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A CONDITION FOR CYCLIC CHIEF FACTORS OF FINITE GROUPS

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Abstract. In this paper, we find a condition under which every chief factor of G below a normal subgroup H of G is cyclic by using the τ-supplemented subgroups. Some recent results are generalized.

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1. Introduction

In this paper, G always denotes a finite group and p is a prime. A subgroup H of G is called S-quasinormal (or S-permutable) [6] in G if it permutes with every Sylow subgroup of G; H is S-supplemented [13] in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are S-quasinormal in G. In [14], Skiba proved the following significant result:

Theorem 1.1. *Let H be a normal subgroup of a group G. Suppose that for every non-cyclic Sylow subgroup P of H, all maximal subgroups of P or all cyclic subgroups of P of prime order and order 4 are S-supplemented in G. Then every G-chief factor below H is cyclic.*

A subgroup H of G is SS-quasinormal [8] in G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B; H is SS-supplemented [17] in G if there is a subgroup T of G such that $G = HT$ and $H \cap T$ is SS-quasinormal in G. In [17], Yan et al. strengthened Theorem 1.1 as follows:

Theorem 1.2. *Let H be a normal subgroup of a group G. Suppose that for every non-cyclic Sylow subgroup P of H, all maximal subgroups of P or all cyclic subgroups of P of prime order and order 4 are SS-supplemented in G. Then every G-chief factor below H is cyclic.*

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A subgroup H of G is said to be s-semipermutable [15] in G if H permutes with all Sylow q-subgroups of G for the primes q not dividing $|H|$; H is τ-quasinormal [11] in G if H permutes with every Sylow subgroup Q of G such that $(|H|,|Q|)=1$ and $(|H|,|Q^G|)\neq 1$; H is τ-supplemented [10] in G if G has a subgroup T such that $G=HT$ and $H\cap T\leq H_{\tau G}$, where $H_{\tau G}$ is the subgroup generated by all those subgroups of H which are τ-quasinormal in G. It is easy to see that if a subgroup H of G is SS-supplemented in G, then H is τ-supplemented in G. However, the converse does not hold in general. In this paper, we get the following result which is an extension of Theorems 1.1 and 1.2.

Theorem 1.3. *Let H be a normal subgroup of G. Suppose that there exists a normal subgroup X of G such that $F^*(H)\leq X\leq H$ and for every non-cyclic Sylow subgroup P of X, either all maximal subgroups of P lacking supersolvable supplements in G or all cyclic subgroups of P of prime order and order 4 without supersolvable supplements in G are τ-supplemented in G. Then every G-chief factor below H is cyclic.*

Here $F^*(G)$ is the generalized Fitting subgroup of G, i.e., the product of all normal quasinilpotent subgroups of G (see [4, X, 13]).

2. Preliminaries

Lemma 2.1 ([10, Lemma 2.2])**.** *Let H be a τ-supplemented subgroup of a group G.*

(1) *If $H\leq L\leq G$, then H is τ-supplemented in L.*
(2) *If $N\trianglelefteq G$, $N\leq H\leq G$ and H is a p-group for some prime p, then H/N is τ-supplemented in G/N.*
(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is τ-supplemented in G/N.*

Lemma 2.2 ([7, Lemma 2.12])**.** *Let P be a Sylow p-subgroup of a group G, where p is a prime with $(|G|,p-1)=1$. If every maximal subgroup of P has a p-nilpotent supplement in G, then G is p-nilpotent.*

Lemma 2.3 ([2, A, 1.2])**.** *Let U,V, and W be subgroups of a group G. Then the following statements are equivalent:*

(1) $U\cap VW=(U\cap V)(U\cap W)$;
(2) $UV\cap UW=U(V\cap W)$.

Lemma 2.4. *Let H be a p-subgroup of G and N be a normal subgroup of G. If H is τ-quasinormal in G, then $H \cap N$ is also τ-quasinormal in G.*

Proof. Let Q be a Sylow q-subgroup of G and $(p, |Q^G|) \neq 1$, where q is a prime distinct from p. Since $|N|_q = |HN|_q$ and $N \cap Q$ is a Sylow q-subgroup of N, we have that $N \cap Q = HN \cap Q$, i.e., $(H \cap Q)(N \cap Q) = HN \cap Q$. By Lemma 2.3, we have $HQ \cap NQ = (H \cap N)Q$. Thus $H \cap N$ is τ-quasinormal in G. □

Lemma 2.5. *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.*

(1) *If G has cyclic Sylow p-subgroups, then G is p-nilpotent.*
(2) *If G is p-supersolvable , then G is p-nilpotent.*
(3) *If N is a normal subgroup of G with order p and G/N is p-supersolvable, then G is p-nilpotent.*

Proof. (1) and (2) are [9, Lemma 2.6]. (3) follows directly from (2). □

Lemma 2.6 ([11, Lemmas 2.2])**.** *If a subgroup H of G is τ-quasinormal in G and $H \leq O_p(G)$ for some prime p, then H is S-quasinormal in G.*

Lemma 2.7 ([14, Corollary 1.1])**.** *Let H be a normal subgroup of a group G. If every G-chief factor below $F^*(H)$ is cyclic, then every G-chief factor below H is cyclic.*

Lemma 2.8 ([12, Theorem A])**.** *If H is an S-permutable p-subgroup of G for some prime p, then $N_G(H) \geq O^p(G)$.*

Lemma 2.9. *Let H be a normal subgroup of a group G. Suppose that for every non-cyclic Sylow subgroup P of H, all cyclic subgroups of P of prime order and order 4 without supersolvable supplements in G are τ-supplemented in G. Then every G-chief factor below H is cyclic.*

Proof. In fact, there are some typing errors in [10, Theorem 3 and Corollary 1] and "S-supplemented" should be "τ-supplemented". □

3. Proof of Theorem 1.3

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P not having a p-nilpotent supplement in G is τ-supplemented in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order.

(1) $O_{p'}(G) = 1$.

Assume that $R = O_{p'}(G) \neq 1$. Then, obviously, PR/R is a Sylow p-subgroup of G/R. Suppose that M/R is a maximal subgroup of PR/R. Then there exists a maximal subgroup P_1 of P such that $M = P_1R$. If P_1 has a p-nilpotent supplement D in G, then M/R has a p-nilpotent supplement DR/R in G/R. If P_1 is τ-supplemented in G, then M/R is τ-supplemented in G/R by Lemma 2.1(3). Hence G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/R is p-nilpotent and so G is also p-nilpotent, a contradiction.

(2) G is solvable.

If every maximal subgroup of P has a p-nilpotent supplement in G, then G is p-nilpotent by Lemma 2.2. Hence there exists a maximal subgroup V of P such that V is τ-supplemented in G. Then there is a non-p-nilpotent subgroup T of G such that $G = VT$ and $V \cap T \leq V_{\tau G}$. Since $O_{p'}(G) = 1$, we have that $p||Q^G|$ for every non-trivial Sylow q-subgroup Q of G $(p \neq q)$. Hence $V_{\tau G}Q = QV_{\tau G}$. This shows that $V_{\tau G}$ is S-semipermutable in G. If $V_{\tau G} = 1$, then, by Lemma 2.5(1), T is p-nilpotent, a contradiction. Hence $V_{\tau G} \neq 1$. Let L be a minimal normal subgroup of G contained in $(V_{\tau G})^G$. By virtue of [5, Theorem A], L is solvable. Consequently, $L \leq O_p(G)$. By Lemma 2.1(2), it is easy to see that every maximal subgroup of P/L not having a p-nilpotent supplement in G/L is τ-supplemented in G/L. Hence G/L satisfies the hypothesis of the theorem. The minimal choice of G implies that G/L is p-nilpotent. Since $(|G/L|, p-1) = 1$, it follows that G/L is solvable by the well-known Feit–Thompson theorem. Consequently, G is solvable.

(3) The final contradiction.

Let N be a minimal normal subgroup of G. From steps (1) and (2), $N \leq O_p(G)$. Using the same argument as in the proof of step (2), we have G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$. Consequently, G has a maximal subgroup M such that $N \nleq M$. Obviously, $N \cap M$ is normal in G. The minimal normality of N yields that $N \cap M = 1$ and so $G/N \cong M$ is p-nilpotent.

Let V be an arbitrary maximal subgroup of P. Next we shall prove that if V is τ-supplemented in G, then V has a p-nilpotent supplement in G. Assume there is a subgroup T of G such that $G = VT$ and $V \cap T \leq V_{\tau G}$. If $N \cap V_{\tau G} \neq 1$, then, by Lemma 2.4, $N \cap V_{\tau G}$ is τ-quasinormal in G. By virtue of Lemma 2.6, $N \cap V_{\tau G}$ is S-quasinormal in G. Furthermore, $O^p(G) \leq N_G(N \cap V_{\tau G})$ from Lemma 2.8. It follows that $N = (N \cap V_{\tau G})^G = (N \cap V_{\tau G})^{O^p(G)P} = (N \cap V_{\tau G})^P \leq (V_{\tau G})^P \leq V^P = V$. This implies that $G = VM$ and V has the p-nilpotent supplement M in G. If

$N \cap V_{\tau G} = 1$, then $V \cap T \cap N = 1$ and so $|T \cap N| = |T \cap N : V \cap T \cap N| = |(T \cap N)V : V| \leq |NV : V| \leq |P : V| \leq p$. Since $T/T \cap N \cong TN/N \leq G/N$, we have that $T/T \cap N$ is p-nilpotent. In view of Lemma 2.5(3), T is p-nilpotent as desired.

Now we have that every maximal subgroup of P has a p-nilpotent supplement in G. Applying Lemma 2.2, G is p-nilpotent, a contradiction. □

Corollary 3.2. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P not having a p-supersolvable supplement in G is τ-supplemented in G, then G is p-nilpotent.*

Proof. It follows directly from Lemma 2.5(2) and Theorem 3.1. □

Corollary 3.3. *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime dividing $|G|$. If every maximal subgroup of P not having a supersolvable supplement in G is τ-supplemented in G, then G is p-nilpotent.*

Theorem 3.4. *Let H be a normal subgroup of a group G. Suppose that for every non-cyclic Sylow subgroup P of H, each maximal subgroup of P not having a supersolvable supplement in G is τ-supplemented in G. Then every G-chief factor below H is cyclic.*

Proof. Suppose that this theorem is false and let (G, H) be a counterexample for which $|G| + |H|$ is minimal. By Corollary 3.3 and Lemma 2.5, H is p-nilpotent, where p is the smallest prime dividing $|H|$. Let V be a normal p-complement of H. Then V is normal in G since it is characteristic in H. Moreover, by Lemma 2.1, the hypothesis holds for $(G/V, H/V)$. Hence in the case when $V \neq 1$, we have every G/V-chief factor below H/V is cyclic by the choice of (G, H). It is clear that (G, V) also satisfies the hypothesis. Hence each G-chief factor below V is cyclic again by the choice of (G, H). It follows that every G-chief factor below H is cyclic, a contradiction. Hence $V = 1$, which implies that H is a p-group. In view of [10, Theorem 4], every G-chief factor below H is cyclic. This contradiction completes the proof. □

Proof of Theorem 1.3. Applying Theorem 3.4 and Lemma 2.9, every G-chief factor below X is cyclic. Since $F^*(H) \leq X$, we have that every G-chief factor below $F^*(H)$ is cyclic. Consequently, every G-chief factor below H is cyclic by virtue of Lemma 2.7. □

4. Applications

Theorem 4.1. *Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$ and H a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P not having a p-nilpotent supplement in G is τ-supplemented in G, then G is p-nilpotent.*

Proof. By Lemma 2.1(1), every maximal subgroup of P not having a p-nilpotent supplement in H is τ-supplemented in H. Applying Theorem 3.1, H is p-nilpotent. Let V be the normal p-complement of H. Then V is normal in G since it is characteristic in H. If $V \neq 1$, then it is easy to see that G/V satisfies the hypothesis of the theorem by virtue of Lemma 2.1(3). Hence G/V is p-nilpotent by induction. It follows that G is p-nilpotent. Next we assume that $V = 1$, i.e. $H = P$. Since G/P is p-nilpotent, we may let K/P be the normal Hall p'-subgroup of G/P. By Schur-Zassenhaus Theorem, $K = PK_{p'}$. Applying Lemma 2.1(1) and Theorem 3.1 again, $K = P \times K_{p'}$. Obviously, $K_{p'}$ is also a normal p-complement of G and so G is p-nilpotent. □

A subgroup H of G is said to be c-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the largest normal subgroup of G contained in H (see [1]).

Corollary 4.2 ([3, Theorem 3.4])**.** *Let P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-supplemented in G, then G is p-nilpotent.*

Corollary 4.3 ([15, Theorem 3.3])**.** *Let p be the smallest prime divisor of $|G|$ and P a Sylow p-subgroup of G. If every maximal subgroup of P is S-semipermutable in G, then G is p-nilpotent.*

Theorem 4.4. *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersoluble groups $\mathfrak{U}$ and H a normal subgroup of a group G such that $G/H \in \mathfrak{F}$. Let X be a normal subgroup of G with $F^*(H) \leq X \leq H$. Suppose that for every non-cyclic Sylow subgroup P of X, either all maximal subgroups of P lacking supersolvable supplements in G or all cyclic subgroups of P of prime order and order 4 without supersolvable supplements in G are τ-supplemented in G. Then $G \in \mathfrak{F}$.*

Proof. Applying Theorem 1.3, every G-chief factor below H is cyclic. By [2, IV, Proposition 3.11], the chief factors of H are f-central. And since $G/H \in \mathfrak{F}$, the chief factors of G/H are f-central. Thus all chief factors of G are f-central, i.e. $G \in \mathfrak{F}$. □

Corollary 4.5 ([3, Theorem 4.2]). *Let* $\mathfrak{F}$ *be a saturated formation containing* $\mathfrak{U}$ *and* H *a normal subgroup of a group* G *such that* $G/H \in \mathfrak{F}$. *If every maximal subgroup of any Sylow subgroup of* H *is c-supplemented in* G, *then* $G \in \mathfrak{F}$.

Corollary 4.6 ([16, Theorem 1.1]). *Let* $\mathfrak{F}$ *be a saturated formation containing* $\mathfrak{U}$ *and* H *a normal subgroup of a group* G *such that* $G/H \in \mathfrak{F}$. *If all maximal subgroups of any Sylow subgroup of* $F^*(H)$ *are c-supplemented in* G, *then* $G \in \mathfrak{F}$.

Corollary 4.7 ([16, Theorem 1.2]). *Let* $\mathfrak{F}$ *be a saturated formation containing* $\mathfrak{U}$ *and* H *a normal subgroup of a group* G *such that* $G/H \in \mathfrak{F}$. *If all minimal subgroups and all cyclic subgroups with order* 4 *of* $F^*(H)$ *are c-supplemented in* G, *then* $G \in \mathfrak{F}$.

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