Araştırma Makalesi / Research Article

Semiparametric EIV Regression Model with Unknown Errors in all Variables

Seçil YALAZ[1*], Müjgan TEZ[2]

[1]*Dicle Üniversitesi, İstatistik Bölümü, Diyarbakır*
[2]*Marmara Üniversitesi, İstatistik Bölümü, İstanbul*
(ORCID: 0000-0001-7283-9225) (ORCID: 0000-0002-8633-1980)

Abstract
This paper develops a method for semiparametric partially linear regression model when all variables measured with errors whose densities are unknown. Identification is achieved using the availability of two error-contaminated measurements of the independent variables. This method is likened to kernel deconvolution method which relies on the assumption that measurement errors densities are known. However with this deconvolution method, convergence rates are very slow. Hence, estimating a regression function with super smooth errors is extremely difficult and in literature the authors only have studied the case that the errors are ordinary smooth. We could tackle this problem with the Fourier representation of the Nadaraya-Watson estimator, because this method can handle both of super smooth and ordinary smooth distributions. In literature studying asymptotic normality also has difficulty because of the same smoothing problem. With this study we could manage to show asymptotic normality of parametric part. Monte Carlo experiments demonstrated the performances of $\hat{\beta}$ and $\hat{g}_n(t)$ in the application part.

Keywords: Errors in variables, Kernel deconvolution, Partially linear model, Semiparametric regression, Monte-Carlo Simulation.

Tüm Değişkenleri Bilinmeyen Hatalı Yarı Parametrik Regresyon Modeli

Öz
Bu makale ile değişkenleri hatalı ölçülmüş yarı parametrik kısmi doğrusal regresyon modelinde hataların yoğunlukları bilinmediğinde kullanılabilecek bir yöntem geliştirilmektedir. Bağımsız değişkenlerin hata bulaşmış iki ölçümünün mevcudiyeti tanımlamayı sağlamak için kullanılır. Bu yöntem, ölçüm hataları yoğunluklarının bilindiği varsayımına dayanan kernel dekonvolüsyon yöntemine benzetilir. Bununla birlikte, bu dekonvolüsyon yönteminde, süper düzgün hataların varlığında bir regresyon fonksiyonunun tahmin edilmesi, yakınsama oranları çok yavaş olduğu için son derece zordur. Bu durum nedeniyle, literatürde yazarlar sadece hatanın olağan düzgün dağılıma sahip olduğu durumlarda çalışmışlardır. Bu problemi Nadaraya-Watson tahmin edicisinin Fourier temsiliyle çözebiliriz, çünkü bu yöntem hem süper düzgün hem de olağan düzgün dağılımların üstesinden gelebilir. Literatürde asimptotik normallik gösteriminde de aynı düzleştirme probleminden dolayı zorluk çekilmektedir. Bu çalışma ile parametrik kısmın asimptotik normalliğinin gösterimi de sağlanabilinmiştir. Uygulama bölümünde, Monte Carlo simülasyon denemeleri ile $\hat{\beta}$ ve $\hat{g}_n(t)$'nin performansları incelenmiştir.

Anahtar kelimeler: Değişkenleri hatalı modeller, Kernel dekonvolüsyonu, Kısmi doğrusal model, Yarı parametrik regresyon, Monte-Carlo simülasyonu.

1. Introduction

Measurement error in predictors causes loss of information and biases and even misleading conclusions for inference [1]. Three main effects of measurement error are:

- It causes bias in parameter estimation for statistical models.

*Sorumlu yazar: syalaz@dicle.edu.tr
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- It leads to a loss of power, sometimes profound, for detecting interesting relationship among variables.
- It masks the features of the data, making graphical model analysis difficult [2].

The bias resulting from the presence of measurement error in the explanatory variables is a common problem in regression analysis [3]. Although numerous solutions to this problem have been derived for parametric and nonparametric regression models, the corresponding problem in semiparametric specifications has remained relatively unexplored. In this paper a semiparametric partially linear regression model when possibly all variables measured with errors is considered. This study presents a new semiparametric estimator which expands the classic Nadaraya Watson kernel estimator to enclose the cases of all variables have errors and the error ridden regressors. In literature semiparametric regression model with errors in all variables are only studied when the error has a known function. Contrary to the popular estimators, our estimator do not require any knowledge about the distribution of the measurement error. Beside this, because estimating a regression function with super smooth errors is extremely difficult, as the convergence rates are very slow, the authors only have studied the case that the errors are ordinary smooth. We could tackle this problem with the Fourier representation of the Nadaraya-Watson estimator because this method can handle both of super smooth and ordinary smooth distributions. In literature studying asymptotic normality also has difficulty because of the same smoothing problem. With this study it could be seen that $\hat{\beta}_n$ and $\hat{\sigma}_n^2$ are asymptotically normal. The average values of 15 replicates of $\hat{\beta}_n$ and $\hat{\sigma}_n^2$ are handle with the simulation study through Monte Carlo experiments. The performances of the estimator $\hat{\beta}_n$ of β and the estimate $\hat{g}_n$ of g are encouraging.

2. Material and Method

2.1. Construction of Estimators

In this section, we firstly consider semiparametric regression model for n observations.

$$y = X^T\beta + g(x^*) + \Delta y, \tag{1}$$

where $E[\Delta y|x^*] = 0$, X is a random vector, x^* is a random variable described in $[0,1]$, $g(.)$ is an unknown function and Δy is an error term which is independent from other variables and have zero mean.

Due to the nature of the environment or the measurement technique, sometimes variables X and X^* could be measured with errors. So X and X^* could be observed as follows:

$$\begin{aligned} X^+ &= X + \Delta x^+, \\ \chi &= X^* + \Delta\chi, \end{aligned} \tag{2}$$

where, Δx^+ and $\Delta\chi$ are random errors, Δx^+, $\Delta\chi$ and $(X^T, X^*, \Delta y)^T$ mutually independent, X^* has an unknown density as $f(x^*)$, $\Delta\chi$ is an error function has an unknown distribution. Let us define a new model for the semiparametric regression function in the case of measurement errors in both the linear and nonparametric parts by using information of the probability of defining the density of this variable when the density of two error-contaminated measurements of an unobserved random variable is available. Then we can describe a semiparametric errors in all variables model for the variables Y, X and X^* as follows:

$$\begin{cases} y = X^T\beta + g(x^*) + \Delta y, \\ x^+ = x + \Delta x^+, \\ x^- = x + \Delta x^-, \\ \chi = x^* + \Delta\chi, \\ z = x^* + \Delta z. \end{cases} \tag{3}$$

Let us denote that (X^*, X, Y) is a triple of random vectors and assume

$$E(Y|X^*, X) = X^T\beta + g(x^*), \tag{4}$$

which shows the conditional expectation of the univariate Y given (X^*, X), where X is p, X^* is 1 dimensional, β is an unknown $(p \times 1)$ dimensional parameter vector and $g(.)$ is an unknown function.

Assumption 1: Δz and x^* are mutually independent;

$$\begin{aligned} &E[\Delta y|x^*, \Delta z] = 0, \\ &E[\Delta x|x^*, \Delta z] = 0. \end{aligned} \tag{5}$$

Assumption 2: $E[|x^*|], E[\Delta x]$ and $E[|y|]$ are finite.
Assumption 3: $E\left[y^k h^{-1} K\left(h^{-1}(x^* - \tilde{x}^*)\right)\right] < \infty$, for all $\tilde{x}^*$, any $h > 0$ and $k = 0,1$.
Assumption 4: The Fourier transform of the kernel, $\kappa(\xi)$, is

i. Bounded,
ii. Compactly supported (without losing generality, we consider the support to be $[-1, 1]$).

Assumption 5: $E(\Delta x^+) = 0, Cov(\Delta x^+) = \Sigma_{\Delta x^+}, E(\Delta x^-) = 0, Cov(\Delta x^-) = \Sigma_{\Delta x^-}, E(\Delta y|X, X^*) = 0$ and $Var(\Delta y|X, X^*) = \sigma^2_{\Delta y}$ where $\sigma^2_{\Delta y}$ is unknown and $\Sigma_{\Delta x^+} > 0, \Sigma_{\Delta x^-} > 0$.
Assumption 6: Kernel function $K(x^*)$ provides following presentations for $\gamma_\kappa > 0$:

$$\begin{aligned} &\int_{-\infty}^{\infty} K(x^*)dx^* = 1, \quad \int_{-\infty}^{\infty} (x^*)^j K(x^*)dx^* \begin{cases} = 0 & j = 1, \dots, \gamma_\kappa - 1 \\ \neq 0 & j = \gamma_\kappa \end{cases}, \\ &\int_{-\infty}^{\infty} |x^*|^j |K(x^*)| dx^* < \infty \quad j = 1, \dots, \gamma_\kappa \end{aligned} \tag{6}$$

And Fourier transform of kernel is $\kappa(\xi) = 1$, provided $|\xi| < \bar{\xi}$ for some $\bar{\xi} > 0$.
Because the measurement error is in the both parametric and nonparametric part this problem is more complex. Firstly, serious boundary problem arising from the measurement error that occurs when generating the nonparametric estimator should be solved. Then, the parametric parts' measurement error, which has a very strong effect on estimation step of the nonparametric function g, should be considered.

3. Results and Discussion

3.1. Generation of the Estimators

Let us take

$$\begin{aligned} U(X^+, X^*) &= x^+ - E(X^+|X^*) = x - E(X|X^*) + \Delta x^+, \\ U(X^-, X^*) &= x^- - E(X^-|X^*) = x - E(X|X^*) + \Delta x^-, \\ U(X^\#, X^*) &= \frac{[U(X^+, X^*) + U(X^-, X^*)]}{2} = x - E(X|X^*) + \frac{[\Delta x^+ + \Delta x^-]}{2} \\ &= x - E(X|X^*) + \Delta x^\# \text{ and} \\ U(Y, X^*) &= y - E(Y|X^*) = [x - E(X|X^*)]^T\beta + \Delta y, \end{aligned}$$

where, $X^\#$ is the mean of X^+ and X^- which are the consecutive measurements of X. This variable is added to the model for convenience of presentation. Define the function $w(x^*) \geq 0$ which is in $[a, b]$, where $0 < \inf_{a \leq x^* \leq b} f(x^*) \leq \sup_{a \leq x^* \leq b} f(x^*) < \infty$. This assumption plays an important role in avoiding the boundary problem arising from the denominator of the kernel estimator ([4]). Let us take

$$
\begin{aligned}
S_1 &= E\left[U(X^{\#},X^*)U(X^{\#},X^*)^T w(x^*)\right] \\
&= E\{[x-E(X|X^*)][x-E(X|X^*)]^T w(x^*)\} + Ew(x^*)\Sigma_{\Delta x^{\#}} \\
&\triangleq S + S_3\Sigma_{x^{\#}}, \\
S_2 &= E\left[U(X^{\#},X^*)U(Y,X^*)w(x^*)\right] = S\beta, \\
S_4 &= E\left[\left(\Delta y - \Delta x^{\#T}\beta\right)^2 w(x^*)\right],
\end{aligned} \tag{7}
$$

where, $S = E\{[x-E(X|X^*)][x-E(X|X^*)]^T w(x^*)\}, S_3 = E(w(x^*))$. Let $f(y,x^{\#},x^*)$ denote the density of $(Y,X^{\#},X^*)$ and
$g_1(x^*) = E(Y|X^*=x^*)$,
$g_2(x^*) = E(X^{\#}|X^*=x^*) \triangleq \left(g_{11}(x^*),\dots,g_{1p}(x^*)\right)^T$.
Let us define S as a positive definite matrix ($S>0$), then the formula of $\beta, g(x^*)$ and $\sigma^2_{\Delta y}$ are
$\beta = \left(S_1 - S_3\Sigma_{\Delta x^{\#}}\right)^{-1} S_2$,
$g(x^*) = g_1(x^*) - g_2(x^*)^T\beta$,
$\sigma^2_{\Delta y} = \dfrac{S_4}{S_3} - \beta^T\Sigma_{\Delta x^{\#}}\beta$.
In this way estimations of β, g and $\sigma^2_{\Delta y}$ are turned to the estimation of S_1,S_2,S_3,S_4 and g_1,g_2.
Suppose that

$$
\left\{\begin{matrix} X_j^{\#} = \left(X_{j1}^{\#},X_{j2}^{\#},\dots,X_{jp}^{\#}\right)^T, Y_j, X_j^*, \\ 1\le j\le n \end{matrix}\right\}
$$

is a sample which size is n and is taking from model (3). Then the estimators of $\beta, \sigma^2_{\Delta y}$ and g are acquired by the following procedure:

Step 1: Estimator of $f(x^*)$ is defined as $\hat f(x^*) = \widehat M_0(\tilde x^*, h_n)$, where h_n is bandwidth and,

$$
M_0(\tilde x^*,h) = \frac{1}{2\pi}\int \kappa(h\xi)\phi_0(\xi)\exp(-i\xi\tilde x^*)d\xi
$$

where $\phi_0(\xi) = \exp\left(\int_0^{\xi}\frac{im_{\chi}(\zeta)}{m_1(\zeta)}d\zeta\right)$, where $i=\sqrt{-1}$ and $m_a(\xi) = E[a\exp(i\xi z)]$ for $a = 1,\chi$.

Step 2: Joint density function estimator $f(y,x^{\#},x^*)$ of $(Y,X^{\#},X^*)$ can be defined as:

$$
\hat f(y,x^{\#},x^*) = \frac{1}{nh^{p+2}}\sum_{j=1}^{n}\prod_{k=1}^{p} K\left(\frac{x_k^{\#}-X_{jk}^{\#}}{h}\right)K\left(\frac{y-Y_j}{h}\right)\left[\int \frac{1}{2\pi}\kappa(h_n\xi)\exp[i\xi(x^*-\tilde x^*)]d\xi\right].
$$

Step 3: Estimators of $g_1(x^*)$ and $g_2(x^*)$ defined as

$$
\hat g_{1n}(\tilde x^*,h_n) = \frac{\sum_{j=1}^n K_n\left((x^*-\tilde x^*)/h\right)Y_j}{\sum_{j=1}^n K_n\left((x^*-\tilde x^*)/h\right)} = \frac{E[yh^{-1}K(h^{-1}(x^*-\tilde x^*))]}{E[h^{-1}K(h^{-1}(x^*-\tilde x^*))]} = \frac{\widehat M_1(\tilde x^*,h_n)}{\widehat M_0(\tilde x^*,h_n)},
$$

$$
\hat g_{2n}(\tilde x^*,h_n) = \frac{\sum_{j=1}^n K_n\left((x^*-\tilde x^*)/h\right)X_j^{\#}}{\sum_{j=1}^n K_n\left((x^*-\tilde x^*)/h\right)} = \frac{E[X_j^{\#}h^{-1}K(h^{-1}(x^*-\tilde x^*))]}{E[h^{-1}K(h^{-1}(x^*-\tilde x^*))]} = \frac{\widehat M_2(\tilde x^*,h_n)}{\widehat M_0(\tilde x^*,h_n)},
$$

where $\widehat M_1(\tilde x^*,h_n) = \frac{1}{2\pi}\int\kappa(h\xi)\phi_1(\xi)\exp(-i\xi\tilde x^*)d\xi$, $\phi_1(\xi) = \phi_0(\xi)\frac{m_{y^*}(\xi)}{m_1(\xi)}$ and

$m_a(\xi) = E[a\exp(i\xi z)]$ for $a = 1, y^*$. Moreover, for $\phi_2(\xi) = \phi_0(\xi)\frac{m_{X_j^{\#}}(\xi)}{m_1(\xi)} = E\left[X_j^{\#}\exp(i\xi x^*)\right]$

$$
\begin{aligned}
M_2(\tilde x^*,h) &= \frac{1}{2\pi}\int\kappa(h\xi)\phi_2(\xi)\exp(-i\xi\tilde x^*)d\xi \\
&= \frac{1}{2\pi}\int\kappa(h\xi)\phi_0(\xi)\frac{m_{X_j^{\#}}(\xi)}{m_1(\xi)}\exp(-i\xi\tilde x^*)d\xi \\
&= \frac{1}{2\pi}\int\kappa(h\xi)E[\exp(i\xi x^*)]\frac{E\left[X_j^{\#}\exp(i\xi z)\right]}{E[\exp(i\xi z)]}\exp(-i\xi\tilde x^*)d\xi \\
&= \frac{1}{2\pi}\int\kappa(h\xi)E[\exp(i\xi x^*)]\frac{E\left[X_j^{\#}\right]E[\exp(i\xi z)]}{E[\exp(i\xi z)]}\exp(-i\xi\tilde x^*)d\xi
\end{aligned}
$$

$$= \frac{1}{2\pi}\int \kappa(h\xi)E\left[X_j^{\#}\exp(i\xi x^*)\right]\exp(-i\xi\tilde{x}^*)d\xi$$
$$= \frac{1}{2\pi}\int \kappa(h\xi)\left(\int E\left[X_j^{\#}\middle|x^*\right]f(x^*)\exp(i\xi x^*)dx^*\right)\exp(-i\xi\tilde{x}^*)d\xi$$
$$= \iint \frac{1}{2\pi}\kappa(h\xi)\exp[i\xi(x^* - \tilde{x}^*)]E\left[X_j^{\#}\middle|x^*\right]f(x^*)d\xi d\, x^*$$
$$\left(\begin{array}{c}\text{from Parseval's equation;}\\ \int \frac{1}{2\pi}\kappa(\xi)\exp[i\xi h^{-1}(x^* - \tilde{x}^*)]d\xi = K\left(h^{-1}(x^* - \tilde{x}^*)\right)\end{array}\right)$$
$$= \int h^{-1}K\left(h^{-1}(x^* - \tilde{x}^*)\right)\left(E\left[X_j^{\#}\middle|x^*\right]f(x^*)\right)dx^*$$
$$= E\left[h^{-1}K\left(h^{-1}(x^* - \tilde{x}^*)\right)E\left[X_j^{\#}\middle|x^*\right]\right] = E\left[X_j^{\#}h^{-1}K\left(h^{-1}(x^* - \tilde{x}^*)\right)\right].$$

Step 4: Estimations of S_q $(q = 1,2,3), \beta$ and $g(.)$ are obtained as:

$$\begin{cases} \hat{S}_{1n} = \int\limits_{\mathbb{R}^p}\int\limits_{\mathbb{R}^1}\int\limits_{\mathbb{R}^1}\left(x^{\#} - \hat{g}_{2n}(\tilde{x}^*, h_n)\right)\left(x^{\#} - \hat{g}_{2n}(\tilde{x}^*, h_n)\right)^T w(x^*)\hat{f}\left(y, x^{\#}, x^*\right)dx^{\#}dydx^*, \\ \hat{S}_{2n} = \int\limits_{\mathbb{R}^p}\int\limits_{\mathbb{R}^1}\int\limits_{\mathbb{R}^1}\left(x^{\#} - \hat{g}_{2n}(\tilde{x}^*, h_n)\right)\left(y - \hat{g}_{1n}(\tilde{x}^*, h_n)\right)w(x^*)\hat{f}\left(y, x^{\#}, x^*\right)dx^{\#}dydx^*, \\ \hat{S}_{3n} = \int\limits_{\mathbb{R}^1} w(x^*)\hat{f}(x^*)dx^*, \end{cases}$$

$$\hat{\beta}_n = \left(\hat{S}_{1n} - \hat{S}_{3n}\Sigma_{\Delta\#}\right)^{-1}\hat{S}_{2n}, \qquad \hat{g}(\tilde{x}^*, h_n) = \hat{g}_{1n}(\tilde{x}^*, h_n) - \hat{g}_{2n}(\tilde{x}^*, h_n)^T\hat{\beta}_n.$$

Step 5: Estimations of S_4 and $\sigma_{\Delta y}^2$ are obtained as:

$$\hat{S}_{4n} = \int\int\int\left(y - x^{\#T}\hat{\beta}_n - \hat{g}(\tilde{x}^*, h_n)\right)^2 w(x^*)\hat{f}\left(y, x^{\#}, x^*\right)dx^{\#}dydx^*,$$
$$\hat{\sigma}_n^2 = \hat{S}_{4n}/\hat{S}_{3n} - \hat{\beta}_n^T\Sigma_{\Delta x^{\#}}\hat{\beta}_n.$$

3.2. Asymptotic Normality of Parametric Part

Assumption 7:

i. $[y_i^*, \chi_i, z_i, x_i^*, \Delta y_i, \Delta\chi_i, \Delta z_i]$; $i = 1, \dots, n$ is an i.i.d sequence.
ii. $E\left[y^{*2-j}|z|^j\right] < \infty, E\left[\chi^{2-j}|z|^j\right] < \infty$; $j = 0,1$.
iii. The density of x^* is nonzero at $x^* = \tilde{x}^*$.
iv. The functions $\phi_0(\zeta) = E\left[e^{i\zeta x^*}\right], \phi'_0(\zeta) \equiv \frac{d\phi_0(\zeta)}{d\zeta}, \phi_1(\zeta) = E\left[y^* e^{i\zeta x^*}\right],\ m_1(\zeta) = E\left[e^{i\zeta z}\right]$ satisfy

$$\left|\frac{\phi'_0(\zeta)}{\phi_0(\zeta)}\right| \preccurlyeq (1 + |\zeta|)^{\gamma_r} \tag{8}$$

for some $\gamma_r \geq 0$ and

$$\max\{|\phi_0(\zeta)|, |\phi_1(\zeta)|\} \preccurlyeq (1 + |\zeta|)^{\gamma_\phi}\exp\left(\alpha_\phi|\zeta|^{\beta_\phi}\right), \tag{9}$$

$$|m_1(\zeta)| \succcurlyeq (1 + |\zeta|)^{\gamma_m}\exp\left(\alpha_m|\zeta|^{\beta_m}\right), \tag{10}$$

for some $\gamma_\phi, \gamma_m \in \mathbb{R}, \alpha_\phi \leq 0, \alpha_m \leq 0, \beta_\phi \geq 0, \beta_m \geq 0$ such that $\gamma_\phi\beta_\phi \geq 0$ and $\gamma_m\beta_m \geq 0$.

Assumption 8:

i. $E(\|X\|^2|X^* = x^*)$ is a bounded function of x^*, where $\|X\|$ is L^2 norm of X.
ii. S is a positive definitely $p \times p$ matrix.

Theorem 1: Let assumptions 1-8 are provided for any given $\tilde{x}^*$. Furthermore let us take $E\left(1+\|\Delta x^\#\|^4+|\Delta y|^2\|X\|^2\right)<\infty, \gamma_\kappa>1+2\gamma$ and $nh^{2(1+2\gamma)}\to\infty, nh^{2\gamma_\kappa}\to 0$, where γ_κ ($\gamma_\kappa\geq 3$) is an integer given in assumption 6 and γ is a smoothness parameter. Then
$\sqrt{n}(\hat{\beta}_n-\beta)\to N(0,\mathbf{\Omega}_1)$, $\sqrt{n}(\hat{\sigma}_n^2-\sigma_{\Delta y}^2)\to N(0,\mathbf{\Omega}_2)$,
where
$\mathbf{\Omega}_1=S^{-1}Cov[\xi_1(\beta)]S^{-1}$, $\mathbf{\Omega}_2=Var[\xi_2(\beta,\sigma_{\Delta y}^2)]$
for $\xi_1(\beta)=\partial_{\gamma_\kappa}\left\{\left[\left(x^\#-g_1(x^*)\right)\left(y-g_2(x^*)-\left(x^\#-g_1(x^*)^T\right)\beta\right)+\Sigma_{\Delta x^\#}\beta\right]w(x^*)\right\}$ and
$\xi_2(\beta,\sigma_{\Delta y}^2)=\left\{\partial_{\gamma_\kappa}\left[\left(\Delta y-\Delta x^{\#T}\beta\right)^2 w(x^*)\right]-\left(\sigma_{\Delta y}^2+\beta^T\Sigma_{\Delta x^\#}\beta\right)\right\}\frac{\partial_{\gamma_\kappa}w(x^*)}{S_3}$.

Proof: From [3] and [4] we can demonstrate that

$$
\begin{aligned}
\sqrt{n}(\hat{\beta}_n-\beta) &= \sqrt{n}\left[\left(\hat{S}_{1n}-\hat{S}_{3n}\Sigma_{\Delta x^\#}\right)^{-1}\hat{S}_{2n}-\left(S_1-S_3\Sigma_{\Delta x^\#}\right)^{-1}S_2\right]\\
&= \sqrt{n}\left(\hat{S}_{1n}-\hat{S}_{3n}\Sigma_{\Delta x^\#}\right)^{-1}\left[\hat{S}_{2n}-\left(\hat{S}_{1n}-\hat{S}_{3n}\Sigma_{\Delta x^\#}\right)\left(S_1-S_3\Sigma_{\Delta x^\#}\right)^{-1}S_2\right]\\
&= \sqrt{n}S^{-1}\left[\hat{S}_{2n}-S_2-\left(\hat{S}_{1n}-\hat{S}_{3n}\Sigma_{\Delta x^\#}\right)\left(S_1-S_3\Sigma_{\Delta x^\#}\right)^{-1}S_2+S_2\right]\\
&= \sqrt{n}S^{-1}\left\{\left(\hat{S}_{2n}-S_2\right)-\left(S_1-S_3\Sigma_{\Delta x^\#}\right)^{-1}\left[\left(\hat{S}_{1n}-\hat{S}_{3n}\Sigma_{\Delta x^\#}\right)S_2-\left(S_1-S_3\Sigma_{\Delta x^\#}\right)S_2\right]\right\}\\
&= \sqrt{n}S^{-1}\left\{\left(\hat{S}_{2n}-S_2\right)-\left(S_1-S_3\Sigma_{\Delta x^\#}\right)^{-1}\left[\left(\hat{S}_{1n}-S_1\right)S_2-\left(\hat{S}_{3n}-S_3\right)\Sigma_{\Delta x^\#}S_2\right]\right\}\\
&= \sqrt{n}S^{-1}\left\{\left(\hat{S}_{2n}-S_2\right)-\left(S_1-S_3\Sigma_{\Delta x^\#}\right)^{-1}\left[\left(\hat{S}_{1n}-S_1\right)S_2-\left(\hat{S}_{3n}-S_3\right)\Sigma_{\Delta x^\#}S_2\right]\right\}\\
&= \sqrt{n}S^{-1}\left\{\left(\hat{S}_{2n}-S_2\right)-\left(\hat{S}_{1n}-S_1\right)\left(S_1-S_3\Sigma_{\Delta x^\#}\right)^{-1}S_2+\left(\hat{S}_{3n}-S_3\right)\Sigma_{\Delta x^\#}\left(S_1-S_3\Sigma_{\Delta x^\#}\right)^{-1}S_2\right\}\\
&= \sqrt{n}S^{-1}\left[\left(\hat{S}_{2n}-S_2\right)-\left(\hat{S}_{1n}-S_1\right)\beta+\left(\hat{S}_{3n}-S_3\right)\Sigma_{\Delta x^\#}\beta\right]\\
&= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}S^{-1}\left[\xi_{j1}(\beta)-E\left(\xi_{j1}(\beta)\right)\right]+o\left(\frac{1}{\sqrt{n}h^{(1+2\gamma)}}\right)+O\left(\sqrt{n}h^{\gamma_\kappa}\right)\\
&\to N(0,S^{-1}Cov[\xi_1(\beta)]).
\end{aligned}
$$

We can complete the proof for $\hat{\sigma}_n^2$ similarly.

3.3. Simulation

To simulate our results we thought both supersmooth and ordinary smooth functions. Hence we create 4 different examples. In Table 1, normal distribution and error function $erf(x^*)=\frac{2}{\pi}\int_0^{x^*}e^{-t^2}dt$ show supersmooth functions, laplace distribution, uniform distribution and $S(x^*)=\begin{cases}-1, & x^*<-1\\ x^*, & x^*\in[-1,1]\\ 1, & x^*>1\end{cases}$ function show ordinary smooth functions.

Table 1. Examples

1	2	3	4
$x^*\to N(0,1)$	$x^*\to N(0,1)$	$x^*\to N(0,1)$	$x^*\to Uniform[-2,2]$
$\Delta\chi,\Delta z\to N(0,0.25)$	$\Delta\chi,\Delta z\to N(0,0.25)$	$\Delta\chi,\Delta z\to L(0,0.25)$	$\Delta\chi,\Delta z\to L(0,0.25)$
$\Delta y\to N(0,0.25)$	$\Delta y\to N(0,0.25)$	$\Delta y\to N(0,0.25)$	$\Delta y\to N(0,0.25)$
$g(x^*)=S(x^*)$	$g(x^*)=erf(x^*)$	$g(x^*)=S(x^*)$	$g(x^*)=S(x^*)$
$h_n^{-1}=1.2(\ln n)^{0.25}$	$h_n^{-1}=1.2(\ln n)^{0.25}$	$h_n^{-1}=1.2(\ln n)^{0.25}$	$h_n^{-1}=1.2(\ln n)^{0.25}$

We choose $n=100, p=2, N=15, \beta=\binom{1}{2}, \sigma_{\Delta y}^2=0.25, x^*=1$ and $w(x^*)=I,\{x^*\leq 2\}$. The medial values of fifteen copies of $\hat{\beta}_n$ and $\hat{\sigma}_n^2$ are given in the Table 2. For $\hat{\beta}_n$ values corresponding standart errors are also given. The performance of the $\hat{\beta}_n$ is encouraging. As for the estimate $\hat{g}_n$ of g, the mean square error could be seen in the Table 2. The performance of the $\hat{g}_n$ is also encouraging.

Table 2. Simulation results

1	**2**	**3**	**4**
$\hat{\beta}_n = \begin{pmatrix} 0.9606 \\ 1.6938 \end{pmatrix}$	$\hat{\beta}_n = \begin{pmatrix} 0.8704 \\ 1.7396 \end{pmatrix}$	$\hat{\beta}_n = \begin{pmatrix} 0.9347 \\ 2.0610 \end{pmatrix}$	$\hat{\beta}_n = \begin{pmatrix} 0.8607 \\ 2.0074 \end{pmatrix}$
$S_\beta = \begin{pmatrix} 0.3474 \\ 0.4175 \end{pmatrix}$	$S_\beta = \begin{pmatrix} 0.00004 \\ 0.00006 \end{pmatrix}$	$S_\beta = \begin{pmatrix} 0.00024 \\ 0.00044 \end{pmatrix}$	$S_\beta = \begin{pmatrix} 0.00095 \\ 0.00160 \end{pmatrix}$
$\hat{\sigma}_n^2(x^*) = 0.5083$	$\hat{\sigma}_n^2(x^*) = 0.0333$	$\hat{\sigma}_n^2(x^*) = 0.1505$	$\hat{\sigma}_n^2(x^*) = 1.5850$
$g(x^*) = 1$	$g(x^*) = 0.8427$	$g(x^*) = 1$	$g(x^*) = 1$
$\hat{g}_n(x^*) = 1.1346$	$\hat{g}_n(x^*) = 0.5975$	$\hat{g}_n(x^*) = 1.1372$	$\hat{g}_n(x^*) = 1.5125$
$MSE = 0.7395$	$MSE = 0.4354$	$MSE = 0.2542$	$MSE = 0.3039$

4. Discussion and Conclusion

This study presents a new semiparametric estimator which expands the classic Nadaraya Watson kernel estimator to enclose the cases of all variables have errors and the error ridden regressors. In literature semiparametric regression model with errors in all variables are only studied when the error has a known function. Contrary to the popular estimators, our estimator do not require any knowledge about the distribution of the measurement error. Beside this, because estimating a regression function with super smooth errors is extremly difficult, as the convergence rates are very slow, the authors only have studied the case that the errors are ordinary smooth. We could tackle this problem with the Fourier representation of the Nadaraya-Watson estimator because this method can handle both of super smooth and ordinary smooth distributions. Our results include Schennach (2004) which is a special case where measurement errors are in the nonparametric regression. In literature studying asymptotic normality also has difficulty because of the same smoothing problem. With this study it could be seen that of $\hat{\beta}_n$ and $\hat{g}_n(t)$ are asymptotically normal. The average values of 15 replicates of of $\hat{\beta}_n$ and $\hat{\sigma}_n^2$ are handle with the simulation study. The performances of the estimator $\hat{\beta}_n$ of β and the estimate $\hat{g}_n(t)$ of g are encouraging.

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