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# Existence and stability of a nonlinear fractional differential equation involving a $\psi$ -Caputo operator

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### Abstract

This paper is devoted to the study of the existence and interval of existence, uniqueness of solutions and estimates on solutions of the nonlocal Cauchy problem for nonlinear fractional differential equations involving a Caputo type fractional derivative with respect to another function  $\psi$ . Further, we prove four different types of Ulam stability results of solutions for a given problem. The used tools in this article are the classical technique of Banach fixed point theorem and generalized Gronwall inequality in frame the  $\psi$ -fractional integral. At the end, illustrative examples are presented.

**Keywords:** Fractional differential equations,  $\psi$ -fractional derivative and  $\psi$ -fractional integral, Existence and stability, Fixed point theorem.

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### 1. Introduction

Fractional calculus is a branch of mathematics that studies the differentiation and integration to an arbitrary real or complex order. It has played a significant role in various branches of science such as physics, biology, chemistry, electrical networks, control of dynamic systems, engineering (see, for instance, [13, 14, 15, 16]). Recently, several researchers have tried to propose different types of fractional operators that deal with derivatives and integrals of arbitrary orders and their applications, for more details, (see [6, 8, 9, 13, 19]). For example, Kilbas et al. in [13] introduced the properties of fractional integrals and fractional derivatives

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with respect to another function. Some of generalized fractional integral and differential operators and their properties were introduced by Agrawal in [6]. Very recently, Almeida in [8] presented a new type of fractional differentiation operator so-called  $\psi$ -Caputo fractional operator and extended work of the Caputo [13, 16]. Almeida et al. in [9] investigated the existence and uniqueness results of nonlinear fractional differential equations involving a Caputo-type fractional derivative with respect to another function by means of fixed point theorem and Picard iteration method. Jarad and Abdeljawad in [12] established generalized Laplace transform of  $\psi$ -fractional integral and  $\psi$ -fractional derivatives and applied this transform to solve some ordinary differential equations with  $\psi$ -fractional derivative. On the other hand, Sousa and Oliveira in [20] established generalized Gronwall inequality through the fractional integral with respect to another function and studied the existence, uniqueness and data dependence of solution of Cauchy-type problem with  $\psi$ -Hilfer fractional derivative. We mention here some recent works which dealt with the existence, uniqueness and stability of solutions of fractional differential equations (see [1, 2, 3, 4, 5, 7, 11, 17, 18, 20, 21, 22, 23]) and references therein.

In this paper, we investigate the existence and interval of existence, uniqueness, estimates on solutions, and different types of Ulam stability results of solutions on a subinterval of  $[0, b]$  for the following nonlinear fractional differential equation involving generalized Caputo fractional derivatives with respect to function  $\psi$

$${}^c D_{a^+}^{\alpha, \psi} u(t) = f(t, u(t)), \quad t \in J = [0, b], \tag{1}$$

with the nonlocal condition

$$u(0) + g(u) = u_0, \tag{2}$$

where  $0 < \alpha < 1$ ,  $u_0 \in \mathbb{R}$ ,  ${}^c D_{a^+}^{\alpha, \psi}$  denotes the  $\psi$ -Caputo fractional derivative of order  $\alpha$ ,  $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : C(J, \mathbb{R}) \rightarrow \mathbb{R}$  are nonlinear continuous functions satisfies some assumptions that will be specified in Section 3, and  $u \in C(J, \mathbb{R})$  such that the operator  ${}^c D_{a^+}^{\alpha, \psi}$  exists and  ${}^c D_{a^+}^{\alpha, \psi} u \in C(J, \mathbb{R})$ .

Organization of the paper to make it self-contained. In Section 2, we present some preliminary results from fractional calculus and functional analysis, which will be employed throughout this paper. Section 3 and Section 4 are devoted to the study of existence, uniqueness of solution and estimates on solutions of the equations (1)–(2). In Section 5, we discuss different types of stability results of solutions to equations (1)–(2). As an application of our main results, illustrative examples are given in the last section.

## 2. Preliminaries

Let  $J = [0, b]$  be a finite interval of the real line  $\mathbb{R}$ .  $C(J, \mathbb{R})$  be the Banach space of continuous real function  $h$  with the norm  $\|h\| = \max\{|h(t)| : t \in J\}$ .  $C^n(J, \mathbb{R})$  be the Banach space of  $n$ -times continuously differentiable functions on  $J$ . By  $L^p(J, \mathbb{R})$   $p \geq 1$ , we denote the Banach space of all measurable functions  $h : J \rightarrow \mathbb{R}$  with the norm

$$\|h\|_{L^p} = \left( \int_0^b |h(t)|^p dt \right)^{\frac{1}{p}}.$$

**Definition 2.1.** [13]. Let  $\alpha > 0$  and  $\psi$  be an increasing and positive monotone function on  $J$ , having a continuous derivative  $\psi'$  on  $J$ . Then the left-sided  $\psi$ -Riemann-Liouville fractional integral of order  $\alpha$  for an integrable function  $h : J \rightarrow \mathbb{R}$  is defined by

$$I_{0^+}^{\alpha, \psi} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} h(s) ds, \quad t > 0. \tag{3}$$

**Definition 2.2.** [13]. Let  $\alpha > 0$  is a real,  $h : J \rightarrow \mathbb{R}$  an integrable function and  $\psi \in C^n(J, \mathbb{R})$  an increasing function such that  $\psi'(t) \neq 0$  for all  $t \in J$ . Then the left-sided  $\psi$ -Riemann-Liouville fractional derivative of  $h$  of order  $\alpha$  is defined as follows:

$$D_{0^+}^{\alpha, \psi} h(t) = \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{0^+}^{n-\alpha, \psi} h(t), \quad t > 0.$$

where  $n = [\alpha] + 1$ , and  $[\alpha]$  denotes the integer part of the real number  $\alpha$ .

**Definition 2.3.** [9]. Let  $\alpha > 0$  and  $\psi \in C^n(J, \mathbb{R})$  be a function such that  $\psi$  is increasing and  $\psi'(t) \neq 0$ , for all  $t \in J$ . Given  $h \in C^{n-1}(J, \mathbb{R})$ , the  $\psi$ -Caputo fractional derivative of  $h$  of order  $\alpha$  is defined by

$$({}^c D_{0^+}^{\alpha, \psi} h)(t) = D_{0^+}^{\alpha, \psi} \left[ h(t) - \sum_{k=0}^{n-1} \frac{h_{\psi}^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k \right],$$

where  $n = [\alpha] + 1$  for  $\alpha \notin \mathbb{N}$ ,  $\alpha = n$  for  $\alpha \in \mathbb{N}$ , and  $h_{\psi}^{[k]}(t) = \left[ \frac{1}{\psi'(t)} \frac{d}{dt} \right]^k h(t)$ . In particular, given  $\alpha \in (0, 1)$ , we have

$${}^c D_{0^+}^{\alpha, \psi} h(t) = D_{0^+}^{\alpha, \psi} [h(t) - h(0)].$$

Further, if  $h \in C^n(J, \mathbb{R})$ , the  $\psi$ -Caputo fractional derivative of  $h$  of order  $\alpha$  can be represented by the expression

$${}^c D_{0^+}^{\alpha, \psi} h(t) = I_{0^+}^{n-\alpha; \psi} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n h(t) \quad t > 0.$$

Thus, if  $\alpha = n \in \mathbb{N}$ , we have  ${}^c D_{0^+}^{\alpha, \psi} h(t) = h_{\psi}^{[n]}(t)$ .

For more details on the  $\psi$ -Caputo fractional derivative, we refer the reader to [8, 9].

**Lemma 2.1.** [9]. Let  $\alpha \in (n - 1, n)$ ,  $h \in C^{n-1}(J, \mathbb{R})$ , and  $h^{(n)}$  exists almost everywhere. Then we have

$$I_{0^+}^{\alpha, \psi} {}^c D_{0^+}^{\alpha, \psi} h(t) = h(t) - \sum_{k=0}^{n-1} \frac{h_{\psi}^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k.$$

In particular, given  $\alpha \in (0, 1)$ , we have  $I_{0^+}^{\alpha, \psi} {}^c D_{0^+}^{\alpha, \psi} h(t) = h(t) - h(0)$ .

**Lemma 2.2.** [3, 13]. Let  $\alpha > 0$ ,  $h \in C(J, \mathbb{R})$ , and  $\psi \in C^1(J, \mathbb{R})$ . Then for all  $t \in J$ ,

1.  $I_{0^+}^{\alpha; \psi}(\cdot)$  maps  $C(J, \mathbb{R})$  into  $C(J, \mathbb{R})$ .
2.  $I_{0^+}^{\alpha; \psi} h(0) = \lim_{t \rightarrow 0^+} I_{0^+}^{\alpha; \psi} h(t) = 0$ .

**Lemma 2.3.** [8],[13]. Let  $\alpha, \beta > 0$ . Then we have

1.  $I_{0^+}^{\alpha, \psi} (\psi(t) - \psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} [\psi(t) - \psi(0)]^{\alpha+\beta-1}$ .
2.  ${}^c D_{0^+}^{\alpha, \psi} (\psi(t) - \psi(0))^k = 0, \forall k \in \{0, 1, \dots, n - 1\}, n \in \mathbb{N}$ .

**Lemma 2.4.** [2]. Let  $\alpha > 0$  and  $\beta > 0$ . Then the relation

$$I_{0^+}^{\alpha, \psi} I_{0^+}^{\beta, \psi} h(t) = I_{0^+}^{\alpha+\beta, \psi} h(t)$$

holds almost everywhere for  $t \in J$ , for  $h \in L^p(J, \mathbb{R})$  and  $p \geq 1$ . If  $\alpha + \beta > 1$ , then the relation holds at any point of  $J$ .

**Lemma 2.5.** [2]. Let  $0 < \alpha < 1$ ,  $\psi \in C^1(J, \mathbb{R})$ , and  $h \in C(J, \mathbb{R})$ . Then

$${}^c D_{0^+}^{\alpha, \psi} I_{0^+}^{\alpha, \psi} h(t) = h(t), \text{ a.e.}$$

**Theorem 2.1.** [10]. (Banach Fixed Point Theorem) Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  be a contraction mapping. Then  $T$  has a unique fixed point in  $X$ .

**Lemma 2.6.** [20]. (Generalized Gronwall’s Inequality) Let  $x, y$ , be two integrable functions and  $z$  continuous, with domain  $J$ . Let  $\psi \in C(J, \mathbb{R})$  an increasing function such that  $\psi'(t) \neq 0, \forall t \in J$ . Assume that  $x$  and  $y$  are nonnegative and  $z$  is nonnegative and nondecreasing. If

$$x(t) \leq y(t) + z(t) \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} x(s) ds,$$

then, for all  $t \in J$ , we have

$$x(t) \leq y(t) + \int_0^t \sum_{k=1}^{\infty} \frac{[z(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(s)(\psi(t) - \psi(s))^{\alpha k-1} y(s) ds.$$

**Remark 2.1.** Under the assumptions of Lemma 2.6, let  $y(t)$  be a nondecreasing function on  $\Omega$ . Then we have

$$x(t) \leq y(t) E_{\alpha}(z(t)\Gamma(\alpha)(\psi(t) - \psi(0))^{\alpha}),$$

where  $E_{\alpha}(\cdot)$  is the Mittag-Leffler function defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad z \in \mathbb{C}. \tag{4}$$

### 3. Existence Results and Interval of Existence

In this partition, we prove the existence of a subinterval where the solution of the given problem exists and unique. We first present the following important result through which we can prove our major results.

**Corollary 3.1.** Let  $0 < \alpha < 1$ ,  $\psi \in C^1(J, \mathbb{R})$  and  $h : J \rightarrow \mathbb{R}$  is a continuous function. Then the Cauchy type problem for a fractional differential equation

$$\begin{cases} {}^c D_{0+}^{\alpha, \psi} u(t) = h(t), \text{ a.e. } t \in J, \\ u(0) = u_0. \end{cases} \tag{5}$$

is equivalent to the fractional integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} h(s) ds, \quad t \in J. \tag{6}$$

*Proof.* Let  $u(t)$  be a solution of the problem (5). From Lemma 2.1, we have

$${}^c D_{0+}^{\alpha, \psi} u(t) = D_{0+}^{\alpha, \psi} [u(t) - u(0)],$$

It follows from the first equation of (5) that

$$h(t) = D_{0+}^{\alpha, \psi} [u(t) - u_0] = D_{0+}^{1, \psi} I_{0+}^{1-\alpha, \psi} [u(t) - u_0],$$

which implies

$$I_{0+}^{1, \psi} h(t) = I_{0+}^{1-\alpha, \psi} [u(t) - u_0]. \tag{7}$$

Operating  $\psi$ -Riemann-Liouville fractional differential operator  $D_{0+}^{1-\alpha, \psi}$  on both sides of (7), we obtain

$$\begin{aligned} u(t) &= u_0 + D_{0+}^{1-\alpha, \psi} I_{0+}^{1, \psi} h(t) \\ &= u_0 + D_{0+}^{1, \psi} I_{0+}^{\alpha, \psi} I_{0+}^{1, \psi} h(t) \\ &= u_0 + D_{0+}^{1, \psi} I_{0+}^{1+\alpha, \psi} h(t) \\ &= u_0 + I_{0+}^{\alpha, \psi} h(t), \end{aligned}$$

which shows that the equation (6) is satisfied.

On the other hand, suppose that  $u(t)$  is the solution of the fractional integral equation (6). Then, it can be written as

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} h(s) ds, \quad t \in J. \tag{8}$$

Applying the  $\psi$ -Caputo fractional derivative operator  ${}^c D_{0+}^{\alpha, \psi}$  on both sides of the equation (8), and using Lemmas 2.3, 2.5, we get

$$\begin{aligned} {}^c D_{0+}^{\alpha, \psi} u(t) &= {}^c D_{0+}^{\alpha, \psi} I_{0+}^{\alpha, \psi} h(t) \\ &= h(t), \quad a.e. \ t \in J. \end{aligned}$$

Now, taking the limit  $t \rightarrow 0^+$  of the equation (8) and using Lemma 2.2, we obtain

$$u(0) = u_0.$$

□

**Corollary 3.2.** *Let  $0 < \alpha < 1$ ,  $f : J \times \mathbb{R} \rightarrow \mathbb{R}$  and  $g : C(J, \mathbb{R}) \rightarrow \mathbb{R}$  are continuous functions. Then the solution  $u$  of the problem (1)–(2) is given by*

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s)) ds. \tag{9}$$

Next, we introduce the following assumptions:

(H<sub>1</sub>)  $f : J \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous function and there exist  $L_1 \in (0, +\infty)$  such that,

$$|f(t, x) - f(t, y)| \leq L_1 |x - y|, \quad \forall t \in J \text{ and } x, y \in \mathbb{R}.$$

(H<sub>2</sub>) The nonlocal term  $g : C(J, \mathbb{R}) \rightarrow \mathbb{R}$  is a given function, and there exists  $L_2 \in (0, 1)$  such that

$$|g(x) - g(y)| \leq L_2 |x - y|, \quad \forall x, y \in C(J, \mathbb{R}).$$

(H<sub>3</sub>)  $\psi \in C^1(J, \mathbb{R})$  an increasing bijective function and there exists  $0 < \chi < b$  such that

$$\chi < \min \left\{ b, \psi^{-1} \left[ \psi(0) + \left( \frac{\Gamma(\alpha + 1)}{L_2 + L_1} \right)^{\frac{1}{\alpha}} \right] \right\}. \tag{10}$$

**Theorem 3.1.** *Assume that (H<sub>1</sub>), (H<sub>2</sub>) and (H<sub>3</sub>) are satisfied. Then the problem (1)–(2) has a local unique solution on the subinterval  $[0, \chi] \subset J$ .*

*Proof.* Define the operator  $T : C([0, \chi], \mathbb{R}) \rightarrow C([0, \chi], \mathbb{R})$  as follows

$$(Tu)(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s)) ds.$$

For each  $t \in [0, \chi]$  and for any  $u, u^* \in C([0, \chi], \mathbb{R})$ , we have

$$\begin{aligned} &|Tu(t) - Tu^*(t)| \\ &\leq |g(u) - g(u^*)| + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \\ &\quad \times |f(s, u(s)) - f(s, u^*(s))| ds \\ &\leq L_2 \|u - u^*\| + \frac{L_1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \|u(s) - u^*(s)\| ds \\ &\leq (L_2 + L_1) \|u - u^*\| \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} ds \\ &= (L_2 + L_1) \|u - u^*\| \frac{(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha + 1)}. \end{aligned} \tag{11}$$

Therefore,

$$\|Tu - Tu^*\| \leq \frac{(L_2 + L_1)}{\Gamma(\alpha + 1)} [\psi(\chi) - \psi(0)]^\alpha \|u - u^*\|.$$

in view of the assumption  $(H_3)$ , we have

$$\chi < \psi^{-1} \left[ \psi(0) + \left( \frac{\Gamma(\alpha + 1)}{L_2 + L_1} \right)^{\frac{1}{\alpha}} \right]. \quad (12)$$

Additionally, since  $\psi \in C^1(J, \mathbb{R})$  is a bijective function,  $\psi^{-1} : \mathbb{R} \rightarrow J$  exists and from the inequality (12), we have

$$(\psi(\chi) - \psi(0))^\alpha < \left( \frac{\Gamma(\alpha + 1)}{L_2 + L_1} \right),$$

which implies

$$\frac{(L_2 + L_1)}{\Gamma(\alpha + 1)} [\psi(\chi) - \psi(0)]^\alpha < 1.$$

Therefore,

$$\|Tu - Tu^*\| \leq \sigma \|u - u^*\|, \quad 0 < \sigma < 1,$$

where

$$\sigma := \frac{(L_2 + L_1)}{\Gamma(\alpha + 1)} [\psi(\chi) - \psi(0)]^\alpha.$$

This shows that  $T$  is a contraction mapping in  $C([0, \chi], \mathbb{R})$ . Hence, we can use the Banach fixed point Theorem 2.1 to get a local unique solution  $u \in C([0, \chi], \mathbb{R})$ . The proof is completed.  $\square$

#### 4. Estimates on the solutions

**Theorem 4.1.** Assume that  $f : J \times \mathbb{R} \rightarrow \mathbb{R}$  and  $g : C(J, \mathbb{R}) \rightarrow \mathbb{R}$  satisfy the hypotheses  $(H_1)$  and  $(H_2)$ , let  $\psi$  be given as in Definition 2.3. If  $u(t)$ ,  $t \in J$  any solution of the problem (1)-(2), then

$$|u(t)| \leq \frac{1}{1 - L_2} \left[ |u_0| + Q + \frac{[\psi(b) - \psi(0)]^\alpha}{\Gamma(\alpha + 1)} \rho \right] E_\alpha \left( \frac{L_1}{(1 - L_2)} (\psi(b) - \psi(0))^\alpha \right), \quad (13)$$

where  $\rho = \sup\{|f(t, 0)| : t \in J\}$ ,  $Q = |g(0)|$  and  $E_\alpha(\cdot)$  is the Mittag-Leffler function defined by (4).

*Proof.* In view of Corollary 3.2, we have

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s, u(s)) ds.$$

Thus, by assumptions  $(H_1)$ ,  $(H_2)$  and for any  $t \in [0, b]$ , we have

$$\begin{aligned} |u(t)| &\leq |u_0| + |g(u) - g(0)| + |g(0)| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \\ &\quad \times (|f(s, u(s)) - f(s, 0)| + \sup_{s \in [0, b]} |f(s, 0)|) ds \\ &\leq |u_0| + L_2 |u(t)| + Q + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} \\ &\quad \times (L_1 |u(s)| + \rho) ds, \end{aligned}$$

which implies

$$|u(t)| \leq \frac{1}{1-L_2} \left[ |u_0| + Q + \frac{[\psi(b) - \psi(0)]^\alpha}{\Gamma(\alpha + 1)} \rho \right] + \frac{L_1}{(1-L_2)\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |u(s)| ds$$

Applying Lemma 2.6, we obtain

$$\begin{aligned} |u(t)| &\leq \frac{1}{1-L_2} \left[ |u_0| + Q + \frac{[\psi(b) - \psi(0)]^\alpha}{\Gamma(\alpha + 1)} \rho \right] \\ &\times \left[ 1 + \int_0^t \sum_{k=1}^\infty \frac{\left[ \frac{L_1}{(1-L_2)} \right]^k}{\Gamma(\alpha k)} \psi'(s)(\psi(t) - \psi(s))^{\alpha k - 1} ds \right] \\ &= \frac{1}{1-L_2} \left[ |u_0| + Q + \frac{[\psi(b) - \psi(0)]^\alpha}{\Gamma(\alpha + 1)} \rho \right] \\ &\times \left[ 1 + \sum_{k=1}^\infty \frac{\left[ \frac{L_1}{(1-L_2)} (\psi(b) - \psi(0))^\alpha \right]^k}{\Gamma(\alpha k + 1)} \right] \\ &= \frac{1}{1-L_2} \left[ |u_0| + Q + \frac{[\psi(b) - \psi(0)]^\alpha}{\Gamma(\alpha + 1)} \rho \right] E_\alpha \left( \frac{L_1}{(1-L_2)} (\psi(b) - \psi(0))^\alpha \right). \end{aligned}$$

The proof is complete. □

### 5. Stability Analysis

In this section, we investigate the UH stability, GUH stability, UHR stability, and GUHR stability of solutions to the fractional differential equation involving Caputo derivative operator with respect to another function  $\psi$ .

**Definition 5.1.** *The problem (1)-(2) is UH stable if there exists a real number  $\lambda_f > 0$  such that for every  $\epsilon > 0$  and for each solution  $\tilde{u} \in C([0, \chi], \mathbb{R})$  of the following inequality*

$$\left| {}^c D_{a^+}^{\alpha; \psi} \tilde{u}(t) - f(t, \tilde{u}(t)) \right| \leq \epsilon, \tag{14}$$

*there exists a solution  $u \in C([0, \chi], \mathbb{R})$  satisfying the problem (1)-(2) with*

$$|\tilde{u}(t) - u(t)| \leq \lambda_f \epsilon, \quad t \in [0, \chi].$$

**Definition 5.2.** *The problem (1)-(2) is GUH stable if there exists  $\varphi_f \in C(\mathbb{R}^+, \mathbb{R}^+)$  with  $\varphi_f(0) = 0$  such that for each  $\epsilon > 0$  and for each solution  $\tilde{u} \in C([0, \chi], \mathbb{R})$  of the inequality (14) there exists a solution  $u \in C([0, \chi], \mathbb{R})$  of problem (1)-(2) with*

$$|\tilde{u}(t) - u(t)| \leq \varphi_f(\epsilon), \quad t \in [0, \chi].$$

**Definition 5.3.** *The problem (1)-(2) is UHR stable with respect to  $\varphi \in C([0, \chi], \mathbb{R})$ , if there exists a real number  $C_{f, \varphi} > 0$  such that for each  $\epsilon > 0$  and for each solution  $\tilde{u} \in C([0, \chi], \mathbb{R})$  of the inequality*

$$|{}^c D_{a^+}^{\alpha; \psi} \tilde{u}(t) - f(t, \tilde{u}(t))| \leq \epsilon \varphi(t), \quad t \in [0, \chi], \tag{15}$$

*there exists a solution  $u \in C([0, \chi], \mathbb{R})$  of problem (1)-(2) such that*

$$|\tilde{u}(t) - u(t)| \leq C_{f, \varphi} \epsilon \varphi(t), \quad t \in [0, \chi].$$

**Definition 5.4.** The problem (1)-(2) is GUHR stable with respect to  $\varphi \in C([0, \chi], \mathbb{R})$ , if there exists a real number  $C_{f,\varphi} > 0$  such that for each solution  $\tilde{u} \in C([0, \chi], \mathbb{R})$  of the inequality

$$|{}^c D_{a^+}^{\alpha;\psi} \tilde{u}(t) - f(t, \tilde{u}(t))| \leq \varphi(t), \quad t \in [0, \chi], \tag{16}$$

there exists a solution  $u \in C([0, \chi], \mathbb{R})$  of problem (1)-(2) such that

$$|\tilde{u}(t) - u(t)| \leq C_{f,\varphi} \varphi(t), \quad t \in [0, \chi].$$

**Remark 5.1.** A function  $\tilde{u} \in C([0, \chi], \mathbb{R})$  is a solution of the inequality (14) if and only if there exist a function  $h \in C([0, \chi], \mathbb{R})$  (where  $h$  depends on solution  $\tilde{u}$ ) such that

$$(i) |h(t)| \leq \epsilon \text{ for all } t \in [0, \chi], \quad (ii) {}^c D_{a^+}^{\alpha;\psi} \tilde{u}(t) = f(t, \tilde{u}(t)) + h(t), \quad t \in [0, \chi].$$

**Theorem 5.1.** Under the assumptions of Theorem 3.1. If the inequality (14) is satisfied, then the problem (1)-(2) is UH stable in  $C([0, \chi], \mathbb{R})$ .

*Proof.* Let  $\epsilon > 0$  and let  $\tilde{u} \in C([0, \chi], \mathbb{R})$  be a function which satisfies the inequality (14), and let  $u \in C([0, \chi], \mathbb{R})$  be a local unique solution of the following problem

$${}^c D_{a^+}^{\alpha;\psi} u(t) = f(t, u(t)), \quad t \in [0, \eta], \tag{17}$$

$$u(0) + g(u) = \tilde{u}(0) + g(\tilde{u}) = u_0 \tag{18}$$

In view of Theorem 3.1, we have

$$u(t) = \mathcal{A}_u + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s)) ds, \quad t \in [0, \chi].$$

where  $\mathcal{A}_u := u_0 - g(u)$ . Since  $\tilde{u} \in C([0, \chi], \mathbb{R})$  is a function satisfies the inequality (14). It follows from Remark 5.1 that

$$\left| \tilde{u}(t) - \mathcal{A}_{\tilde{u}} - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, \tilde{u}(s)) ds \right| \leq \frac{[\psi(\chi) - \psi(0)]^\alpha}{\Gamma(\alpha + 1)} \epsilon, \tag{19}$$

where  $\mathcal{A}_{\tilde{u}} := u_0 - g(\tilde{u})$ . Due to (18),  $\mathcal{A}_{\tilde{u}} = \mathcal{A}_u$ . Hence

$$u(t) = \mathcal{A}_{\tilde{u}} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s)) ds.$$

From our assumption, we obtain

$$\begin{aligned} & |\tilde{u}(t) - u(t)| \\ & \leq \left| \tilde{u}(t) - \mathcal{A}_{\tilde{u}} - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, \tilde{u}(s)) ds \right| \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |f(s, \tilde{u}(s)) - f(s, u(s))| ds \\ & \leq \frac{[\psi(\chi) - \psi(0)]^\alpha}{\Gamma(\alpha + 1)} \epsilon + \frac{L_1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |\tilde{u}(s) - u(s)| ds. \end{aligned}$$

It follows from Lemma 2.6 that

$$\begin{aligned} |\tilde{u}(t) - u(t)| & \leq \frac{[\psi(\chi) - \psi(0)]^\alpha}{\Gamma(\alpha + 1)} \epsilon \left( 1 + \int_0^t \sum_{k=1}^{\infty} \frac{L_1^k}{\Gamma(\alpha k)} \psi'(s)(\psi(t) - \psi(s))^{\alpha k - 1} ds \right) \\ & = \frac{[\psi(\chi) - \psi(0)]^\alpha}{\Gamma(\alpha + 1)} \epsilon \left( 1 + \sum_{k=1}^{\infty} L_1^k \frac{1}{\Gamma(\alpha k)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha k - 1} ds \right) \\ & \leq \frac{[\psi(\chi) - \psi(0)]^\alpha}{\Gamma(\alpha + 1)} \epsilon \left( 1 + \sum_{k=1}^{\infty} \frac{L_1^k}{\Gamma(\alpha k + 1)} [\psi(\chi) - \psi(a)]^{\alpha k} \right) \\ & = \frac{[\psi(\chi) - \psi(0)]^\alpha}{\Gamma(\alpha + 1)} \epsilon E_\alpha (L_1 (\psi(\chi) - \psi(a))^\alpha), \end{aligned}$$



for  $\lambda_f = \frac{[\psi(\chi) - \psi(0)]^\alpha}{\Gamma(\alpha+1)} E_\alpha (L_1(\psi(\chi) - \psi(a))^\alpha)$  with  $t \in [0, \chi]$ , we get

$$|\tilde{u}(t) - u(t)| \leq \lambda_f \epsilon.$$

This means that the problem (1)-(2) is UH stable. □

**Theorem 5.2.** *Let the hypotheses of Theorem 5.1 hold. If there exists  $\varphi_f \in C(\mathbb{R}^+, \mathbb{R}^+)$  with  $\varphi_f(0) = 0$ . Then problem (1)-(2) has GUH stability.*

*Proof.* By the same fashion of Theorem 5.1 with putting  $\varphi_f(\epsilon) = \lambda_f \epsilon$  and  $\varphi_f(0) = 0$ , we get

$$|\tilde{u}(t) - u(t)| \leq \varphi_f(\epsilon).$$

□

**Remark 5.2.** *A function  $\tilde{u} \in C([0, \chi], \mathbb{R})$ . is a solution of inequality (15) if and only if there exist two functions  $h \in C([0, \chi], \mathbb{R})$  and  $\varphi \in C([0, \chi], \mathbb{R})$  (where  $h$  depends on solution  $\tilde{u}$ ) such that*

- (i)  $|h(t)| \leq \epsilon \varphi(t)$  for all  $t \in [0, \chi]$ ,
- (ii)  ${}^c D_{0+}^{\alpha; \psi} \tilde{u}(t) = f(t, \tilde{u}(t) + h(t), \quad t \in [0, \chi]$ .

First, we introduce the following hypothesis:

(H<sub>4</sub>) There exists an increasing function  $\varphi \in C([0, \chi], \mathbb{R})$ . and there exists  $\lambda_\varphi > 0$  such that for any  $t \in [0, \chi]$ ,

$$I_{a+}^{\alpha; \psi} \varphi(t) \leq \lambda_\varphi \varphi(t).$$

**Lemma 5.1.** *Let  $\tilde{u} \in C([0, \chi], \mathbb{R})$  is a solution of inequality (15). Then  $\tilde{u}$  is a solution of the following integral inequality:*

$$\begin{aligned} & \left| \tilde{u}(t) - \mathcal{A}_{\tilde{u}} - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s, \tilde{u}(s)) ds \right| \\ & \leq \epsilon \lambda_\varphi \varphi(t), \end{aligned}$$

where  $\mathcal{A}_{\tilde{u}}$  is defined as above.

*Proof.* Indeed, by Remark 5.2, we have that

$${}^c D_{0+}^{\alpha; \psi} \tilde{u}(t) = f(t, \tilde{u}(t) + h(t), \quad t \in [0, \chi].$$

In view of Theorem 3.1, and using Remark 5.2, we have

$$\begin{aligned} \tilde{u}(t) &= \mathcal{A}_{\tilde{u}} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s, \tilde{u}(s)) ds \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} h(s) ds. \end{aligned}$$

It follows from (H<sub>4</sub>) that

$$\begin{aligned} & \left| \tilde{u}(t) - \mathcal{A}_{\tilde{u}} - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} f(s, \tilde{u}(s)) ds \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) [\psi(t) - \psi(s)]^{\alpha-1} |h(s)| ds \\ & \leq \frac{\epsilon}{\Gamma(\alpha)} \int_0^t \psi'(s) [\psi(t) - \psi(s)]^{\alpha-1} \varphi(s) ds \\ & \leq \epsilon \lambda_\varphi \varphi(t). \end{aligned}$$

□

**Theorem 5.3.** Assume that  $(H_1)$ - $(H_4)$  hold. If (15) is satisfied and  $L_1\lambda_\varphi \neq 1$ , then the problem (1)-(2) is UHR and GUHR stable.

*Proof.* Let  $\epsilon > 0$ , and  $\tilde{u} \in C([0, \chi], \mathbb{R})$  be a function which satisfies inequality (15), and let  $u \in C([0, \chi], \mathbb{R})$  is a local unique solution of the problem (17)-(18). Using Theorem 3.1, then from fact that  $\mathcal{A}_u = \mathcal{A}_{\tilde{u}}$ , we have

$$u(t) = \mathcal{A}_{\tilde{u}} + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s)) ds, \quad t \in [0, \chi].$$

By  $(H_1)$  and Lemma 5.1, we get

$$\begin{aligned} & |\tilde{u}(t) - u(t)| \\ & \leq \left| \tilde{u}(t) - \mathcal{A}_{\tilde{u}} - \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, \tilde{u}(s)) ds \right| \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |f(s, \tilde{u}(s)) - f(s, u(s))| ds, \\ & \leq \epsilon \lambda_\varphi \varphi(t) + \frac{L_1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |\tilde{u}(s) - u(s)| ds. \end{aligned}$$

Now, we apply Lemma 2.6, Remark 2.1, and  $(H_4)$  to obtain

$$\begin{aligned} & |\tilde{u}(t) - u(t)| \\ & \leq \epsilon \lambda_\varphi \varphi(t) + \epsilon \lambda_\varphi \int_0^t \sum_{k=1}^\infty \frac{L_1^k}{\Gamma(\alpha k)} \psi'(s)(\psi(t) - \psi(s))^{\alpha k - 1} \varphi(s) ds \\ & = \epsilon \lambda_\varphi \varphi(t) + \epsilon \lambda_\varphi \left[ \int_0^t \frac{L_1}{\Gamma(\alpha)} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(s) ds \right. \\ & \quad \left. + \int_0^t \frac{L_1^2}{\Gamma(2\alpha)} \psi'(s)(\psi(t) - \psi(s))^{2\alpha-1} \varphi(s) ds + \dots \right] \\ & = \epsilon \lambda_\varphi \varphi(t) + \epsilon \lambda_\varphi \left[ L_1 I_{0+}^{\alpha; \psi} \varphi(t) + L_1^2 I_{0+}^{2\alpha; \psi} \varphi(t) + \dots \right] \\ & \leq \epsilon \lambda_\varphi \varphi(t) + \epsilon \lambda_\varphi \left[ L_1 \lambda_\varphi \varphi(t) + (L_1 \lambda_\varphi)^2 \varphi(t) + \dots \right] \\ & = \epsilon \lambda_\varphi \varphi(t) \sum_{k=0}^\infty (L_1 \lambda_\varphi)^k = \frac{\lambda_\varphi}{1 - L_1 \lambda_\varphi} \epsilon \varphi(t). \end{aligned}$$

Then, for  $C_{f,\varphi} = \frac{\lambda_\varphi}{1 - L_1 \lambda_\varphi}$ , we conclude that

$$|\tilde{u}(t) - u(t)| \leq C_{f,\varphi} \epsilon \varphi(t), \quad t \in [0, \chi].$$

This means that problem (1)-(2) is UHR stable. Further, by same fashion above, with putting  $\epsilon = 1$ , we can easily find that

$$|\tilde{y}(t) - y(t)| \leq C_{f,\varphi} \varphi(t), \quad t \in [0, \chi].$$

This yields that problem (1)-(2) has GUHR stability. The proof is completed. □

### 6. Examples

In this section, we present some examples to illustrate our results.

**Example 6.1.** Consider the following nonlocal problem of fractional order with respect to another function  $\psi$

$${}^c D_{0^+}^{\alpha, \psi} u(t) = t + \frac{1}{6} \cos u(t), \quad t \in [0, 1], \tag{20}$$

$$u(0) + \frac{1}{4} u\left(\frac{1}{3}\right) = 0. \tag{21}$$

Here  $u_0 = 0$ ,  $\alpha = \frac{1}{3}$ ,  $f(t, u(t)) = t + \frac{1}{6} \cos u(t)$ , and  $g(u) = \frac{1}{4} u\left(\frac{1}{3}\right)$ . For  $u, v \in \mathbb{R}$  and for  $t \in [0, 1]$ , we find that  $|f(t, u) - f(t, v)| \leq \frac{1}{6} |u - v|$ . Also for  $u, v \in C(J, \mathbb{R}^+)$ , we get  $|g(u) - g(v)| \leq \frac{1}{4} |u - v|$ . Hence the conditions  $(H_1)$  and  $(H_2)$  hold with  $L_1 = \frac{1}{6}$ , and  $L_2 = \frac{1}{4}$ . Now, we will check that  $\left[ \frac{L_1 + L_2}{\Gamma(\alpha + 1)} [\psi(\chi) - \psi(0)]^\alpha \right] < 1$ . Set  $\psi(t) := \sqrt{1+t}$  for all  $t \in [0, 1]$ , we can choose  $\chi = \frac{1}{3}$ . So, some elementary computations gives us

$$\left[ \frac{L_1 + L_2}{\Gamma(\alpha + 1)} [\psi(\chi) - \psi(0)]^\alpha \right] = \frac{5 \sqrt[3]{\frac{2}{\sqrt{3}} - 1}}{12\Gamma(\frac{4}{3})} \approx 0.25 < 1.$$

Moreover, we have  $\psi^{-1}(t) = t^2 - 1$  for all  $t \in [0, 1]$ , it follows that

$$\begin{aligned} \frac{1}{3} &= \chi < \min \left\{ 1, \psi^{-1} \left[ \psi(0) + \left( \frac{\Gamma(\alpha + 1)}{L_2 + L_1} \right)^{\frac{1}{\alpha}} \right] \right\} \\ &= \min \left\{ 1, \left[ \psi(0) + \left( \frac{\Gamma(\alpha + 1)}{L_2 + L_1} \right)^{\frac{1}{\alpha}} \right]^2 - 1 \right\} \\ &= \min \left\{ 1, \left[ 1 + \frac{1728\Gamma(\frac{4}{3})^3}{125} \right]^2 - 1 \right\} = 1. \end{aligned}$$

Hence, the condition  $(H_3)$  holds. Thus all the conditions of Theorem 3.1 are satisfied. So, by the conclusion of Theorem 3.1, the problem (20)-(21) has a local unique solution on  $[0, 1]$ .

On the other hand, we have  $Q = |g(0)| = 0$  and  $\rho = \sup\{ |t + \frac{1}{6}| : t \in [0, 1] \} = \frac{7}{6}$ . Take  $\psi(t) = t$ , for  $t \in [0, 1]$ . So, by Theorem 4.1, we obtain

$$|u(t)| \leq \frac{14}{9\Gamma(\frac{4}{3})} E_{\frac{1}{3}}\left(\frac{2}{9}\right), \quad \forall t \in [0, 1].$$

**Example 6.2.** Consider the nonlocal fractional differential equation involving the Caputo-Hadamard fractional derivative

$${}^{CH} D_{1^+}^{\frac{1}{2}, \ln t} u(t) = \frac{1}{20} \ln(\sqrt{t}) \cos(t) u(t), \quad t \in [1, e], \tag{22}$$

$$u(1) = \frac{1}{2} u\left(\frac{3}{2}\right), \tag{23}$$

where  $\alpha = \frac{1}{2}$  and  $u_0 = 0$ ,  $g(u) = \frac{1}{2} u\left(\frac{3}{2}\right)$ ,  $\psi(t) = \ln t$  for all  $t \in [1, e]$ , and

$$f(t, u) = \frac{1}{20} \ln(\sqrt{t}) \cos(t) u, \quad t \in [1, e], \quad u \in \mathbb{R}.$$

Let  $u, v \in \mathbb{R}$  and  $t \in [1, \frac{3}{2}]$  ( $\chi = \frac{3}{2}$ ), it is easy to see that  $|f(t, u) - f(t, v)| \leq \frac{1}{20} |u - v|$ , and for  $u, v \in C([1, \frac{3}{2}], \mathbb{R}^+)$   $|g(u) - g(v)| \leq \frac{1}{2} |u - v|$ .

Hence, the hypothesis  $(H_1)$  hold with  $L_1 = \frac{1}{20}$   $L_2 = \frac{1}{2}$ . Now, we will check that  $\left[ \frac{L_1 + L_2}{\Gamma(\alpha + 1)} [\psi(\chi) - \psi(0)]^\alpha \right] < 1$ . Some elementary computations give us

$$\left[ \frac{L_1 + L_2}{\Gamma(\alpha + 1)} [\psi(\chi) - \psi(0)]^\alpha \right] = \frac{11}{10} \sqrt{\frac{\ln\left(\frac{3}{2}\right)}{\pi}} \approx 0.4 < 1.$$

Thus, by Theorem 3.1, the problem (22)-(23) has a local unique solution in  $C([1, e], \mathbb{R})$ .

Further, as shown in Theorem 5.3, for every  $\epsilon = \frac{1}{2} > 0$  if  $\tilde{u} \in C([1, \frac{3}{2}], \mathbb{R})$  satisfies

$$\left| {}^{CH}D_{1+}^{\alpha; \ln} \tilde{u}(t) - f(t, \tilde{u}(t)) \right| \leq \frac{1}{2}, \quad t \in [1, \frac{3}{2}],$$

there exists a local unique solution  $u \in C([1, \frac{3}{2}], \mathbb{R})$  such that

$$|\tilde{u}(t) - u(t)| \leq \frac{1}{2} \lambda_f$$

where

$$\begin{aligned} \lambda_f &= \frac{[\ln(\frac{3}{2})]^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} E_{\frac{1}{2}} \left( \frac{1}{20} \left( \ln(\frac{3}{2}) \right)^{\frac{1}{2}} \right) \\ &= \frac{[\ln(\frac{3}{2})]^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} e^{[\frac{1}{400} \ln(\frac{3}{2})]} \left[ 1 + \operatorname{erf} \left( \frac{1}{20} \left( \ln(\frac{3}{2}) \right)^{\frac{1}{2}} \right) \right] \simeq 0.7 > 0. \end{aligned}$$

Hence the problem (22)-(23) is UH stable on  $[1, \frac{3}{2}]$ .

Finally, we consider  $\varphi(t) = (\ln t)^{\frac{1}{2}}$ . To verify the condition  $I_{1+}^{\alpha; \ln t} \varphi(t) \leq \lambda_{\varphi} \varphi(t)$ ,  $\lambda_{\varphi} > 0$ , by the Hadamard fractional integral and simple computation, we get

$$\begin{aligned} I_{1+}^{\frac{1}{2}; \ln t} (\ln t)^{\frac{1}{2}} &= \frac{1}{\Gamma(\frac{1}{2})} \int_1^t (\ln \frac{t}{s})^{-\frac{1}{2}} (\ln s)^{\frac{1}{2}} \frac{ds}{s} \\ &\leq (\ln t)^{\frac{1}{2}} \frac{1}{\Gamma(\frac{1}{2})} \int_1^t (\ln \frac{t}{s})^{-\frac{1}{2}} \frac{ds}{s} \\ &\leq \frac{2}{\sqrt{\pi}} \left( \ln \frac{3}{2} \right)^{\frac{1}{2}} \varphi(t). \end{aligned}$$

Thus, the hypothesis (H<sub>4</sub>) is satisfied with  $\lambda_{\varphi} = \frac{2}{\sqrt{\pi}} (\ln \frac{3}{2})^{\frac{1}{2}} > 0$ . And for every  $\epsilon = \frac{1}{2} > 0$  if  $\tilde{u} \in C([1, \frac{3}{2}], \mathbb{R})$  satisfies

$$\left| {}^{CH}D_{1+}^{\alpha; \ln t} \tilde{u}(t) - f(t, \tilde{u}(t)) \right| \leq \frac{1}{2} (\ln t)^{\frac{1}{2}}, \quad t \in [1, \frac{3}{2}],$$

there exists a unique solution  $u \in C([1, \frac{3}{2}], \mathbb{R})$  such that

$$|\tilde{u}(t) - u(t)| \leq \frac{1}{2} C_{f, \varphi} (\ln t)^{\frac{1}{2}}, \quad t \in [1, \frac{3}{2}],$$

where

$$C_{f, \varphi} = \frac{\lambda_{\varphi}}{1 - L_1 \lambda_{\varphi}} = \frac{\frac{2}{\sqrt{\pi}} (\ln \frac{3}{2})^{\frac{1}{2}}}{1 - \frac{1}{10\sqrt{\pi}} (\ln \frac{3}{2})^{\frac{1}{2}}} \simeq 0.7 > 0$$

and  $L_1 \lambda_{\varphi} \approx 0.04 \neq 1$ . Therefore, the problem (22)-(23) is UHR stable. Finally, taking  $\epsilon = 1$ , we get

$$|\tilde{u}(t) - u(t)| \leq \frac{1}{2} C_{f, \varphi} (\ln t)^{\frac{1}{2}}, \quad t \in [1, \frac{3}{2}].$$

Thus, the conclusion of Theorem 5.3 applies to the problem (22)-(23).

## Conclusions

We can conclude that the most results of this paper were successfully achieved, that is, through Banach fixed point theorem, and extremely significant results within the mathematical analysis, we had a tendency to investigate the existence and interval of existence, uniqueness, and estimates on solutions of the non-local Cauchy problem for nonlinear fractional differential equation introduced by the  $\psi$ -Caputo fractional derivative. On the opposite hand, as an application, we have proved the Ulam-Hyers (UH) stability, generalized Ulam-Hyers (GUH) stability, Ulam-Hyers-Rassias (UHR) stability, and generalized Ulam-Hyers-Rassias (GUHR) stability of solutions of a given problem, via generalized Gronwall inequality. Also the main results have been verified by suitable examples. As future work, one can generalize existence results to an impulsive fractional problem, a neutral time delay problem, and a time-delay problem involving a Caputo type fractional derivative with respect to another function  $\psi$ .

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