668	6.1334
	$\left[\beta_3^\alpha\right]$	[14.6901 24.2969]	19.4935	9.6068	4.8034
	$\left[\beta_4^\alpha\right]$	[-14.0774 -3.8062]	-8.9418	10.2712	5.1356
	$\left[\beta_5^\alpha\right]$	[-10.9550 2.545]	-4.205	13.5	6.75
0.50	$\left[\beta_0^\alpha\right]$	[80.1948 88.885]	84.5399	8.6902	4.3451
	$\left[\beta_1^\alpha\right]$	[10.4214 18.4886]	14.4550	8.0672	4.0336
	$\left[\beta_2^\alpha\right]$	[-2.4037 7.8186]	2.7075	10.2223	5.1112
	$\left[\beta_3^\alpha\right]$	[15.5728 23.5784]	19.5756	8.0056	4.0028
	$\left[\beta_4^\alpha\right]$	[-13.1282 -4.5689]	-8.8485	8.5593	4.2797
	$\left[\beta_5^\alpha\right]$	[-9.7562 1.4938]	-4.1312	11.25	5.625
0.60	$\left[\beta_0^\alpha\right]$	[81.1995 88.1516]	84.6756	6.9522	3.4761
	$\left[\beta_1^\alpha\right]$	[11.2662 17.7199]	14.4931	6.4538	3.2269
	$\left[\beta_2^\alpha\right]$	[-1.2949 6.8829]	2.7940	8.1779	4.0889
	$\left[\beta_3^\alpha\right]$	[16.4555 22.86]	19.6577	6.4045	3.2023
	$\left[\beta_4^\alpha\right]$	[-12.1790 -5.3315]	-8.7552	6.8475	3.4237
	$\left[\beta_5^\alpha\right]$	[-8.5575 0.4425]	-4.0575	9	4.5
	$\left[\beta_0^\alpha\right]$	[82.2042 87.4183]	84.8113	5.2141	2.6071
	$\left[\beta_1^\alpha\right]$	[12.1110 16.9513]	14.5311	4.8403	2.4202
	$\left[\beta_2^\alpha\right]$	[-0.1862 5.9472]	2.8805	6.1334	3.0667

0.70	$\left[\beta_3^\alpha\right]$	[17.3382 22.1415]	19.7399	4.8034	2.4017
	$\left[\beta_4^\alpha\right]$	[-11.2297 -6.0941]	-8.6619	5.1356	2.5678
	$\left[\beta_5^\alpha\right]$	[-7.3588 -0.6087]	-3.9837	6.75	3.375
0.80	$\left[\beta_0^\alpha\right]$	[83.2089 86.685]	84.947	3.4761	1.738
	$\left[\beta_1^\alpha\right]$	[12.9558 16.1827]	14.5692	3.2269	1.6134
	$\left[\beta_2^\alpha\right]$	[0.9226 5.0115]	2.9670	4.0889	2.0445
	$\left[\beta_3^\alpha\right]$	[18.2208 21.4231]	19.8220	3.2023	1.6011
	$\left[\beta_4^\alpha\right]$	[-10.2805 -6.8567]	-8.5686	3.4237	1.7119
	$\left[\beta_5^\alpha\right]$	[-6.1600 -1.66]	-3.9100	4.5	2.25
0.90	$\left[\beta_0^\alpha\right]$	[84.2136 85.9517]	85.0826	1.738	0.869
	$\left[\beta_1^\alpha\right]$	[13.8006 15.4141]	14.6073	1.6134	0.8067
	$\left[\beta_2^\alpha\right]$	[2.0313 4.0758]	3.0535	2.0445	1.0222
	$\left[\beta_3^\alpha\right]$	[19.1035 20.7046]	19.9041	1.6011	0.8006
	$\left[\beta_4^\alpha\right]$	[-9.3312 -7.6193]	-8.4753	1.7119	0.8559
	$\left[\beta_5^\alpha\right]$	[-4.9612 -2.7112]	-3.8362	2.25	1.125
0.95	$\left[\beta_0^\alpha\right]$	[84.716 85.585]	85.1505	0.869	0.4345
	$\left[\beta_1^\alpha\right]$	[14.2230 15.0297]	14.6264	0.8067	0.4034
	$\left[\beta_2^\alpha\right]$	[2.5857 3.6079]	3.0968	1.0222	0.5111
	$\left[\beta_3^\alpha\right]$	[19.5449 20.3454]	19.9451	0.8006	0.4003
	$\left[\beta_4^\alpha\right]$	[-8.8566 -8.0006]	-8.4286	0.8559	0.428
	$\left[\beta_5^\alpha\right]$	[-4.3619 -3.2369]	-3.7994	1.125	0.5625

It can be easily seen from Table 4 that the larger the alpha-cut levels, the smaller the w metric values. The small value of w metric means that the variation of parameter estimates is small. In order to define the similarities between interval type parameter estimates of Bayesian model, given in Table 3, and interval type parameter estimates of fuzzy model, given in Table 4, Hausdorff metric is calculated. The obtained results are presented in Table 5.

Table 5. *Hausdorff metric of interval type parameter estimates for Roman Catapult data set*

alpha-cut levels	H_1	H_2	H_3	H_4	H_5	H_6	H_7
0	2.1363	1.7899	5.2169	1.8698	3.6981	3.4243	18.1353
0.05	1.6339	1.4056	4.7491	1.5106	3.3168	2.8987	15.5147
0.10	1.1316	1.0213	4.2812	1.1513	2.9355	2.3731	12.8940
0.20	***0.2966***	***0.6629***	3.3455	***0.9417***	***2.1729***	***1.3218***	***8.7414***
0.30	1.0300	1.5077	***2.4098***	1.8243	2.2730	1.9219	10.9668
0.40	1.8826	2.3525	2.5251	2.7070	3.2223	3.1207	15.8101
0.50	2.8873	3.1974	3.6338	3.5897	4.1715	4.3194	21.7991
0.60	3.8920	4.0422	4.7426	4.4724	5.1207	5.5182	27.7880

0.70	4.8967	4.8870	5.8513	5.3551	6.0700	6.7169	33.7770
0.80	5.9014	5.7318	6.9601	6.2377	7.0192	7.9157	39.7660
0.90	6.9061	6.5766	8.0688	7.1204	7.9685	9.1144	45.7549
0.95	7.4085	6.9990	8.6232	7.5618	8.4431	9.7138	48.7494

From Table 5, it is possible to say that the small H values are obtained for alpha-cut level is between 0.20 and 0.30 roughly. The smaller the H values, the higher the similarity of interval type parameter estimates. According to the H metric values in Table 5, someone can prefer to use fuzzy modeling approach for $\alpha \in [0.20, 0.30]$- cut level instead of using Bayesian approach for 95% confident without making any assumptions for interval type parameter estimates.

Printing Ink Data Set: Printing Ink data set is originally used in the studies of [30] to model printing machine ability. The printing ink process is designed with three inputs (speed - X_1, pressure - X_2, distance - X_3) and three replicated response measures by using 3^3 full factorial design. In this study, the same interval type Bayesian parameter estimates, which are obtained 95% confident with HPD credible interval, are used as in the study of [11]. The previously obtained Bayesian interval estimates of model parameters with midpoint, width and radius metrics are summarized in Table 6.

Table 6. *Interval type parameter estimates of Bayesian model with midpoint, width and radius metrics for Printing Ink data set*

Interval type parameters $[\boldsymbol{\beta}]$	**Interval estimates of parameters** $[\underline{\boldsymbol{\beta}}, \overline{\boldsymbol{\beta}}]$	**Midpoint** $mid(\boldsymbol{\beta})$	**Width** $w(\boldsymbol{\beta})$	**Radius** $Rad(\boldsymbol{\beta})$
$[\beta_0]$	[274.8511 354.4823]	314.6667	79.6312	39.8156
$[\beta_1]$	[128.236 225.764]	177	97.528	48.764
$[\beta_2]$	[60.662 158.1899]	109.426	97.5279	48.7639
$[\beta_3]$	[82.699 180.2269]	131.463	97.5279	48.764
$[\beta_4]$	[6.3044 125.7512]	66.0278	119.4468	59.7234
$[\beta_5]$	[15.7488 135.1956]	75.4722	119.4468	59.7234
$[\beta_6]$	[-16.1401 103.3067]	43.5833	119.4468	59.7234

In order to understand the data structure of the replicated measures, box-plots of the responses were plotted. In this study, fuzzification rule, given in Equation (5), is used for replicated response measures since the replicated measures have skewed distribution, presented with box-plots in Figure 2 for each unit.

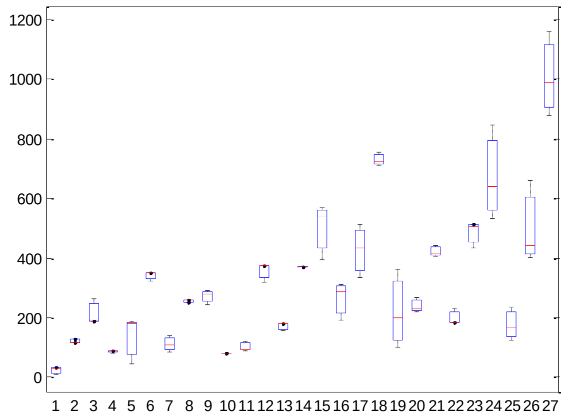

Figure 2. *Box-plots of replicated response measures for each unit of Printing Ink data set.*

The printing ink data set with fuzzy response values is given in Table 7.

Table 7. *The Printing Ink data set with fuzzy responses*

No	X_1	X_2	X_3	$\tilde{\mathbf{Y}}$
1	-1	-1	-1	(10, 24.8, 34)
2	0	-1	-1	(115, 124.3, 130)
3	1	-1	-1	(186, 233.6, 263)
4	-1	0	-1	(82, 85.7, 88)
5	0	0	-1	(44, 133, 188)
6	1	0	-1	(322, 339.3, 350)
7	-1	1	-1	(86, 120, 141)
8	0	1	-1	(251, 255.9, 259)
9	1	1	-1	(245, 272.8, 290)
10	-1	-1	0	(81, 81, 81)
11	0	-1	0	(90, 109.8, 122)
12	1	-1	0	(319, 354.2, 376)
13	-1	0	0	(154, 170.1, 180)
14	0	0	0	(372, 372, 372)
15	1	0	0	(396, 502.3, 568)
16	-1	1	0	(192, 266.2, 312)
17	0	1	0	(336, 445.4, 513)
18	1	1	0	(713, 738.3, 754)
19	-1	-1	1	(99, 262.8, 364)
20	0	-1	1	(221, 248.8, 266)
21	1	-1	1	(408, 429.6, 443)
22	-1	0	1	(182, 213.5, 233)
23	0	0	1	(434, 484.1, 515)
24	1	0	1	(535, 727.2, 846)
25	-1	1	1	(126, 194, 236)
26	0	1	1	(403, 561.8, 660)
27	1	1	1	(878, 1052.9, 1161)

The predicted fuzzy response model, with alpha-cut parameter estimates, is obtained as

$$\begin{aligned}\hat{\tilde{Y}} &= \left[269.6296+56.4211\alpha, 360.9259-34.8752\alpha\right]+\left[129.6111+49.9587\alpha, 224.3889-44.8191\alpha\right]X_1 \\ &+\left[63.9444+49.3016\alpha, 155.3889-42.1429\alpha\right]X_2+\left[85.7222+57.9027\alpha, 187.9444-44.3196\alpha\right]X_3 \\ &+\left[21.25+48.3393\alpha, 113.9167-44.3273\alpha\right]X_1X_2+\left[\ 21.9167+55.107\alpha, 129.4167-52.393\alpha\right]X_1X_3 \\ &+\left[-3.75+53.8695\alpha, 97.8333-47.7138\alpha\right]X_2X_3.\end{aligned} \tag{31}$$

The obtained interval type fuzzy parameter estimates, midpoint, width and radius metrics are presented in Table 8 for each alpha-cut levels.

Table 8. *Interval type parameter estimates of fuzzy model with midpoint, width and radius metrics for Printing Ink data set*

alpha-cut levels	**Interval type parameters** $\left[\boldsymbol{\beta}^{\alpha}\right]$	**Interval estimates of model parameters** $\left[\underline{\boldsymbol{\beta}}^{\alpha}, \overline{\boldsymbol{\beta}}^{\alpha}\right]$	**Midpoint** $mid\left(\boldsymbol{\beta}^{\alpha}\right)$	**Width** $w\left(\boldsymbol{\beta}^{\alpha}\right)$	**Radius** $Rad\left(\boldsymbol{\beta}^{\alpha}\right)$
0	$\left[\beta_0^{\alpha}\right]$	[269.6296 360.9259]	315.2778	91.2963	45.6481
	$\left[\beta_1^{\alpha}\right]$	[129.6111 224.3889]	177.0000	94.7778	47.3889
	$\left[\beta_2^{\alpha}\right]$	[63.9444 155.3889]	109.6667	91.4444	45.7222
	$\left[\beta_3^{\alpha}\right]$	[85.7222 187.9444]	136.8333	102.2222	51.1111
	$\left[\beta_4^{\alpha}\right]$	[21.25 113.9167]	67.5833	92.6667	46.3333
	$\left[\beta_5^{\alpha}\right]$	[21.9167 129.4167]	75.6667	107.5000	53.7500
	$\left[\beta_6^{\alpha}\right]$	[-3.75 97.8333]	47.0417	101.5833	50.7917
0.05	$\left[\beta_0^{\alpha}\right]$	[272.4507 359.1822]	315.8164	86.7315	43.3657
	$\left[\beta_1^{\alpha}\right]$	[132.109 222.1479]	177.1285	90.0389	45.0194
	$\left[\beta_2^{\alpha}\right]$	[66.4095 153.2817]	109.8456	86.8722	43.4361
	$\left[\beta_3^{\alpha}\right]$	[88.6174 185.7285]	137.1729	97.1111	48.5556
	$\left[\beta_4^{\alpha}\right]$	[23.667 111.7003]	67.6836	88.0333	44.0167
	$\left[\beta_5^{\alpha}\right]$	[24.672 126.797]	75.7345	102.1250	51.0625
	$\left[\beta_6^{\alpha}\right]$	[-1.0565 95.4476]	47.1956	96.5042	48.2521
0.10	$\left[\beta_0^{\alpha}\right]$	[275.2717 357.4384]	316.3551	82.1667	41.0833
	$\left[\beta_1^{\alpha}\right]$	[134.607 219.907]	177.2570	85.3000	42.6500
	$\left[\beta_2^{\alpha}\right]$	[68.8746 151.1746]	110.0246	82.3000	41.1500
	$\left[\beta_3^{\alpha}\right]$	[91.5125 183.5125]	137.5125	92.0000	46.0000
	$\left[\beta_4^{\alpha}\right]$	[26.0839 109.4839]	67.7839	83.4000	41.7000
	$\left[\beta_5^{\alpha}\right]$	[27.4274 124.1774]	75.8024	96.7500	48.3750
	$\left[\beta_6^{\alpha}\right]$	[1.6369 93.062]	47.3494	91.4250	45.7125

	$\left[\beta_0^\alpha\right]$	[280.9139 353.9509]	317.4324	73.0370	36.5185
	$\left[\beta_1^\alpha\right]$	[139.6028 215.4251]	177.5140	75.8222	37.9111
	$\left[\beta_2^\alpha\right]$	[73.8048 146.9603]	110.3825	73.1556	36.5778
0.20	$\left[\beta_3^\alpha\right]$	[97.3028 179.0805]	138.1916	81.7778	40.8889
	$\left[\beta_4^\alpha\right]$	[30.9179 105.0512]	67.9845	74.1333	37.0667
	$\left[\beta_5^\alpha\right]$	[32.9381 118.9381]	75.9381	86.0000	43.0000
	$\left[\beta_6^\alpha\right]$	[7.0239 88.2906]	47.6572	81.2667	40.6333
	$\left[\beta_0^\alpha\right]$	[286.556 350.4634]	318.5097	63.9074	31.9537
	$\left[\beta_1^\alpha\right]$	[144.5987 210.9432]	177.7709	66.3444	33.1722
	$\left[\beta_2^\alpha\right]$	[78.7349 142.746]	110.7405	64.0111	32.0056
0.30	$\left[\beta_3^\alpha\right]$	[103.093 174.6486]	138.8708	71.5556	35.7778
	$\left[\beta_4^\alpha\right]$	[35.7518 100.6185]	68.1851	64.8667	32.4333
	$\left[\beta_5^\alpha\right]$	[38.4488 113.6988]	76.0738	75.2500	37.6250
	$\left[\beta_6^\alpha\right]$	[12.4108 83.5192]	47.9650	71.1083	35.5542
	$\left[\beta_0^\alpha\right]$	[292.1981 346.9759]	319.5870	54.7778	27.3889
	$\left[\beta_1^\alpha\right]$	[149.5946 206.4612]	178.0279	56.8667	28.4333
	$\left[\beta_2^\alpha\right]$	[83.6651 138.5317]	111.0984	54.8667	27.4333
0.40	$\left[\beta_3^\alpha\right]$	[108.8833 170.2166]	139.5500	61.3333	30.6667
	$\left[\beta_4^\alpha\right]$	[40.5857 96.1857]	68.3857	55.6000	27.8000
	$\left[\beta_5^\alpha\right]$	[43.9595 108.4595]	76.2095	64.5000	32.2500
	$\left[\beta_6^\alpha\right]$	[17.7978 78.7478]	48.2728	60.9500	30.4750
	$\left[\beta_0^\alpha\right]$	[297.8402 343.4883]	320.6643	45.6481	22.8241
	$\left[\beta_1^\alpha\right]$	[154.5904 201.9793]	178.2849	47.3889	23.6944
	$\left[\beta_2^\alpha\right]$	[88.5952 134.3174]	111.4563	45.7222	22.8611
0.50	$\left[\beta_3^\alpha\right]$	[114.6736 165.7847]	140.2291	51.1111	25.5556
	$\left[\beta_4^\alpha\right]$	[45.4197 91.753]	68.5863	46.3333	23.1667
	$\left[\beta_5^\alpha\right]$	[49.4702 103.2202]	76.3452	53.7500	26.8750
	$\left[\beta_6^\alpha\right]$	[23.1847 73.9764]	48.5806	50.7917	25.3958
	$\left[\beta_0^\alpha\right]$	[303.4823 340.0008]	321.7416	36.5185	18.2593
	$\left[\beta_1^\alpha\right]$	[159.5863 197.4974]	178.5419	37.9111	18.9556
0.60	$\left[\beta_2^\alpha\right]$	[93.5254 130.1032]	111.8143	36.5778	18.2889

	$\left[\beta_3^\alpha\right]$	[120.4638 161.3527]	140.9083	40.8889	20.4444
	$\left[\beta_4^\alpha\right]$	[50.2536 87.3203]	68.7869	37.0667	18.5333
	$\left[\beta_5^\alpha\right]$	[54.9809 97.9809]	76.4809	43.0000	21.5000
	$\left[\beta_6^\alpha\right]$	[28.5717 69.205]	48.8884	40.6333	20.3167
0.70	$\left[\beta_0^\alpha\right]$	[309.1244 336.5133]	322.8189	27.3889	13.6944
	$\left[\beta_1^\alpha\right]$	[164.5822 193.0155]	178.7988	28.4333	14.2167
	$\left[\beta_2^\alpha\right]$	[98.4555 125.8889]	112.1722	27.4333	13.7167
	$\left[\beta_3^\alpha\right]$	[126.2541 156.9208]	141.5874	30.6667	15.3333
	$\left[\beta_4^\alpha\right]$	[55.0875 82.8875]	68.9875	27.8000	13.9000
	$\left[\beta_5^\alpha\right]$	[60.4916 92.7416]	76.6166	32.2500	16.1250
	$\left[\beta_6^\alpha\right]$	[33.9586 64.4337]	49.1961	30.4750	15.2375
0.80	$\left[\beta_0^\alpha\right]$	[314.7665 333.0258]	323.8961	18.2593	9.1296
	$\left[\beta_1^\alpha\right]$	[169.578 188.5336]	179.0558	18.9556	9.4778
	$\left[\beta_2^\alpha\right]$	[103.3857 121.6746]	112.5301	18.2889	9.1444
	$\left[\beta_3^\alpha\right]$	[132.0444 152.4888]	142.2666	20.4444	10.2222
	$\left[\beta_4^\alpha\right]$	[59.9215 78.4548]	69.1881	18.5333	9.2667
	$\left[\beta_5^\alpha\right]$	[66.0023 87.5023]	76.7523	21.5000	10.7500
	$\left[\beta_6^\alpha\right]$	[39.3456 59.6623]	49.5039	20.3167	10.1583
0.90	$\left[\beta_0^\alpha\right]$	[320.4086 329.5383]	324.9734	9.1296	4.5648
	$\left[\beta_1^\alpha\right]$	[174.5739 184.0517]	179.3128	9.4778	4.7389
	$\left[\beta_2^\alpha\right]$	[108.3158 117.4603]	112.8881	9.1444	4.5722
	$\left[\beta_3^\alpha\right]$	[137.8346 148.0568]	142.9457	10.2222	5.1111
	$\left[\beta_4^\alpha\right]$	[64.7554 74.0221]	69.3887	9.2667	4.6333
	$\left[\beta_5^\alpha\right]$	[71.513 82.263]	76.8880	10.7500	5.3750
	$\left[\beta_6^\alpha\right]$	[44.7326 54.8909]	49.8117	10.1583	5.0792
0.95	$\left[\beta_0^\alpha\right]$	[323.2297 327.7945]	325.5121	4.5648	2.2824
	$\left[\beta_1^\alpha\right]$	[177.0718 181.8107]	179.4413	4.7389	2.3694
	$\left[\beta_2^\alpha\right]$	[110.7809 115.3531]	113.0670	4.5722	2.2861
	$\left[\beta_3^\alpha\right]$	[140.7298 145.8409]	143.2853	5.1111	2.5556
	$\left[\beta_4^\alpha\right]$	[67.1724 71.8057]	69.4890	4.6333	2.3167
	$\left[\beta_5^\alpha\right]$	[74.2683 79.6433]	76.9558	5.3750	2.6875

$\left[\beta_6^{\alpha}\right]$	[47.426 52.5052]	49.9656	5.0792	2.5396

The Hausdorff metric calculation results are presented in Table 9.

***Table 9**. Hausdorff metric of interval type parameter estimates Printing Ink data set*

alpha-cut levels	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
0	6.4436	***1.3751***	***3.2824***	7.7175	***14.9456***	***6.1679***	***12.3901***	***52.3222***
0.05	4.6999	3.8730	5.7475	***5.9184***	17.3626	8.9232	15.0836	61.6082
0.10	***2.9561***	6.3710	8.2126	8.8135	19.7795	11.6786	17.7770	75.5883
0.20	6.0628	11.3668	13.1428	14.6038	24.6135	17.1893	23.1640	110.1430
0.30	11.7049	16.3627	18.0729	20.3940	29.4474	22.7000	28.5510	147.2329
0.40	17.3470	21.3586	23.0031	26.1843	34.2813	28.2107	33.9379	184.3229
0.50	22.9891	26.3544	27.9332	31.9746	39.1153	33.7214	39.3248	221.4128
0.60	28.6312	31.3503	32.8634	37.7648	43.9492	39.2321	44.7118	258.5028
0.70	34.2733	36.3462	37.7935	43.5551	48.7831	44.7428	50.0987	295.5927
0.80	39.9154	41.3420	42.7237	49.3454	53.6171	50.2535	55.4857	332.6828
0.90	45.5575	46.3379	47.6538	55.1356	58.4510	55.7642	60.8727	369.7727
0.95	48.3786	48.8358	50.1189	58.0308	60.8680	58.5195	63.5661	388.3177

It can be easily seen from Table 9 that the similarity of fuzzy and Bayesian model parameter estimates are high at $\alpha \in [0, 0.10]$- cut levels according to the H metric. From Table 9, it is possible to prefer fuzzy modeling approach for $\alpha \in [0, 0.10]$- cut levels alternatively modeling with Bayesian approach for interval type parameter estimates.

5. CONCLUSION

This study presents the alternatively preferability of possibilistic modeling approach to probabilistic modeling approach for RRM data sets. For this purpose, fuzzy and Bayesian linear modeling approaches are applied to RRM data sets. After obtaining model parameter estimates, it is aimed to compare the uncertainty of model parameters. Therefore, it is needed to define the estimated values of parameters on the same domain. This is achieved with interval type presentation of parameter estimates through alpha-cut level intervals and HPD credible intervals for fuzzy and Bayesian approaches, respectively. $mid([\boldsymbol{\beta}])$, $w([\boldsymbol{\beta}])$, $rad([\boldsymbol{\beta}])$ and H metrics of intervals are used as comparison metric tools for both deterministic closed intervals of fuzzy and Bayesian models. It should be noted here that the comparison is achieved between Bayesian interval type estimates with fuzzy interval type estimates for each defined alpha-cut levels, $\alpha \in [0, 1]$. It is seen from the results that the H metric values of interval type estimates are roughly similar for $\alpha \in [0.20, 0.30]$-cut and $\alpha \in [0, 0.10]$-cut levels of fuzzy approach with 95 % confident of Bayesian approach for Roman Catapult data set and Printing Ink data set, respectively. Thus, someone can prefer to use fuzzy linear modeling approach instead of Bayesian to model the RRM data set and to analyse the uncertainty of unknown model parameters with great flexibility and without any strict modeling assumptions. It is possible to say that the fuzzy modeling could be directly applied to RRM data set with $\alpha \in [0, 0.30]$-cut levels without making comparison analysis with Bayesian approach. For future work, different types of fuzzy numbers and different confidence levels are planned to apply to the RRM data set for fuzzy modeling and Bayesian modeling, respectively.

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

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