A High-Order Accurate Numerical Algorithm for the Numerical Solution of Burgers' Equation

Melis ZORŞAHİN GÖRGÜLÜ[1]

[1] Department of Matematics-Computer, Faculty of Science and Letter, Eskişehir Osmangazi University, Eskişehir, Turkey

✉: mzorsahin@ogu.edu.tr 0000-0001-7506-4162

Received (Geliş): 18.09.2019 Revision (Düzeltme):30.10.2019 Accepted (Kabul): 03.12.2019

ABSTRACT

In this study, a high accurate numerical solution of the Burgers' equation is obtained. For this, the collocation method based on quintic B-spline functions for space discretization and the fourth-order single step method for time discretization are used. In order to see the efficiency of the algorithm, a test problem with an analytical solution is discussed and compared with the numerical solution obtained.

Keywords: Burgers' equation, Collocation method, Fourth order single step method, Quintic B-spline functions

Burgers Denkleminin Nümerik Çözümü için Yüksek Dereceden Doğruluklu Nümerik Algoritma

ÖZ

Bu çalışmada, Burgers denkleminin yüksek doğruluklu bir sayısal çözümü elde edilmiştir. Bunun için, konum ayrıştırmasında quintic B-spline fonksiyonlarını temel alan kolokasyon yöntemi ve zaman ayrıştırmasında dördüncü dereceden tek adımlı yöntem kullanılmıştır. Algoritmanın etkinliğini görmek için, analitik çözüme sahip bir test problemi ele alınmış ve elde edilen sayısal çözümler ile karşılaştırılmıştır.

Anahtar Kelimeler: Burgers denklemi, Dördüncü dereceden tek adımlı yöntem, Kolokasyon metodu, Kuintik B-spline fonksiyonları

INTRODUCTION

The Burgers equation which is frequently encountered in many sciences such as engineering, theoritical and environmental is as follows:

$$u_t + uu_x - \nu u_{xx} = 0. \tag{1}$$

Although this equation was first proposed by Bateman [1], it is known as Burgers because of his work [2]. Numerical solutions of this equation, used in modelling the solitary wave and travelling waves, have been studied by the researchers due to the limitations in analytical solutions. Numerical solutions, which have been investigated recently, have been obtained by least squares, splitting, homotopy perturbation, finite difference and quadrature methods [3-10]. In parallel with these studies, we used the quintic B-spline collocation method for space and the various-order single step methods for time discretization that were not implemented before. The aim of this study is to see the effectiveness of time discretization for the most accurate numerical solution of the Burgers' equation.

Application of the Methods

The time and space steps are denoted with Δt and h, respectively. The exact solution is represented by

$$u(x_m, t_n) = u_m^n;\ m = 0,1,\dots,N; n = 0,1,2,\dots.$$

where $x_m = a + mh$, $t_n = n\Delta t$ and the numerical value of u_m^n is shown by U_m^n at the grid points.

Time discretization

Consider the Burgers' equation of the form

$$u_t = \nu u_{xx} - uu_x \tag{2}$$

$$\begin{aligned} u_{tt} &= (\nu u_{xx} - uu_x)_t \\ &= 2uu_xu_x + (-4\nu u_x + uu)u_{xx} - 2\nu uu_{xxx} \\ &\quad + \nu^2 u_{xxxx} \end{aligned} \tag{3}$$

and the following fourth order single step method

$$u^{n+1} = u^n + \theta_1 u_t^{n+1} + \theta_2 u_t^n + \theta_3 u_{tt}^{n+1} + \theta_4 u_{tt}^n. \tag{4}$$

By choosing $\theta_1 = \theta_2 = \frac{\Delta t}{2}, \theta_3 = \theta_4 = 0$ in (4), we get the method 1, which is of order 2, known as Crank-Nicolson method. By changing the values as $\theta_1 = \theta_2 = \frac{\Delta t}{2}, \theta_3 = -\frac{\Delta t^2}{12}, \theta_4 = \frac{\Delta t^2}{12}$, the method 2 which is of order 4 is obtained. Using Eqs. (2) and (3) in (4), the discretized Burgers' equation in time is obtained as

$$\begin{aligned} & u^{n+1} + [\theta_1 u^{n+1} - 2\theta_3 u^{n+1}(u_x)^{n+1}](u_x)^{n+1} \\ & + [4\nu\theta_3(u_x)^{n+1} - \theta_1\nu - \theta_3 u^{n+1}u^{n+1}](u_{xx})^{n+1} \\ & + 2\nu\theta_3 u^{n+1}(u_{xxx})^{n+1} - \theta_3\nu^2(u_{xxxx})^{n+1} \\ & = u^n + [-\theta_2 u^n + 2\theta_4 u^n(u_x)^n](u_x)^n \end{aligned} \tag{5}$$

Cite as: *Gorgulu, M.Z. (2019). A High-Order Accurate Numerical Algorithm for the Numerical Solution of Burgers' Equation. Mus Alparslan University Journal of Science, 7 (2), 653-656.*

$$+[-4\nu\theta_4(u_x)^n + \theta_2\nu + \theta_4 u^n u^n](u_{xx})^n$$
$$-2\nu\theta_4 u^n(u_{xxx})^n + \theta_4\nu^2(u_{xxxx})^n.$$

Space discretization

The quintic B-spline function is defined as

$$Q_m(x) = \frac{1}{h^5}\begin{cases} (x-x_{m-3})^5, & [x_{m-3},x_{m-2}) \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5, & [x_{m-2},x_{m-1}) \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 \\ \quad +15(x-x_{m-1})^5, & [x_{m-1},x_m) \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 \\ \quad +15(x-x_{m-1})^5 - 20(x-x_m)^5, & [x_m,x_{m+1}) \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 \\ \quad +15(x-x_{m-1})^5 - 20(x-x_m)^5 \\ \quad +15(x-x_{m+1})^5, & [x_{m+1},x_{m+2}) \\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 \\ \quad +15(x-x_{m-1})^5 - 20(x-x_m)^5 \\ \quad +15(x-x_{m+1})^5 - 6(x-x_{m+2})^5, & [x_{m+2},x_{m+3}) \\ 0, & otherwise. \end{cases} \tag{6}$$

The approximate solutions $U_N(x,t)$ can be expressed in terms of the quintic B-spline functions as

$$U_N(x,t) = \sum_{j=-2}^{N+2} \delta_j(t) Q_j(x) \tag{7}$$

where $\delta_j, j = -2,-1,0,1,\dots,N+2$ are unknowns time depend parameters to be determined from collocation form of Eq. (5). Over the element $[x_m, x_{m+1}]$, the approximation can be rewritten as

$$U(x,t) = \sum_{j=m-2}^{m+3} \delta_j(t) Q_j(x). \tag{8}$$

The approximate solution and their derivatives at the knots can be found from the Eqs. (6) and (8) as

$$\begin{aligned} U_m &= U(x_m) = \delta_{m+2} + 26\delta_{m+1} + 66\delta_m + 26\delta_{m-1} + \delta_{m-2}, \\ U'_m &= U'(x_m) = \frac{5}{h}(\delta_{m+2} + 10\delta_{m+1} - 10\delta_{m-1} - \delta_{m-2}), \\ U''_m &= U''(x_m) = \frac{20}{h^2}(\delta_{m+2} + 2\delta_{m+1} - 6\delta_m + 2\delta_{m-1} \\ &\quad + \delta_{m-2}), \\ U'''_m &= U'''(x_m) = \frac{60}{h^3}(\delta_{m+2} - 2\delta_{m+1} + 2\delta_{m-1} - \delta_{m-2}), \\ U^{(4)}_m &= U^{(4)}(x_m) = \frac{120}{h^4}(\delta_{m+2} - 4\delta_{m+1} + 6\delta_m - 4\delta_{m-1} \\ &\quad + \delta_{m-2}). \end{aligned} \tag{9}$$

Substituting (9) into the time discretized form of the proposed equation, we obtain;

$$\begin{aligned} &\left[1 + \beta_1\left(\frac{-5}{h}\right) + \beta_2\left(\frac{20}{h^2}\right) + \beta_3\left(\frac{-60}{h^3}\right) + \beta_4\left(\frac{120}{h^4}\right)\right]\delta_{m-2}^{n+1} \\ &+ \left[26 + \beta_1\left(\frac{-50}{h}\right) + \beta_2\left(\frac{40}{h^2}\right) + \beta_3\left(\frac{120}{h^3}\right) + \beta_4\left(\frac{-480}{h^4}\right)\right]\delta_{m-1}^{n+1} \\ &+ \left[66 + \beta_2\left(\frac{-120}{h^2}\right) + \beta_4\left(\frac{720}{h^4}\right)\right]\delta_m^{n+1} \\ &+ \left[26 + \beta_1\left(\frac{50}{h}\right) + \beta_2\left(\frac{40}{h^2}\right) + \beta_3\left(\frac{-120}{h^3}\right) + \beta_4\left(\frac{-480}{h^4}\right)\right]\delta_{m+1}^{n+1} \\ &+ \left[1 + \beta_1\left(\frac{5}{h}\right) + \beta_2\left(\frac{20}{h^2}\right) + \beta_3\left(\frac{60}{h^3}\right) + \beta_4\left(\frac{120}{h^4}\right)\right]\delta_{m+2}^{n+1} \\ &= \\ &\left[1 + \beta_5\left(\frac{-5}{h}\right) + \beta_6\left(\frac{20}{h^2}\right) + \beta_7\left(\frac{-60}{h^3}\right) + \beta_8\left(\frac{120}{h^4}\right)\right]\delta_{m-2}^{n} \\ &+ \left[26 + \beta_5\left(\frac{-50}{h}\right) + \beta_6\left(\frac{40}{h^2}\right) + \beta_7\left(\frac{120}{h^3}\right) + \beta_8\left(\frac{-480}{h^4}\right)\right]\delta_{m-1}^{n} \\ &+ \left[66 + \beta_6\left(\frac{-120}{h^2}\right) + \beta_8\left(\frac{720}{h^4}\right)\right]\delta_m^{n} \\ &+ \left[26 + \beta_5\left(\frac{50}{h}\right) + \beta_6\left(\frac{40}{h^2}\right) + \beta_7\left(\frac{-120}{h^3}\right) + \beta_8\left(\frac{-480}{h^4}\right)\right]\delta_{m+1}^{n} \\ &+ \left[1 + \beta_5\left(\frac{5}{h}\right) + \beta_6\left(\frac{20}{h^2}\right) + \beta_7\left(\frac{60}{h^3}\right) + \beta_8\left(\frac{120}{h^4}\right)\right]\delta_{m+2}^{n} \end{aligned} \tag{10}$$

When the above expressions are associated, the system of linear equations, which is $N+1$ algebraic equations with $N+5$ unknowns, is obtained. By the help of the conditions $u(a,t) = u_x(a,t) = 0$ and $u(b,t) = u_x(b,t) = 0$, the parameters $\delta_{-2}, \delta_{-1}, \delta_{N+1}$ and δ_{N+2} can be eliminated from the system and the obtained solvable (N+1)×(N+1) matrix system is solved easily by using Matlab packet program. To start the iteration of system, δ^0 can be determined by using the initial and boundary conditions which will be given in numerical experiment, so then we can obtain the δ^n at time $t_n = n\Delta t$.

Numerical Example

In this part, we applied the proposed methods to one example of nonlinear Burgers' equation. To compute the maximum error L_∞, we used the following formula:

$$L_\infty = \|u - U_N\|_\infty = \max_j |u_j - (U_N)_j|.$$

The order of convergence is obtained by the following formula:

$$order = \frac{\log\left|\frac{u - U_{\Delta t_n}}{u - U_{\Delta t_{n+1}}}\right|}{\log\left|\frac{\Delta t_n}{\Delta t_{n+1}}\right|}$$

where u is the exact solution and $U_{\Delta t_n}$ is the numerical solution with time step Δt_n.

The exact solution, which models a shock propagation of the Burgers' equation is as

$$u(x,t) = \frac{x/t}{1 + \sqrt{t/t_0 exp\left(x^2/(4\nu t)\right)}}, t \geq 1, \tag{11}$$

where $t_0 = \exp\left(\frac{1}{8\nu}\right)$. With this solution of the Burgers' equation, the sharpness of the shock waves can be simulated by choice of various viscosity values. The initial shock is obtained from Eq. (11) with the choice of $t = 1$. The boundary conditions are taken as $u(0,t) = 0$ and $u(1,t) = u_x(1,t) = 0$. The computations are performed with the parameters $\nu = 0.005, 0.0005$, $h = 0.001$ and $\Delta t = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002, 0.001$ over the solution domain $[-2,2]$. The propagation of the shock wave is simulated in Fig. 1 for $\nu = 0.005, h = 0.001$ and $\nu = 0.0005, h = 0.001$ by method 2. The figure shows that the obtained wave for the parameter $\nu = 0.0005$, is steeper than the other.

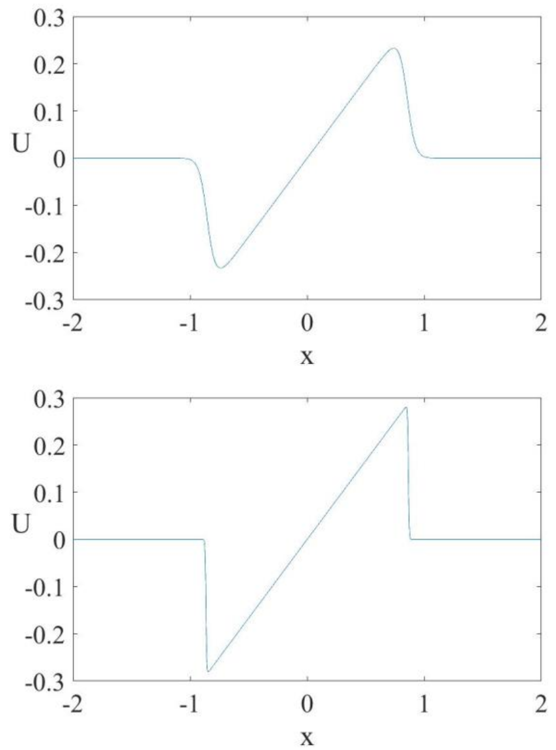

Figure 1. Solutions for $v = 0.005, 0.0005;\ \Delta t = h = 0.001$.

The absolute errors profiles of the proposed methods at time $t = 3$ is given in Fig. 2 for various viscosity coefficients. From these figures, as expected, the absolute error increases at the peak of the waves. For this problem, by calculating the L_∞ error norm and the rate of convergence of algorithms, the accuracy of the proposed methods can be seen and compared each other in Table 1. From this table we conclude that the method 2, for finding the shock wave solution of the Burgers' equation, gives better accuracy than the method 1. The rate of convergence for time is around 2 for method 1 and around 4 for method 2.

CONCLUSION

In this paper, an effective algorithm is proposed for the numerical solution of the Burgers' equation. This algorithm is constructed by using the quintic B-spline based collocation method for space discretization and fourth order single-step method for the time discretization. To compare the effect of the time discretization methods, the single step methods which has two different order as 2 and 4 is chosen and the obtained solutions are compared with the those of a test problem which has an exact solution. By this study it is concluded that the fourth order single step method which used for the time discretization gives the better solutions than the Crank-Nicolson method.

ACKNOWLEGMENT

This paper is presented at the Recent Advances in Mathematical and Engineering Sciences- International Mathematics and Engineering Conference (RAIMS-LVIV).

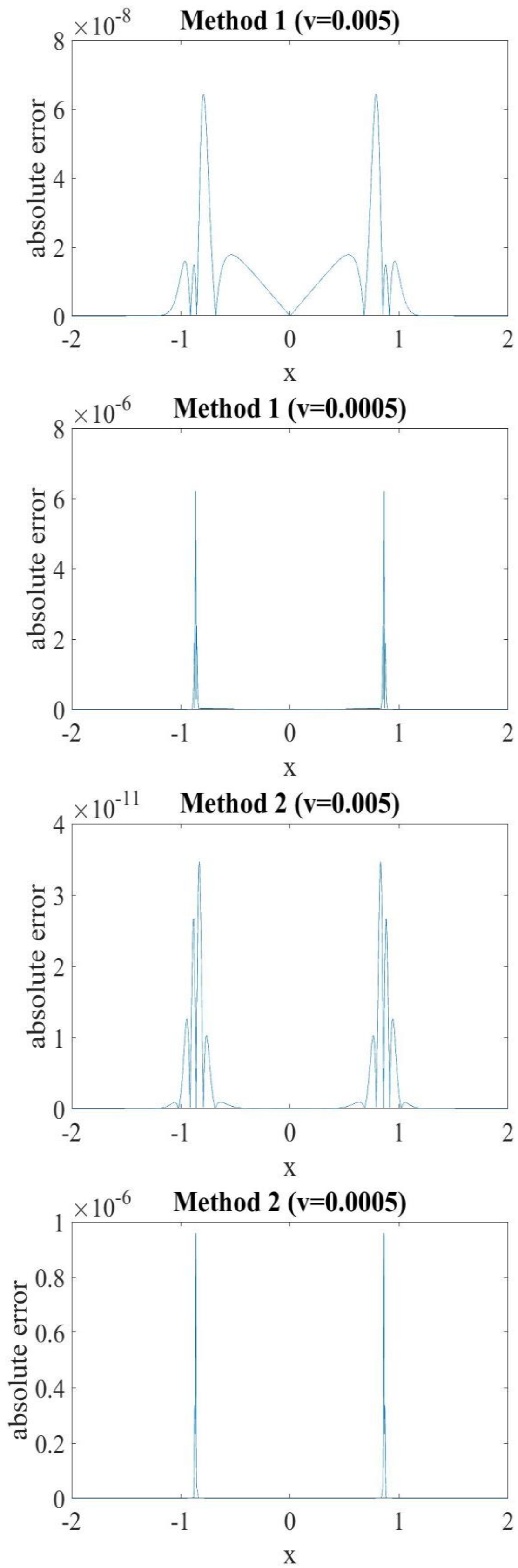

Figure 2. Errors for $v = 0.005, 0.0005;\ \Delta t = h = 0.001$

Table 1. Comparison of numerical results at different times for $h = 0.001$.

	Method 1		Method 2	
$\nu = 0.005$				
Δt	L_∞	order	L_∞	order
0.1	5.76×10^{-4}		1.20×10^{-4}	
0.05	1.54×10^{-4}	1.908	1.38×10^{-5}	3.123
0.02	2.55×10^{-5}	1.960	5.48×10^{-7}	3.521
0.01	6.42×10^{-6}	1.989	4.08×10^{-8}	3.748
0.005	1.61×10^{-6}	1.997	2.89×10^{-9}	3.820
0.002	2.58×10^{-7}	1.999	1.08×10^{-10}	3.590
0.001	6.44×10^{-8}	2.000	3.46×10^{-11}	1.637
$\nu = 0.0005$				
Δt	L_∞	order	L_∞	order
0.1	2.32×10^{-1}		2.05×10^{-1}	
0.05	1.44×10^{-1}	0.692	9.36×10^{-2}	1.132
0.02	2.36×10^{-2}	1.973	1.53×10^{-2}	1.974
0.01	3.39×10^{-3}	2.800	2.32×10^{-3}	2.725
0.005	4.17×10^{-4}	3.024	2.52×10^{-4}	3.203
0.002	3.18×10^{-5}	2.807	9.79×10^{-6}	3.544
0.001	6.20×10^{-6}	2.360	9.59×10^{-7}	3.352

REFERENCES

[1] Bateman H. Some recent researches on the motion of the fluids, Monthly Weather Review. 43 163-70, 1915.

[2] Burgers J.M. A mathematical model illustrating the theory of turbulence, Advances in Applied Mechanics. 1 171-199, 1948.

[3] Yang X., Ge Y., Zhang L. A class of high-order compact difference schemes for solving the Burgers' equations, Applied Mathematics and Computation. 358 394-417, 2019.

[4] Fu F., Li J., Lin J., Guan Y., Gao F., Zhang C., Chen L. Moving least squares particle hydrodynamics method for Burgers' equation, Applied Mathematics and Computation. 356 362-378, 2019.

[5] Sari M., Tunc H., Seydaoglu M. Higher order splitting approaches in analysis of the Burgers equation, Kuwait Journal of Science. 46 1-14, 2019.

[6] Rathore S., Saraswat G.K. Homotopy perturbation approach to the solution of non-linear Burgers's equation, International Journal of Mathematics and its Applications. 6 203-211, 2018.

[7] Mohamed N.A. Fully implicit scheme for solving Burgers' equation based on finite difference method. The Egyptian International Journal of Engineering Sciences Technology. 26 38-44, 2018

[8] Aswin V.S., Awasthi A., Rashidi M.M. A differential quadrature based numerical method for highly accurate solutions of Burgers' equation, Numerical Methods for Partial Differential Equations. 33 2023-2042, 2017.

[9] Tamsir M., Srivastava V.K., Jiwari R. An algorithm based on exponential modified cubic B-spline differential quadrature method for nonlinear Burgers' equation, Applied Mathematics and Computation. 290 111-124, 2016.

[10] Zorsahin Gorgulu M., Dag I., Irk D. Wave Propagation by Way of Exponential B-Spline Galerkin Method, Journal of Physics: Conference Series. 766 1-6, 2016.