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GORENSTEIN $\pi[T]$-PROJECTIVITY WITH RESPECT TO A TILTING MODULE

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Abstract. Let T be a tilting module. In this paper, Gorenstein $\pi[T]$-projective modules are introduced and some of their basic properties are studied. Moreover, some characterizations of rings over which all modules are Gorenstein $\pi[T]$-projective are given. For instance, on the T-cocoherent rings, it is proved that the Gorenstein $\pi[T]$-projectivity of all R-modules is equivalent to the $\pi[T]$-projectivity of $\sigma[T]$-injective as a module.

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1. Introduction

Throughout this paper, R is an associative ring with non-zero identity, all modules are unitary left R-modules. First we recall some known notions and facts needed in the sequel. Let R be a ring and T an R-module. Then

(1) We denote by $ProdT$ (resp. $F.ProdT$), the class of modules isomorphic to direct summands of direct product of copies (resp. finitely many copies) of T.
(2) We denote by $AddT$ (resp. $F.AddT$), the class of modules isomorphic to direct summands of direct sum of copies (resp. finitely many copies) of T.
(3) Following [3], a module T is called tilting (1-tilting) if it satisfies the following conditions:
(a) $\mathrm{pd}(T) \leq 1$, where $pd(T)$ denotes the *projective dimension of* T.
(b) $\mathrm{Ext}^i(T, T^{(\lambda)}) = 0$, for each $i > 0$ and for every cardinal λ.
(c) There exists the exact sequence $0 \to R \to T_0 \to T_1 \to 0$, where $T_0, T_1 \in \mathrm{Add}T$.
(4) By $Copres^nT$ (resp. $F.Copres^nT$) and $Copres^\infty T$ (resp. $F.Copres^\infty T$), we denote the set of all modules M such that there exists exact sequences

$$0 \longrightarrow M \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n$$

and

$$0 \longrightarrow M \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1} \longrightarrow T_n \longrightarrow \cdots,$$

respectively, where $T_i \in \text{Prod}T$ (resp. $T_i \in \text{F.Prod}T$), for every $i \geq 0$.

(5) A module M is said to be *cogenerated*, by T, denoted by $M \in CogenT$, (resp. *generated*, denoted $M \in GenT$) by T if there exists an exact sequence $0 \to M \to T^n$ (resp. $T^{(n)} \to M \to 0$), for some positive integer n.

(6) Let $\mathcal{C}$ be a class of modules and M be a module. A *right (resp. left) $\mathcal{C}$-resolution* of M is a long exact sequence $0 \to M \to C_0 \to C_1 \to \cdots$ (resp. $\cdots \to C_1 \to C_0 \to M \to 0$), where $C_i \in \mathcal{C}$, for all $i \geq 0$. It is said that a module M has right $\mathcal{C}$*-dimension* n (briefly, $\mathcal{C}.dim(M) = n$) if n is the least non-negative integer such that there exists a long exact sequence

$$0 \longrightarrow M \longrightarrow C_0 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow C_{n-1} \longrightarrow C_n \longrightarrow 0$$

with $C_i \in \mathcal{C}$, for each $i \geq 0$. In particular, the Prod$T$-dimension of M is called *T-injective dimension* of M and is denoted by $T.i.dim(M)$. Note that for any tilting module M, if $M \in \text{Cogen}T$, then [6, Proposition 2.1] implies that $\text{Cogen}T = \text{Copres}^{\infty}T$. This shows that any module cogenerated by T has an Prod$T$-resolution. The Prod$T$-resolutions and the relative homological dimension were studied by Nikmehr and Shaveisi in [6].

(7) For any homomorphism f, we denote by $kerf$ and imf, the kernel and image of f, respectively. Let A and $M \in \text{Cogen}T$ be two modules. We define the functor

$$\mathcal{E}_T^n(A, M) := \frac{\ker\delta_*^n}{\text{im}\delta_*^{n-1}},$$

where

$$0 \longrightarrow M \xrightarrow{\delta_0} T_0 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_n} T_n \longrightarrow \cdots$$

Prod$T$-resolution of M and $\delta_*^n = \text{Hom}(id_B, \delta_n)$, for every $i \geq 0$. See [6,9] for more details.

(8) Let $M \in \text{Cogen}T$ and N be two modules. A similar proof to that of [7, Lemma 2.11] shows that $\mathcal{E}_T^0(N, M) \cong \text{Hom}(N, M)$. Moreover, $\mathcal{E}_T^1(-, M) = 0$ implies that $M \in \text{Prod}T$, and if $M \in \text{Gen}T$, then $\mathcal{E}_T^1(M, -) = 0$ implies that $M \in \text{Add}T$. It is clear that $\text{T.i.dim}(M) = n$ if and only if n is the least non-negative integer such that $\mathcal{E}_T^{n+1}(A, M) = 0$, for any module A, see [6, Remark 2.2] for more details. So, $\text{T.i.dim}(M) = n$ if and only if $\mathcal{E}_T^{n+i}(A, M) = 0$ for every module A and every $i \geq 1$. A module with zero T-injective dimension (resp. T-projective dimension) is called *T-injective*

(*resp.* T*-projective*). A similar proof to that of [7, Proposition 2.3] shows that the definition of $\mathcal{E}_T^n(C, M)$ is independent from the choice of Prod$T$-resolutions. For unexplained concepts and notations, we refer the reader to [2,6,8].

(9) For a module T, we denote by $\pi[T]$, the full subcategory of modules whose objects are of the form $\frac{B}{A} \leq \frac{T^I}{A}$, for some cardinal I and some modules $A \leq B \leq T^I$. Also, the full subcategory $\sigma[T]$ of modules subgenerated by a given module T (see [10]).

(10) G is called Gorenstein $\sigma[T]$-injective if there exists an exact sequence of $\sigma[T]$-injective modules

$$\mathbf{A} = \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$$

with $G = \ker(A^0 \to A^1)$ such that $\mathrm{Hom}(U, -)$ leaves this sequence exact whenever $U \in \mathrm{Pres}^1 T$ with T.p.dim$(U) < \infty$ (see [9]).

(11) M is said to be *finitely cogenerated* [2] if for every family $\{V_k\}_J$ of submodules of M with $\bigcap_J V_k = 0$, there is a finite subset $I \subset J$ with $\bigcap_I V_k = 0$.

(12) M is said to be *finitely copresented* if there is an exact sequence of R-modules $0 \to M \to E^0 \to E^1$, where each E^i is a finitely cogenerated injective module, see [1,11,12].

Let T be a tilting module. In this paper, we introduce the $\pi[T]$*-projective* modules, *the* $\pi[T]$*-projective dimension* and *Gorenstein* $\pi[T]$*-projective* modules.

Let $M \in \mathrm{Gen}T$. Then, M is called $\pi[T]$-projective if the functor $\mathcal{E}_T^1(M, -)$ vanishes on $\pi[T]$. Also, the $\pi[T]$-projective dimension of M is defined to be

$$\pi[T].pd(M) = \inf\{n : \ \mathcal{E}_T^{n+1}(M, N) = 0 \text{ for every } N \in \pi[T]\}.$$

We define a module G to be Gorenstein $\pi[T]$-projective (GT-projective for short), if there exists an exact sequence of $\pi[T]$-projective modules

$$\mathbf{B} = \cdots \longrightarrow B_1 \longrightarrow B_0 \longrightarrow B^0 \longrightarrow B^1 \longrightarrow \cdots$$

with $G = \ker(B^0 \to B^1)$ such that $\mathrm{Hom}(-, U)$ leaves this sequence exact whenever $U \in \mathrm{F.Copres}^1 T$ with T.i.dim$(U) < \infty$. In this paper, the GT-projective dimension of a module G is denoted by GT-$pd(G)$.

In Section 2, we study some basic properties of the Gorenstein $\pi[T]$-projective modules. Recall that a ring R is said to be *cocoherent* if every finitely cogenerated module is finitely copresented. So, R is a cocoherent ring if and only if $\mathrm{Copres}^0 R = \mathrm{Copres}^1 R$. For more information about the cocoherent rings, we refer the reader to [5]. As a cogeneralization of this concept, we call a ring R to be T-*cocoherent* if $\text{F.Copres}^0T = \text{F.Copres}^1T$.

Section 3 is devoted to some characterizations of T-cocoherent rings over which all modules are Gorenstein $\pi[T]$-projective. For instance, it is proved that every module is Gorenstein $\pi[T]$-projective if and only if every T-injective module is $\pi[T]$-projective if and only if every $\sigma[T]$-injective module is Gorenstein $\pi[T]$-projective. Finally, we give a sufficient condition under which every Gorenstein $\pi[T]$-projective module is $\pi[T]$-projective.

2. Gorenstein $\pi[T]$-projectivity

We start with the following definition.

Definition 2.1. Let T be a tilting module. Then

(1) M is called $\pi[T]$-projective if $\mathcal{E}^1_T(M,N)=0$, for every $N\in\pi[T]$.
(2) Let $G\in \text{Gen}T$. Then, G is called Gorenstein $\pi[T]$-projective if there exists an exact sequence of $\pi[T]$-projective modules

$$\mathbf{B}=\cdots\longrightarrow B_1\longrightarrow B_0\longrightarrow B^0\longrightarrow B^1\longrightarrow\cdots$$

with $G=\ker(B^0\to B^1)$ such that $\text{Hom}(-,U)$ leaves this sequence exact whenever $U\in\text{F.Copres}^1T$ with $\text{T.i.dim}(U)<\infty$.

Remark 2.2. Let T be a tilting module. Then

(1) $\mathcal{E}^1_T(N,M)=0$ for any $\pi[T]$-projective module N and any $M\in\text{Copres}^0T$.
(2) If $A\in\text{Add}T$, then A is $\pi[T]$-projective.

Lemma 2.3. *Let* $0\to A\overset{f}{\to} B\overset{g}{\to} C\to 0$ *be an exact sequence. Then*

(1) *If A is T-injective and $A,B,C\in\text{Cogen}T$, then $B=A\oplus C$.*
(2) *If $A\in\text{F.Copres}^nT$ and $C\in\text{F.Copres}^nT$, then $B\in\text{F.Copres}^nT$.*
(3) *If $C\in\text{F.Copres}^nT$ and $B\in\text{F.Copres}^{n+1}T$, then $A\in\text{F.Copres}^{n+1}T$.*
(4) *If $B\in\text{F.Copres}^nT$ and $A\in\text{F.Copres}^{n+1}T$, then $C\in\text{F.Copres}^nT$.*

Proof. (1) If A is T-injective and $A,B,C\in\text{Cogen}T$, then we deduce that the sequence

$$0\longrightarrow\text{Hom}(C,A)\overset{g^*}{\longrightarrow}\text{Hom}(B,A)\overset{f^*}{\longrightarrow}\text{Hom}(A,A)\longrightarrow\mathcal{E}^1_T(C,A)=0$$

is exact. So, there exists $h:B\to A$ such that $hf=1_A$.

(2) We prove the assertion by induction on n. If $n = 0$, then the commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
o & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow h_0^{'} & & \downarrow h_0 & & \downarrow h_0^{''} & & \\
0 & \longrightarrow & T_0^{'} & \xrightarrow{i_0} & T_0^{'} \oplus T_0^{''} & \xrightarrow{\pi_0} & T_0^{''} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

exists, where $T_0', T_0'' \in$ F.Prod$T$, i_0 is the inclusion map, π_0 is a canonical epimorphism and $h_0 = i_0 h_0'$ is endomorphism, by Five Lemma. Let $K_1' = \operatorname{coker}(h_0')$, $K_1 = \operatorname{coker}(h_0)$ and $K_1'' = \operatorname{coker}(h_0'')$. It is clear that $(T_0^{'} \oplus T^{''}) \in$ F.Prod$T$ and $K_1', K_1'' \in$ F.Copres^{n-1}T; so, the induction implies that $K_1 \in$ F.Copres^{n-1}T. Hence $B \in$ F.Copres^{n}T.

(3) Let $B \in$ F.Pres^{n+1}T and $C \in$ F.Pres^{n}T, then the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A & = & A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & T_0 & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where $T_0 \in$ F.Prod$T$ and $L \in$ F.Copres^{n}T. By (2), $D \in$ F.Copres^{n}T. So, we deduce that $A \in$ F.Copres^{n+1}T.

(4) Let $A \in$ F.Pres^{n+1}T and $B \in$ F.Pres^{n}T, then the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & A & \longrightarrow & T_0^{'} & \longrightarrow & L^{'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & B & \longrightarrow & T_0 & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

where $T_0, T_0^{'} \in \text{F.Prod}T$ and $L \in \text{F.Copres}^{n-1}T$. Since $T_0^{'}$ is T-injective, we have that $T_0 = T_0^{'} \oplus D$ By (1), and $D \in \text{Cogen}T$. Thus for any $N \in \text{Cogen}T$, we have

$$\mathcal{E}_T^1(T_0, N) = \mathcal{E}_T^1(T_0^{'} \oplus D, N) = \mathcal{E}_T^1(T_0^{'}, N) \oplus \mathcal{E}_T^1(D, N) = 0.$$

Hence $D \in \text{F.Prod}T$. On the other hand, $L \in \text{F.Copres}^{n-1}T$. Therefore, we conclude that $C \in \text{F.Copres}^n T$. □

In the following theorem, we show that in the case of T-cocoherent rings, the existence of $\pi[T]$-projective complex of a module is sufficient to be Gorenstein $\pi[T]$-projective.

Theorem 2.4. *Let R be a T-cocoherent ring and $G \in \text{Gen}T$ be a module. Then G is Gorenstein $\pi[T]$-projective if and only if there is an exact sequence*

$$\mathbf{B} = \cdots \longrightarrow B_1 \longrightarrow B_0 \longrightarrow B^0 \longrightarrow B^1 \longrightarrow \cdots$$

of $\pi[T]$-projective modules such that $G = \ker(B^0 \to B^1)$.

Proof. $(\Rightarrow)$: This is a direct consequence of definition.

$(\Leftarrow)$: By definition, it suffices to show that $\text{Hom}(\mathbf{B}, U)$ is exact for every module $U \in \text{F.Copres}^1 T$ with $\text{T.i.dim}(U) = m < \infty$. To prove this, we use the induction on m. The case $m = 0$ is clear. Assume that $m \geq 1$. Since $U \in \text{F.Copres}^1 T$, there exists an exact sequence $0 \to U \to T_0 \to I \to 0$ with $T_0 \in \text{F.Prod}T \subseteq \text{F.Copres}^0 T$. Now, from the T-cocoherence of R and Lemma 2.3, we deduce that $I, T_0 \in \text{F.Copres}^1 T$. Also, $\text{T.i.dim}(I) \leq m-1$ and $\text{T.i.dim}(T_0) = 0$. Thus by Remark 2.2, the following short exact sequence of complexes exists:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(B^1,U) & \longrightarrow & \mathrm{Hom}(B^1,T_0) & \longrightarrow & \mathrm{Hom}(B^1,I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(B^0,U) & \longrightarrow & \mathrm{Hom}(B^0,T_0) & \longrightarrow & \mathrm{Hom}(B^0,I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(B_0,U) & \longrightarrow & \mathrm{Hom}(B_0,T_0) & \longrightarrow & \mathrm{Hom}(B_0,I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(B_1,U) & \longrightarrow & \mathrm{Hom}(B_1,T_0) & \longrightarrow & \mathrm{Hom}(B_1,I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathbf{B},U) & \longrightarrow & \mathrm{Hom}(\mathbf{B},T_0) & \longrightarrow & \mathrm{Hom}(\mathbf{B},I) & \longrightarrow & 0.
\end{array}$$

By induction, $\mathrm{Hom}(\mathbf{B}, T_0)$ and $\mathrm{Hom}(\mathbf{B}, I)$ are exact, hence $\mathrm{Hom}(\mathbf{B}, U)$ is exact by [8, Theorem 6.10]. Therefore, G is Gorenstein $\pi[T]$-projective. $\square$

It is worthy to mention that the notion of T-injectivity (T-projectivity) is different from the notion of an M-injective (M-projective) module in [2].

Corollary 2.5. *Let R be a T-cocoherent ring and $G \in \mathrm{Gen}T$ be a module. Then the following assertions are equivalent:*

(1) *G is Gorenstein $\pi[T]$-projective;*
(2) *There is an exact sequence $0 \to G \to B^0 \to B^1 \to \cdots$ of modules, where every B^i is $\pi[T]$-projective;*
(3) *There is a short exact sequence $0 \to G \to M \to I \to 0$ of modules, where M is $\pi[T]$-projective and I is Gorenstein $\pi[T]$-projective.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follow from definition.

(2) $\Rightarrow$ (1) For module $G \in \mathrm{Gen}T$, [6, Proposition 2.1] implies that $\mathrm{Gen}T = \mathrm{Pres}^{\infty}T$. So, there is an exact sequence

$$\cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow G \longrightarrow 0$$

where any T_i is $\pi[T]$-projective by Remark 2.2. Thus, the exact sequence

$$\cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow B^0 \longrightarrow B^1 \longrightarrow \cdots$$

of $\pi[T]$-projective modules exists, where $G = \ker(B^0 \to B^1)$. Therefore, G is Gorenstein $\pi[T]$-projective, by Theorem 2.4.

(3) $\Rightarrow$ (2) Assume that the exact sequence

$$0 \longrightarrow G \longrightarrow M \longrightarrow I \longrightarrow 0 \quad (1)$$

exists, where M is $\pi[T]$-projective and I is Gorenstein $\pi[T]$-projective. Since I is Gorenstein $\pi[T]$-projective, there is an exact sequence

$$0 \to I \to C^0 \to C^1 \to \cdots \quad (2)$$

where every C^i is $\pi[T]$-projective. Assembling the sequences (1) and (2), we get the exact sequence

$$0 \to G \to M \to C^0 \to C^1 \to \cdots,$$

where M and every C^i are $\pi[T]$-projective, as desired. □

Proposition 2.6. *For any module $G \in \mathrm{Gen}T$, the following statements hold.*

(1) *If G is Gorenstein $\pi[T]$-projective, then $\mathcal{E}^i_T(G,U) = 0$ for all $i > 0$ and every module $U \in \mathrm{F.Copres}^1T$ with $\mathrm{T.i.dim}(U) < \infty$.*

(2) *If $0 \to N \to G_{n-1} \to \cdots \to G_0 \to G \to 0$ is an exact sequence of modules where every G_i is a Gorenstein $\pi[T]$-projective and $G_i \in \mathrm{Gen}T$, then $\mathcal{E}^i_T(N,U) = \mathcal{E}^{n+i}_T(G,U)$ for any $i > 0$ and any module $U \in \mathrm{F.Copres}^1T$ with $\mathrm{T.i.dim}(U) < \infty$.*

Proof. (1) Let G be a Gorenstein $\pi[T]$-projective module, and $\mathrm{T.i.dim}(U) = m < \infty$. Then by hypothesis, the following $\pi[T]$-projective resolution of G exists:

$$0 \to G \to B^0 \to \cdots \to B^{m-1} \to N \to 0.$$

By Remark 2.2, $\mathcal{E}^i_T(B_j,U) = 0$ for every $i > 0$ and every $0 \le j \le m-1$. Since $\mathrm{T.i.dim}(U) = m$, we deduce that $\mathcal{E}^i_T(G,U) \cong \mathcal{E}^{m+i}_T(N,U) = 0$.

(2) Setting $G_n = N$ and $K_j = \ker(G_j \to G_{j-1})$, for every $0 \le j \le n$, the short exact sequence $0 \to K_j \to G_j \to K_{j-1} \to 0$ exists. Thus by (1), the induced exact sequences

$$0 = \mathcal{E}^r_T(G_j,U) \to \mathcal{E}^r_T(K_j,U) \to \mathcal{E}^{r+1}_T(K_{j-1},U) \to \mathcal{E}^{r+1}_T(G_j,U) = 0$$

exists and so $\mathcal{E}^r_T(K_j,U) \cong \mathcal{E}^{r+1}_T(K_{j-1},U)$, for every $r \ge 0$. Since $K_{n-1} = N$, we have

$$\mathcal{E}^{n+i}_T(G,U) \cong \mathcal{E}^{n+i-1}_T(K_0,U) \cong \cdots \cong \mathcal{E}^i_T(N,U),$$

as desired. □

Next, we study the Gorenstein $\pi[T]$-projectivity of modules on T-cocoherent rings, in short exact sequences.

Proposition 2.7. *Let R be T-cocoherent and consider the exact sequence $0 \to N \to B \to G \to 0$, where B is $\pi[T]$-projective. Then* $\text{GT-pd}(G) \leq \text{GT-pd}(N) + 1$. *In particular, if G is Gorenstein $\pi[T]$-projective, so is N.*

Proof. We shall show that $\text{GT-pd}(G) \leq \text{GT-pd}(N) + 1$. In fact, we may assume that $\text{GT-pd}(N) = n < \infty$. Then, by definition, N admits a Gorenstein $\pi[T]$-projective resolution:

$$0 \to B_n \to B_{n-1} \to \cdots \to B_0 \to N \to 0.$$

Assembling this sequence and the short exact sequence $0 \to N \to B \to G \to 0$, the following commutative diagram is obtained:

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & B_n & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & B_0 & \longrightarrow & B & \longrightarrow & G & \longrightarrow & 0 \\
 & & & & & & & & \downarrow & & \uparrow & & & & \\
 & & & & & & & & N & = & N & & & & \\
 & & & & & & & & \downarrow & & \uparrow & & & & \\
 & & & & & & & & 0 & & 0 & & & &
\end{array}$$

which shows that $\text{GT-pd}(G) \leq n + 1$. The particular case follows from Corollary 2.5. □

Proposition 2.8. *Let R be a T-cocoherent ring and $0 \to N \to G \to B \to 0$ be an exact sequence, where $N, B \in \text{Gen}T$. If N is Gorenstein $\pi[T]$-projective and B is $\pi[T]$-projective, then G is Gorenstein $\pi[T]$-projective.*

Proof. Since N is Gorenstein $\pi[T]$-projective, by Corollary 2.5, there exists an exact sequence of $0 \to N \to B^{'} \to K \to 0$, where $B^{'}$ is $\pi[T]$-projective and K is Gorenstein $\pi[T]$-projective. Now, we consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B' & \longrightarrow & D & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The exactness of the middle horizontal sequence with B and $B^{'}$, $\pi[T]$-projective, implies that D is $\pi[T]$-projective. Hence from the middle vertical sequence and Corollary 2.5, we deduce that G is Gorenstein $\pi[T]$-projective. □

3. Gorensetein $\pi[T]$-projective modules on T-cocoherent rings

This section is devoted to T-cocoherent rings over which every module is Gorenstein $\pi[T]$-projective.

Lemma 3.1. *Let T be a tilting module and $G \in GenT$. Then, $G \in CogenT$.*

Proof. Let $G \in GenT$. Then, the short exact sequence $0 \to K \to T^{(I)} \to G \to 0$ exists. We have $K \subseteq T^{(I)} \subseteq T^I$. So, $K \in \mathrm{Cogen}T$. By [6, Proposition 2.1], $\mathrm{Cogen}T = \mathrm{Copres}^{\infty}T$, since T is tilting. Thus by Lemma 2.3, $G \in \mathrm{Copres}^m T$, and hence $G \in CogenT$. □

Proposition 3.2. *Let R be a ring. The following assertions are equivalent:*

(1) *Every module belong* $\mathrm{Gen}T$*, is Gorenstein $\pi[T]$-projective;*

(2) *The ring satisfies the following two conditions:*

(i) *Every T-injective module is $\pi[T]$-projective.*

(ii) *$\mathcal{E}_T^1(N, U) = 0$ for any $N \in \mathrm{Gen}T$ and any $U \in \mathrm{F.Copres}^n T$ with* $\mathrm{T.i.dim}(U) < \infty$*.*

Proof. (1) $\Rightarrow$ (2) The condition (i) follows from this fact that every T-injective module M is Gorenstein $\pi[T]$-projective. So, the following $\pi[T]$-projective resolution of M exists:

$$0 \to M \to B^0 \to B^1 \to \cdots .$$

Since M is T-injective, M is $\pi[T]$-projective as a direct summand of B^0. Also, Proposition 2.6(1) and (1) imply that $\mathcal{E}_T^1(N, U) = 0$ for any module $N \in \mathrm{Gen}T$ and any module $U \in \mathrm{F.Copres}^1 T$ with finite T-injective dimension. So the condition (ii) follows.

(2) $\Rightarrow$ (1) Let $G \in \mathrm{Gen}T$. Then by Lemma 3.1, $G \in CogenT$. So, a $\mathrm{Add}T$-resolution $\cdots \to T_1 \to T_0 \to G \to 0$ and a $\mathrm{Prod}T$- resolution $0 \to G \to T^0 \to T^1 \to \cdots$ of G exists. By Remark 2.2, any T_i is $\pi[T]$-projective and any T^i is T-injective. Hence by (2), every T^i is $\pi[T]$-projective. Assembling these resolutions, we get the following exact sequence of $\pi[T]$-projective modules:

$$\mathbf{B} = \cdots \to T_1 \to T_0 \to T^0 \to T^1 \to \cdots ,$$

where $G = \ker(T^0 \to T^1)$. So by (2)(ii), $\mathrm{Hom}(\mathbf{B}, U)$ is exact for any module $U \in \mathrm{F.Copres}^1 T$ with finite T-injective dimension. Hence G is Gorenstein $\pi[T]$-projective. □

The next theorem shows that if R is a T-cocoherent ring and every $\sigma[T]$-injective module is Gorenstein $\pi[T]$-projective, then every module is Gorenstein $\pi[T]$-projective.

Theorem 3.3. *Let R be a T-cocoherent ring. Then the following are equivalent:*

(1) *Every module is Gorenstein $\pi[T]$-projective;*
(2) *Every Gorenstein $\sigma[T]$-injective module is Gorenstein $\pi[T]$-projective;*
(3) *Every $\sigma[T]$-injective module is Gorenstein $\pi[T]$-projective;*
(4) *Every T-injective module is $\pi[T]$-projective.*

Proof. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) Let G be a $\sigma[T]$-injective module. Every $\sigma[T]$-injective module is Gorenstein $\sigma[T]$-injective (see,[9]). Since G is Gorenstein $\sigma[T]$-injective, we deduce that G is Gorenstein $\pi[T]$-projective by hypothesis.

(3) $\Rightarrow$ (4) Let G be a T-injective module. Then G is $\sigma[T]$-injective, and so G is Gorenstein $\pi[T]$-projective by hypothesis. By Corollary 2.5, there exists an exact sequence $0 \to G \to B \to N \to 0$, where B is $\pi[T]$-projective. Thus the sequence splits. Hence G is $\pi[T]$-projective as a direct summand of B.

(4) $\Rightarrow$ (1) Let $G \in \mathrm{Gen}T$. Then by Lemma 3.1, there is an exact sequence

$$0 \longrightarrow G \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots,$$

where any T^i is T-injective. Then by (5), every T^i is $\pi[T]$-projective. Hence Corollary 2.5 completes the proof. □

We denote the right $\pi[T]$-projective dimension of any module M by $\pi[T].pd(M)$, and $\pi[T].pd(M) = \inf\{n : \mathcal{E}_T^{n+1}(M,N) = 0 \text{ for every } N \in \pi[T]\}$.

Example 3.4. Let R be a 1-Gorenstein ring and $0 \to R \to E^0 \to E^1 \to 0$ be the minimal injective resolution of R. Then, $\pi[T].pd(E^0) = \pi[T].pd(E^1) = 0$. Since by [4], $T = E_0 \oplus E_1$ is a tilting module. So, any E^i is $\pi[T]$-projective and hence, any E^i is Gorenstein $\pi[T]$-projective for $i = 0, 1$.

Definition 3.5. We define the *global* $\pi[T]$*-projective dimension* of any ring R to be:

$$gl.\pi[T].pd(R) = \sup\{\pi[T].pd(M) |\ M \text{ is a module}\}.$$

Clearly, every $\pi[T]$-projective module is Gorenstein $\pi[T]$-projective. But the converse is not true in general. We finish this paper with the following theorem which determines a sufficient condition under which the converse holds.

Theorem 3.6. *If $gl.\pi[T].pd(R) < \infty$, then every Gorenstein $\pi[T]$-projective module is $\pi[T]$-projective.*

Proof. Suppose that $gl.\pi[T].\mathrm{pd}(R) = m < \infty$, and G is a Gorenstein $\pi[T]$-projective module. If $m = 0$, then $\mathcal{E}^1_T(M, N) = 0$ for any $N \in \pi[T]$, and hence G is $\pi[T]$-projective. For $m \geq 1$, since G is Gorenstein $\pi[T]$-projective, there exists an exact sequence $0 \to G \to B^0 \to B^1 \to \cdots$ with each B^i is $\pi[T]$-projective. Let $L = \mathrm{coker}(B^{m-2} \to B^{m-1})$. Then

$$0 \longrightarrow G \longrightarrow B^0 \longrightarrow B^1 \longrightarrow \cdots \longrightarrow B^{m-2} \longrightarrow B^{m-1} \longrightarrow L \longrightarrow 0$$

is exact, and hence G is $\pi[T]$-projective since $\pi[T].\mathrm{pd}(L) \leq m$. □

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