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ON CO-FILTERS IN CO-QUASIORDERED RESIDUATED SYSTEMS

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Abstract

Residuated relational systems have been the focus of many researchers in the past decade. In this article, as a continuation of [9], we focused on residuated relational systems $\langle A, \cdot, \rightarrow, 1, \not\preccurlyeq \rangle$ ordered under co-quasiorder relation $' \not\preccurlyeq '$ within the Bishop's constructive framework. In this report we give some new results on co-filters in such relational systems by more depth and deeper analyzing of the connection between the internal operation $' \cdot '$ and $' \rightarrow '$ with the co-quasiorder relation.

Keywords: Bishop's constructive mathematics; Set with apartness; Co-quasiordered residuated system; Co-filter.

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1 Introduction

Although in the last decade the concept of residual relational systems is in the focus of many researchers (for example, [3, 4]), there are still not many research reports on such algebraic structures.

Definition 1.1. ([4], Definition 2.1) A residuated relational system is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R \rangle$, where $\langle A, \cdot, \rightarrow, 1 \rangle$ is an algebra of type $\langle 2, 2, 0 \rangle$ and R is a binary relation on A and satisfying the following properties:

(1) $\langle A, \cdot, 1 \rangle$ is a commutative monoid;

(2) $(\forall x \in A)((x, 1) \in R)$;

(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.

They referred to the operation $' \cdot '$ as multiplication, to $' \rightarrow '$ as its residuum and to condition (3) as residuation.

The concept of residual relational system ordered under a quasi-order relation can be found in Bonzio's dissertation [3] from 2015 and in one of his articles [4] from 2018 (done together with I. Chajda). In the forthcoming articles [11, 12] this author introduced and analyzed concepts of ideals and filters in such systems. In the aforementioned texts, authors observed the relational system $\langle A, \cdot, \rightarrow, 1, R \rangle$ where R was a quasi-order relation.

In our article [9], we are developed this concept within the Bishop's constructive framework [1, 2, 5, 6, 13]. Observed and analyzed is residuated relational system with a set with apartness as the carrier of the algebraic construction, and additionally R was a co-quasiorder relation on the set A. With this article, as a continuation of our article [9, 10], we complements our researches on algebraic structures within Bishop's principled-philosophical orientation (see, for example [7, 8]).

The Constructive algebra abounds in specific behavior of algebraic structures determined on sets with apartness. Additionally, the ordered algebraic structures constructed on sets with apartness are also very interesting. Particularly, there is a possibility that an algebraic structure is ordered under a co-order (under a co-quasiorder) relation instead an order (or a quasi-order) relation.

In this article we continue our analysis of co-quasiordered residuated systems launched in [9] and [10]. Second, we continue to analyze the concept of co-filters in such systems and proved some new properties of this concept.

2 Preliminaries

2.1 The research framework

The setting of this research is the Bishop's constructive mathematics [**Bish**] in the seance of the following books [1], [2], [5], [6] and [13] - a mathematics based on the Intuitionistic logic [**IL**] (See [13]) and principled-philosophical orientation on Bishop's constructive mathematics.

Let $(S, =, \neq)$ be a constructive set in the sense of Bishop [1], Mines et all. [6], Troelstra and van Dalen [13]. On set $S = (S, =, \neq)$ in this mathematics we look as on a relational system with an one binary relation extensive with respect to the equality in the following sense

$$= \circ \neq \subseteq \neq \text{ and } \neq \circ = \subseteq \neq,$$

where $'\circ'$ is the standard operation between relations. The relation $\neq$ is a binary relation on S with the following properties:

$$\neg(x \neq x), x \neq y \Longrightarrow y \neq x, x \neq z \Longrightarrow x \neq y \vee y \neq z,$$
$$x \neq y \wedge y = z \Longrightarrow x \neq z.$$

It is called *apartness*. Let S and T be two sets with apartness, then the relation $\neq$ on $S \times T$ is defined by

$$(x, y) \neq (u, v) \Longleftrightarrow (x \neq u \vee y \neq v)$$

for any $x, u \in S$ and any $y, v \in T$.

Let Y be a subset of S and $x \in S$. We put it the following notation $\lhd$ as a relation between an element x and subset Y with (For more details on this relation, the readers can see the following texts [7, 8])

$$x \lhd Y \Longleftrightarrow (\forall y \in Y)(\, x \neq y).$$

Following the orientation in books [1], [2], [5] we define a subset

$$Y^{\lhd} = \{x \in S : x \lhd Y\}$$

of S called the *complement of Y in S*.

For subset Y of S we say that it is a *strongly extensional subset* if

$$(\forall x, y \in S)(y \in Y \Longrightarrow x \neq y \;\vee\; x \in Y).$$

For a relation R on S it is called a strongly extensional if

$$(\forall x, y, z, u \in S)((x, y) \in R \Longrightarrow ((x, y) \neq (z, u) \vee (z, u) \in R))$$

holds. For example, for a mapping $f : S \longrightarrow T$ it is called a strongly extensional (shortly: *se-mapping*) if holds

$$(\forall x, y \in S)(f(x) \neq f(y) \Longrightarrow x \neq y).$$

2.2 Co-quasiorder relation

The constructive notion of a co-quasiorder relation is the dual notion to the classical notion of a quasi-order relation. Let $(S, =, \neq)$ be a set with apartness. A consistent and co-transitive relation $\not\leqslant$ defined on S is called a *co-quasiorder* ([7, 8]):

$(\forall c, y \in S)(x \not\leqslant y \implies x \neq y)$ (consistency)

$(\forall x, y, z \in S)(x \not\leqslant z \implies (x \not\leqslant y \vee y \not\leqslant z))$ (co-transitivity).

We accept that the empty set $\emptyset$ is also a co-quasiorder relation on set S. The strong complement $\not\leqslant^{\lhd}$ of a co-quasiorder $\not\leqslant$ has the well known property.

Lemma 2.1. ([7], Lemma 2.2) If $\not\leqslant$ is a co-quasiorder on S, then the relation $\not\leqslant^{\lhd} = \{(x, y) \in S \times S : (x, y) \lhd \sigma\}$ is a quasi-order on S.

2.3 Co-quasiordered residuated systems

In our papers [9, 10], following the ideas of Bonzio [3] and Bonzio and Chajda [4], we introduced and analyzed the notion of residuated relational systems ordered under a co-quasiorder - a residuated relational systems $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ where R is a co-quasiorder relation on set $(A, =, \neq)$. In the article [9] we introduced and analyzed the concept of co-filters in such systems, and in the text [10] we introduced and analyzed the concept of co-ideals.

If R is a co-quasiorder relation on set $(A, =, \neq)$, then the axiom (2) in Definition 1.1 gives $(1, 1) \in R \subseteq \neq$ which is a contradiction. That is why we transformed this axiom into the next formula

(2') $(\forall x \in A)(x \neq 1 \Longrightarrow (x, 1) \in R)$.

Let $(A, =, \neq)$ be a set with apartness. A co-quasiordered residuated system is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where the axiom (2') is replaced by (2) and where R is a co-quasiorder on A.

Definition 2.1. ([9], Definition 2.1) A co-quasiordered residuated relational system is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \not\leqslant\rangle$, where $A = (A, =, \neq)$ is a set with apartness and where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and $\not\leqslant$ is a co-quasiorder relation on A and satisfying the following properties:

(1) $\langle A, \cdot, 1\rangle$ is a commutative monoid;

(2') $(\forall x \in A)(x \neq 1 \Longrightarrow x \not\leqslant 1)$;

(3) $(\forall x, y, z \in A)(x \cdot y \not\leqslant z \iff x \not\leqslant y \rightarrow z)$.

We will refer to the operation $' \cdot '$ as multiplication, to $' \rightarrow '$ as its residuum and to condition (3) as residuation.

Apart from the difference in the carrier of this constructed algebraic structure, the difference between the residuated relational system in our definition and the definition in texts [3, 4] is in the strong extensionality of the internal binary operations in A. Let us note that the internal operations $' \cdot '$ and $' \rightarrow '$ are total strongly extensional function from $A \times A$ into A:

$(\forall a, b, a', b' \in A)(a \cdot b \neq a' \cdot b' \Longrightarrow (a, b) \neq (a', b'))$,

$(\forall a, b, a', b' \in A)(a \rightarrow b \neq a' \rightarrow b' \Longrightarrow (a, b) \neq (a', b'))$.

Proposition 2.1. ([9], Proposition 2.3) Let $\mathfrak{A}$ be a co-quasiordered residuated relational system. Then

$$(\forall x, y \in A)(x \not\leqslant y \iff 1 \neq x \rightarrow y).$$

In the following theorem we shown that the co-quasiorder $' \not\leqslant '$ is compatible with the internal operation $' \cdot '$.

Theorem 2.1. ([9], Theorem 2.1) Let $\mathfrak{A}$ be a co-quasiordered residuated system. Then

$$(\forall x, y, a, b \in A)((a \cdot x \not\leqslant a \cdot y \vee x \cdot b \not\leqslant y \cdot b) \implies x \not\leqslant y).$$

In the following theorem we shown that the co-quasiorder $' \not\leqslant '$ is left compatible and right anti-compatible with the internal operation $' \rightarrow '$.

Theorem 2.2. ([9], Theorem 2.2) Let $\mathfrak{A}$ be a co-quasiordered residuated system. Then

(a) $(\forall x, y, a \in A)(a \to x \not\preceq a \to y \implies x \not\preceq y)$.

(b) $(\forall x, y, b \in A)(y \to b \not\preceq x \to b \implies x \not\preceq y)$.

Speaking by the language of classical algebra, when we speak of the compatibility of the internal binary operations $' \cdot '$ and $' \to '$ with the relation $' \not\preceq '$, we mean on the cancellativity of these operations with respect to $' \not\preceq '$.

The algebraic system ordered under co-quasiorder relation thus determined was in the focus of our forthcoming work [10], also.

3 Further developing the idea of co-filters

The following is valid

Lemma 3.1. Let $\langle A, \cdot, \to, 1, \not\preceq \rangle$ be a co-quasiordere resuated system. The relation $\not\preceq^{\lhd}$ is a quasi-order on the monoid $(A, \cdot)$ compatible with the internal operation in A.

Proof. As is known (see, for example [7], Lemma 2.1), $\not\preceq^{\lhd}$ is a quasi-order relation on the set A. Let $x, y, a, u, v \in A$ be arbitrary elements such that $x \not\preceq^{\lhd} y$ and $u \not\preceq v$. Then

$$u \not\preceq a \cdot x \vee a \cdot x \not\preceq a \cdot y \vee a \cdot y \not\preceq u$$

by co-transitivity of $\not\preceq$. Thus $u \neq a \cdot x \vee a \cdot y \neq v$ because the option $a \cdot x \not\preceq a \cdot y$ implies $x \not\preceq y$ by Theorem 2.1 and according to consistency of $\not\preceq$. So, we have $(a \cdot x, a \cdot y) \neq (u, v) \in \not\preceq$. This means $a \cdot x \not\preceq^{\lhd} a \cdot y$. Therefore, the relation $\not\preceq^{\lhd}$ is left compatible with the internal operation in A.
The implication of $x \not\preceq^{\lhd} y \implies x \cdot a \not\preceq^{\lhd} y \cdot a$ can be prove by analogy with the previous evidence. ■

Corollary 3.1. If $\not\preceq \cap \not\preceq^{-1} = \emptyset$, then

(4) $(\forall x \in A)(1 \not\preceq^{\lhd} x)$ and

(5) $(\forall x, y \in A)(x \not\preceq^{\lhd} x \cdot y$ and $y \not\preceq^{\lhd} x \cdot y)$.

Proof. Let $x, u, v \in A$ be arbitrary elements such that $u \not\preceq v$ and $x \neq 1$. Then $x \not\preceq 1$ by (2') and

$$u \not\preceq v \implies (u \not\preceq 1 \vee 1 \not\preceq x \vee x \not\preceq v).$$

Since the second option is impossible because $x \not\preceq 1$ and $\not\preceq \cap \not\preceq^{-1} = \emptyset$, we have $(1, x) \neq (u, v) \in \not\preceq$. So, it means $1 \not\preceq^{\lhd} x$.
Since $1 \not\preceq^{\lhd} x$ by the first evidence of this proof, it follows $1 \cdot y \not\preceq^{\lhd} x \cdot y$ by Lemma 3.1. So, $y \not\preceq^{\lhd} x \cdot y$ holds. The claim $y \not\preceq^{\lhd} x \cdot y$ can prove by analogy to the previous claim. ■

It should be noted here that the condition $\not\preceq \cap \not\preceq^{-1} = \emptyset$ is not always satisfied. In what follows, we will always assume that this condition is fulfilled.

It is shown in [9], Proposition 2.1, that condition (3) implies condition

(6) $(\forall x, yz \in A)(x \cdot y \not\preceq^{\lhd} z \iff x \not\preceq^{\lhd} y \to z)$.

Naturally, the reverse implication does not valid in general case.

In our forthcoming article [10], Proposition 5, is proven.

Proposition 3.1. Classes $L_{\not\preceq}(a) = \{y \in A : a \not\preceq y\}$ $(a \in A)$ are strongly extensional subsets of A such that $a \lhd L_{\not\preceq}(a)$, $1 \in L_{\not\preceq}(a)$ and following formula is valid

(L) $(\forall u, v \in A)(v \in L_{\not\preceq}(a) \implies (u \not\preceq v \vee u \in L_{\not\preceq}(a)))$.

In addition, these left classes of the relation $\not\preceq$ have the following properties:

Proposition 3.2. Let $\langle A, \cdot, \to, 1, \not\preceq \rangle$ be a co-quasiordere resuated system with $\not\preceq \cap \not\preceq^{-1} = \emptyset$ and $a, b \in A$. Then

(7) $(\forall x, y \in A)(x \cdot y \in L_{\not\preceq}(a) \implies (x \in L_{\not\preceq}(a) \wedge y \in L_{\not\preceq}(a)))$;

(8) $(\forall x, y \in A)(x \not\preceq y \implies x \to y \in L_{\not\preceq}(a))$;

(9) $L_{\not\preceq}(a) \cup L_{\not\preceq}(b) \subseteq L_{\not\preceq}(a \cdot b)$.

Proof. (7) Let $x, y \in A$ be arbitrary elements such that $x \cdot y \in L_{\not\preccurlyeq}(a)$. Then $a \not\preccurlyeq x \cdot y$. Thus $a \not\preccurlyeq x \vee x \not\preccurlyeq x \cdot y$ and $a \not\preccurlyeq y \vee y \not\preccurlyeq x \cdot y$ by co-transitivity of $\not\preccurlyeq$. Since the second option is impossible by (5), we have $x \in L_{\not\preccurlyeq}(a)$ and $y \in L_{\not\preccurlyeq}(a)$.
(8) Let $x, y \in A$ arbitrary elements such that $x \not\preccurlyeq y$. Then $x \not\preccurlyeq a \cdot x \vee a \cdot x \not\preccurlyeq y$ by co-transitivity of $\not\preccurlyeq$. Thus $a \cdot x \not\preccurlyeq y$ because the first option is impossible by (5). So, $a \not\preccurlyeq x \to y$ by (3). Therefore, $x \to y \in L_{\not\preccurlyeq}(a)$.
(9) If $t \in L_{\not\preccurlyeq}(a)$, then $a \not\preccurlyeq t$. Thus $a \not\preccurlyeq a \cdot b \vee a \cdot b \not\preccurlyeq y$. So, we have $t \in L_{\not\preccurlyeq}(a \cdot b)$ by (5). From this follows $L_{\not\preccurlyeq}(a) \cup L_{\not\preccurlyeq}(b) \subseteq L_{\not\preccurlyeq}(a \cdot b)$ immediately. ■

Corollary 3.2. Let $\langle A, \cdot, \to, 1, \not\preccurlyeq\rangle$ be a co-quasiordere resuated system with $\not\preccurlyeq \cap \not\preccurlyeq^{-1} = \emptyset$ and $a \in A$. Then

(10) $(\forall x, y \in A)(y \in L_{\not\preccurlyeq}(a) \implies (x \to y \in L_{\not\preccurlyeq}(a) \vee x \in L_{\not\preccurlyeq}(a)))$.

Proof. Let $x, y \in A$ be arbitrary elements such that $y \in L_{\not\preccurlyeq}(a)$. Then $x \not\preccurlyeq y \vee x \in L_{\not\preccurlyeq}(a)$ by (L). Thus $x \to y \in L_{\not\preccurlyeq}(a) \vee x \in L_{\not\preccurlyeq}(a)$ by (8). ■

In the article [9], we have developed the idea of co-filters in these algebraic systems. In addition, we have shown some of the significant features of these substructures in a residuated relational system ordered under a co-quasiorder.

Definition 3.1. ([9], Definition 2.2) A subset G of A is a co-filter of a residuated system $\mathfrak{A}$ ordered under a co-quasiorder $\not\preccurlyeq$ if the following conditions hold

(G1) $(\forall x, y \in A)(x \cdot y \in G \Longrightarrow x \in G \ \vee \ y \in G)$;

(G2) $(\forall x, y \in A)(y \in G \Longrightarrow (x \not\preccurlyeq y \vee x \in G)$.

Condition (G1) speaks that a co-filter G is a co-subgroupoid in $(A, \cdot)$.

Lemma 3.2. ([9]) Any co-filter G of a co-quasiordered residuated system $\mathfrak{A}$ is a strongly extensional subset in A.

Our first theorem correlate condition (G2) to condition (G1).

Theorem 3.1. Let $\mathfrak{A}$ be a co-quasiordered residuated system and G be a co-filter in $\mathfrak{A}$. Then $(G2) \implies (G1)$.

Proof. Let $x, y \in A$ be arbitrary elements such that $x \cdot y \in G$. Then $x \not\preccurlyeq x \cdot y \vee x \in G$ by (G2). Since the first option is impossible by (5), we have $x \in G$. The second part $x \cdot y \in G \implies y \in G$ of the proof of this theorem can be obtained analogously to the first part. ■

Corollary 3.3. Any co-filter G of a co-quasiordered residuated system $\mathfrak{A} = \langle A, \cdot, 1, \to, \not\preccurlyeq\rangle$ is a consistent subset in A.

Corollary 3.4. If G is a non empty co-filter in a co-quasiordered residuated system $\mathfrak{A}$, then $1 \in G$.

Theorem 3.2. Let A be a co-quasiordered residuated system and G be a subset of A. Then the condition (G2) is equivalent to the condition

(G3) $(\forall x, y, z \in A)(z \in G \implies (x \not\preccurlyeq y \to z \vee x \cdot y \in G))$.

Proof. $(G2) \implies (G3)$: Suppose (G2) holds and let $x, y, z \in A$ be arbitrary element such that $z \in G$. Then $z \in G \implies (x \cdot y \not\preccurlyeq z \vee x \cdot y \in G)$. Thus $x \not\preccurlyeq y \to z \vee x \cdot y \in G$ by (3). So, the condition (G3) is proven.
$(G3) \implies (G2)$. Opposite, let the condition (G3) be a valid formula in $\mathfrak{A}$ and let $x, y \in A$ be arbitrary elements such that $y \in G$. Then $y \in G \implies (x \not\preccurlyeq 1 \to y \vee x \cdot 1 \in G$ by (G3) where we put $z = 1$. Thus $y \in G \implies (x \not\preccurlyeq y \vee x \in G)$ by (1) and (3). So, the condition (G2) is a valid formula in $\mathfrak{A}$. ■

Subsets $L_{\not\preccurlyeq}(a)$ $(a \in A)$ are co-filters in a residuated relational system $\mathfrak{A}$ ordered under a co-quasiorder $\not\preccurlyeq$ according to (L) and (7), Therefore, the family $\mathfrak{G}(A)$ of all co-filters in $\mathfrak{A}$ is not empty.

Theorem 3.3. The family $\mathfrak{G}(A)$ of all co-filters of a co-quasiordered residuated system $\mathfrak{A}$ forms a complete lattice.

Proof. (i) Let $x, y \in A$ be arbitrary elements. Thus

$$\begin{aligned} y \in \bigcup \mathfrak{G} &\Longleftrightarrow (\exists G \in \mathfrak{G})(y \in G) \\ &\Longrightarrow (\exists G \in \mathfrak{G})(x \not\leqslant y \ \vee \ x \in G) \\ &\Longrightarrow x \not\leqslant y \ \vee \ x \in \bigcup \mathfrak{G}. \end{aligned}$$

(ii) Let $\mathfrak{B}$ be the families of all co-ideals contained in $\bigcap \mathfrak{G}$. Then $\bigcup \mathfrak{B}$ is the maximal co-filter contained in $\bigcap \mathfrak{G}$, according to the first part of this evidence.
(iii) If we put $\sqcup \mathfrak{G} = \bigcup \mathfrak{G}$ and $\sqcap \mathfrak{G} = \bigcup \mathfrak{B}$, then $(\mathfrak{G}(A), \sqcup, \sqcap)$ is a complete lattice. ■

Corollary 3.5. For each subset B of A, there is the maximal co-filter of $\mathfrak{A}$ contained in B.

Corollary 3.6. For elements $a_1, ..., a_n \in A$, there is the maximal co-filter K of $\mathfrak{A}$ such that $a_1 \lhd K$, ..., $a_n \lhd K$.

If T is a subset of A, then $\bigcup_{t \in T} L_{\not\leqslant}(t)$ is a co-filter in $\mathfrak{A}$, by Theorem 3.3. We call such a co-filter a *normal* co-filter. We will write $T^U = \bigcup_{t \in T} L_{\not\leqslant}(t)$ in this case.

Proposition 3.3. Let $\mathfrak{A}$ be a co-quasiordered residuated system. Then the union of any family of normal co-filters in $\mathfrak{A}$ is a normal co-filter in $\mathfrak{A}$.

Proof. The assertion of this proposition is a direct consequence of the following equality $(\bigcup_{i \in I} T_i)^U = \bigcup_{i \in I} T_i^U$. ■

Corollary 3.7. The family of all normal co-filters in $\mathfrak{A}$ forms join semi-lattice.
However, the intersection of two normal co-filters is not a co-filter in the general case.

In the following proposition we give one upper measure for a non-empty co-filter.

Proposition 3.4. For any non empty co-filter G in a co-quasiordered residuated system $\mathfrak{A}$ the following $G \subseteq \bigcup_{a \lhd G} L_{\not\leqslant}(a)$ holds.

Proof. Let $a \in A$ be an arbitrary element such that $a \lhd G$. Then from $t \in G$ follows $a \not\leqslant t \vee a \in G$ by (G2). Since the second option is impossible by hypothesis, we have $t \in L_{\not\leqslant}(a)$. Thus $G \subseteq \bigcup_{a \lhd G} L_{\not\leqslant}(a)$.

■

In order to offer one lower measure of a co-filter in a co-quasiprdered residuated system $\mathfrak{A}$, we need the notion of right class $R_{\not\leqslant}(b)$ of relation $\not\leqslant$ generated by the element $b \in A$: $R(b) = \{x \in A : x \not\leqslant b\}$.

Proposition 3.5. For any non empty co-filter G in a co-quasiordered residuated system $\mathfrak{A}$ the following $\bigcup_{b \in G} R(b)^{\lhd} \subseteq G$ holds.

Proof. Let $t \in A$ be an arbitrary element such that $t \in \bigcup_{b \in G} R(b)^{\lhd}$. Then there exists an element $b \in G$ such that $t \lhd R(b)$. Thus from (G2): $b \in G \Longrightarrow (t \not\leqslant b \vee y \in G)$ follows $t \in G$ because $\neg(t \not\leqslant b)$ by the hypothesis. Therefore, we have $\bigcup_{b \in G} R(b)^{\lhd} \subseteq G$. ■

4 Final reflection

Bishop's constructive mathematics includes the following two aspects:
(1) The Intuitionistic logic and
(2) The principled-philosophical orientations of constructivism.
Intuitionistic logic does not accept the TND principle as an axiom. In addition, Intuitionistic logic does not accept the validity of the 'double negation' principle. This makes it possible to have a difference relation in sets which is not a negation of the equality relation. Therefore, we accept that in Bishop's constructive mathematics we consider set S as one relational system $(S, =, \neq)$. In Bishop's constructive algebra we always encounter the following two problems:

(a) How to choose a predicate (or more predicates) between several classically equivalent ones by which an algebraic concept is determined.

(b) Since every predicate has at least one of its duals, how to construct a dual of the algebraic concept defined with a given predicate(s).

In this case, we are faced with the problem of describing a residuated relational system based on a set with apartness as the carrier for constructing an algebraic structure. By our orientation that in this construction, groupoid $(A, \cdot)$ is ordered under a a co-quasiorder relation instead of a quasi-order relation, a significantly different logical-sets framework is formed. In addition to the above, in this report we have described some of the important features of a class of substructures (in this case - the class of co-filers) in residuated relational systems constructed on sets with apartness in which both internal binary operations are strongly extensional functions.

The problem encountered by authors working within Bishop's constructive framework is that when developing concepts of new ideas and defining their interrelationships with respect to the permissible rules of conclusion in [**IL**], they must always strive for the results obtained to be correlated with the corresponding results that exist or can be obtained in the classical case.

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