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Some Results on Nearly Cosymplectic Manifolds

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Abstract

The object of this paper is to study Ricci solitons under some curvature conditions in nearly cosymplectic manifolds.

1. Introduction

Cosympletic manifold is an odd dimensional counterpart of a Kähler manifold which is defined by Lipperman and Blair 1967 [9]. In parallel with Olzak's work [1], [2], Endo investigated the geometry of nearly cosymplectic manifolds [3].
Ricci soliton is a special solution to the Ricci flow introduced by Hamilton [10] in the year 1982. In [12], Sharma initiated the study of Ricci solitons in contact Riemannian geometry. Later, Tripathi [13], Nagaraja et al. [11] and others extensively studied Ricci solitons in contact metric manifolds. Ricci soliton in Riemanian manifold (M,g) is a natural generalization of an Einstein metric and is defined as a triple (g,V,λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$(\mathfrak{L}_V g)(X,Y)+2S(X,Y)+2\lambda g(X,Y)=0 \tag{1.1}$$

where S is the Ricci tensor of M and $\mathfrak{L}_V$ denoted the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively.
In [16], [19], authors studied the properties of generalized recurrent manifolds where as the properties of generalized φ-recurrent manifolds have studied in [8], [16], [17] and [18].
In this paper we study some curvature conditions such that φ-recurrent, pseudo-projective φ-recurrent, concircular φ-recurrent and Ricci recurrent which characterize Ricci solitons in nearly cosymplectic manifolds.

2. Preliminaries

2.1. Nearly Cosymplectic Manifolds

Let (M,φ,ξ,η,g) be an $(2n+1)-$dimensional almost contact Riemannian manifold, where φ is a type of $(1,1)-$tensor field, ξ is the structure vector field, η is a $1-$form and g is the Riemannian metric. It is well known that the $(\varphi,\xi,\eta,g)-$structure satisfies the conditions [7] for any vector fields X and Y on M,

$$\varphi^2 X=-X+\eta(X)\xi, \quad \eta(\xi)=1, \quad g(X,\xi)=\eta(X)$$

$$\eta(\varphi X)=0, \quad \varphi\xi=0, \tag{2.1}$$

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$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y). \tag{2.2}$$

A nearly cosymplectic manifold is an almost contact metric manifold (M,φ,ξ,η,g) such that

$$(\nabla_X\varphi)Y+(\nabla_Y\varphi)X=0, \tag{2.3}$$

for all vector fields X,Y. Clearly, this condition is equivalent to $(\nabla_X\varphi)X=0$. It is known that in a nearly cosymplectic manifold the Reeb vector field ξ is Killing and satisfies $\nabla_\xi\xi=0$ and η is a contact form $\nabla_\xi\eta=0$. The tensor field h of type $(1,1)$ defined by

$$\nabla_X\xi=hX, \tag{2.4}$$

is skew symmetric and anticommutes with φ. It satisfies

$$h\xi=0, \quad \eta\circ\varphi=0, \tag{2.5}$$

and the following formulas hold [3], [4]

$$g((\nabla_X\varphi)Y,hZ)=\eta(Y)g(h^2X,\varphi Z)-\eta(X)g(h^2Y,\varphi Z),$$

$$tr(h^2)=constant,$$

$$R(Y,Z)\xi=\eta(Y)h^2Z-\eta(Z)h^2Y, \tag{2.6}$$

$$S(Z,\xi)=-tr(h^2)\eta(Z), \tag{2.7}$$

where R, S, Q and η are the Riemannian curvature tensor type of $(1,3)$, the Ricci tensor of type $(0,2)$, the Ricci operator defined by $g(QX,Y)=S(X,Y)$.
Let (g,V,λ) be a Ricci soliton in a nearly cosymplectic manifold M. Taking $V=\xi$ then from (2.4) and (1.1), we have

$$S(X,Y)=-\lambda g(X,Y). \tag{2.8}$$

The above equation yields

$$QX=-\lambda X, \tag{2.9}$$

$$S(X,\xi)=\lambda\eta(X), \tag{2.10}$$

$$r=-\lambda n. \tag{2.11}$$

Also by definition of covariant derivative, we have

$$(\nabla_W S)(Y,\xi)=\nabla_W S(Y,\xi)-S(\nabla_W Y,\xi)-S(Y,\nabla_W\xi). \tag{2.12}$$

3. φ-Recurrent Nearly Cosymplectic Manifolds

Definition 3.1. *A nearly cosymplectic manifold is said to be φ-recurrent manifold [14] if there exist a non-zero $1-$form A such that*

$$\varphi^2((\nabla_W R)(X,Y)Z=A(W)R(X,Y)Z \tag{3.1}$$

for arbitrary vector fields X, Y, Z, W.
Let us consider a φ-recurrent nearly cosymplectic manifold. By virtue of (2.1) and (3.1), we have

$$-(\nabla_W R)(X,Y)Z+\eta((\nabla_W R)(X,Y)Z)\xi=A(W)R(X,Y)Z. \tag{3.2}$$

Theorem 3.2. *Let given Ricci soliton on nearly cosymplectic manifolds. Then there is not exist $\varphi-$recurrent nearly cosymplectic manifold.*

Proof. Contracting (3.2) with U, we obtain

$$-g((\nabla_W R)(X,Y)Z,U)+\eta((\nabla_W R)(X,Y)Z)\eta(U)=A(W)g(R(X,Y)Z,U). \tag{3.3}$$

Let e_i $(i=1,2,...,2n+1)$, be an orthonormal basis of the tangent space at any point of the manifold. Taking $X=U=e_i$ in (3.3) and taking summation over i, $1\leq i\leq 2n+1$, we get

$$-(\nabla_W S)(Y,Z)=A(W)S(Y,Z). \tag{3.4}$$

Replacing Z by ξ in (3.4) and using (2.7), we have

$$-(\nabla_W S)(Y,\xi)=-tr(h^2)A(W)\eta(Y). \tag{3.5}$$

Using (2.7) and (2.4) in (2.12), we obtain

$$(\nabla_W S)(Y,\xi)=-[S(Y,hW)+tr(h^2)g(Y,hW)]. \tag{3.6}$$

In view of (3.5) and (3.6), we have

$$S(Y,hW)=-tr(h^2)[g(Y,hW)+A(W)\eta(Y)]. \tag{3.7}$$

Taking $Y=\xi$ in (3.7), we get

$$S(\xi,hW)=-tr(h^2)[g(Y,hW)+A(W)\eta(\xi). \tag{3.8}$$

Using (2.1), (2.5) and (2.8) in (3.8), we find

$$-\lambda g(hW,\xi)=tr(h^2)A(W),$$

$$tr(h^2)A(W)=0,$$

$$A(W)=0.$$

This is a contradiction. □

4. Generalized $\varphi-$Recurrent Nearly Cosymplectic Manifolds

Definition 4.1. *A nearly cosymplectic manifold is said to be generalized φ-recurrent manifold if its curvature tensor R satisfies the relation*

$$\varphi^2((\nabla_W R)(X,Y)Z)=A(W)R(X,Y)Z+B(W)\{g(Y,Z)X-g(X,Z)Y\}, \tag{4.1}$$

where A and B are $1-$forms and non-zero and these are defined by

$$A(W)=g(W,\rho_1), \quad B(W)=g(W,\rho_2),$$

and ρ_1,ρ_2 are unit vector fields associated with $1-$forms A, B respectively.

Theorem 4.2. *In a generalized φ-recurrent strictly nearly cosymplectic manifold (M_n,g), the associated vector fields ρ_1 and ρ_2 of the $1-$forms A and B respectively are co-directional.*

Proof. In consequence of (2.1), equation (4.1) becomes

$$-(\nabla_W R)(X,Y)Z+\eta((\nabla_W R)(X,Y)Z)\xi=A(W)R(X,Y)Z+B(W)\{g(Y,Z)X-g(X,Z)Y\}\ ,$$

from which it follows by taking inner product with U that

$$-g((\nabla_W R)(X,Y)Z,U)+\eta((\nabla_W R)(X,Y)Z)\eta(U)=A(W)g(R(X,Y)Z,U)+B(W)\{g(Y,Z)g(X,U)-g(X,Z)g(Y,U)\}. \tag{4.2}$$

Let $\{e_i\}$, $i=1,2,...,2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (4.2) and taking summation over $i, 1\leq i\leq 2n+1$, we get

$$\begin{aligned}&-(\nabla_W S)(Y,Z)+\sum_{i=1}^{2n+1}\eta((\nabla_W R)(e_i,Y)Z)\eta(e_i)=\\&A(W)S(Y,Z)+2nB(W)g(Y,Z).\end{aligned} \tag{4.3}$$

Again replacing Z by ξ in (4.3) and using (2.1) and (2.7), we get

$$-(\nabla_W S)(Y,\xi)+\sum_{i=1}^{2n+1}\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i)=\{-trh^2A(W)+2nB(W)\}\eta(Y). \tag{4.4}$$

The second term of left hand side in (4.4) with (2.1)takes the form

$$\sum_{i=1}^{2n+1}\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i)=\eta((\nabla_W R)(\xi,Y)\xi)\eta(\xi)=g((\nabla_W R)(\xi,Y)\xi,\xi). \tag{4.5}$$

Using (2.4), (2.5)and (2.6)in (4.5), we obtain

$$g((\nabla_W R)(\xi,Y)\xi,\xi)=0. \tag{4.6}$$

In view of (4.6), (4.4) becomes

$$(\nabla_W S)(Y,\xi)=\{tr(h^2)A(W)-2nB(W)\}\eta(Y). \tag{4.7}$$

The equation (2.12) with (2.4) and (2.7) takes the form

$$(\nabla_W S)(Y,\xi)=-tr(h^2)g(Y,hW)-S(Y,hW). \tag{4.8}$$

From equations (4.7) and (4.8), we find

$$-tr(h^2)g(Y,hW)-S(Y,hW)=(tr(h^2)A(W)-2nB(W))\eta(Y). \tag{4.9}$$

Replacing Y by ξ then using (2.5) in (4.9) we have

$$A(W)=(\frac{2n}{tr(h^2)})B(W).$$

This means that the vector fields ρ_1 and ρ_2 of the $1-$forms are co-directional. □

5. Ricci-Recurrent Nearly Cosymplectic Manifold

Theorem 5.1. *Let given Ricci soliton on nearly cosymplectic manifolds. Then there is not exist Ricci recurrent nearly cosymplectic manifold.*

Proof. A nearly cosymplectic manifold is said to be Ricci-recurrent manifold if there exist a non-zero $1-$form A such that

$$(\nabla_W S)(Y,Z)=A(W)S(Y,Z). \tag{5.1}$$

Replacing Z by ξ in (5.1) and using (2.7), we have

$$(\nabla_W S)(Y,\xi)=-tr(h^2)A(W)\eta(Y). \tag{5.2}$$

Using (2.4) and (2.7) in (2.12), we obtain

$$(\nabla_W S)(Y,\xi)=-[S(Y,hW)+tr(h^2)g(y,hW)]. \tag{5.3}$$

In view of (5.2) and (5.3), we have

$$S(Y,hW)=-tr(h^2)g(Y,hW)+tr(h^2)A(W)\eta(Y). \tag{5.4}$$

Taking $Y=\xi$ in (5.4), we get

$$A(W)=0.$$

It contradicts that $A\neq 0$. Thus, the proof is completed. □

6. Pseudo-projective $\varphi-$recurrent Nearly Cosymplectic Manifold

In a nearly cosymplectic manifold M, the pseudo-projective curvature tensor $\widetilde{P}$ is given by [20]

$$\widetilde{P}(X,Y)Z=aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y]-\frac{r}{2n+1}(\frac{a}{2n}+b)[g(Y,Z)X-g(X,Z)Y] \tag{6.1}$$

where a and b are constants such that $a,b\neq 0$.

Theorem 6.1. *Ricci soliton in a pseudo-projective φ-recurent nearly cosymplectic manifold (M,g) with 1-form non-zero A depends on the sign of $tr(h^2)$.*

Proof. A nearly cosymplectic manifold is said to be pseudo-projective φ-recurrent manifold if there exists a non-zero $1-$form A such that

$$\varphi^2((\nabla_W\widetilde{P})(X,Y)Z)=A(W)\widetilde{P}(X,Y)Z, \tag{6.2}$$

for arbitrary vector fields X, Y, Z, W. Let us consider a pseudo-projective φ-recurrent nearly cosymplectic manifold. By virtue of (2.1) and (6.2), we have

$$-(\nabla_W\widetilde{P})(X,Y)Z)+\eta((\nabla_W\widetilde{P})(X,Y)Z)\xi=A(W)\widetilde{P}(X,Y)Z). \tag{6.3}$$

Contracting (6.3) with U, we obtain

$$-g((\nabla_W\widetilde{P})(X,Y)Z,U)+\eta((\nabla_W\widetilde{P})(X,Y)Z)\eta(U)=A(W)g(\widetilde{P}(X,Y)Z,U). \tag{6.4}$$

Let e_i $(i=1,2,...,2n+1)$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (6.4) and taking summation over i, $1\leq i\leq 2n+1$, we get

$$(\nabla_W S)(Y,Z)=A(W)\{S(Y,Z)-\frac{r}{2n+1}g(Y,Z)\}. \tag{6.5}$$

Replacing Z by ξ in (6.5) and using (2.1) and (2.7), we have

$$(\nabla_W S)(Y,\xi)=-A(W)\{tr(h^2)-\frac{r}{2n+1}\}\eta(Y). \tag{6.6}$$

Using (2.7) and (2.4) in (2.12), we obtain

$$(\nabla_W S)(Y,\xi)=-[S(Y,hX)+tr(h^2)g(Y,hX). \tag{6.7}$$

In view of(6.6) and (6.7), we have

$$S(Y,hX)=A(W)\{tr(h^2)+\frac{r}{2n+1}\}\eta(Y)-tr(h^2)g(Y,hX).$$

Taking $Y=\xi$ and using (2.5), (2.8), (2.11) we get

$$A(W)\{tr(h^2)-\frac{\lambda n}{2n+1}\}=0.$$

for non-zero $A(W)$ we find

$$\lambda=\frac{tr(h^2)(2n+1)}{n}.$$

Hence, the proof is completed. □

7. Concircular $\varphi-$Recurrent Nearly Cosymplectic Manifold

The Concircular curvature tensor of (M,g) is given by [21]

$$\widetilde{C}(X,Y)Z=R(X,Y)Z-\frac{r}{2n(2n+1)}[g(Y,Z)X-g(X,Z)Y]. \tag{7.1}$$

Definition 7.1. *A nearly cosymplectic manifold is said to be concircular $\varphi-$recurrent manifold if there exist a non-zero $1-$form A such that*

$$\varphi^2((\nabla_W\widetilde{C})(X,Y)Z)=A(W)\widetilde{C}(X,Y)Z. \tag{7.2}$$

for arbitrary vector fields X, Y, Z, W.

Theorem 7.2. *Ricci soliton in a concircular $\varphi-$recurrent nearly cosymplectic manifold M with $1-$form non-zero A depends on the sign of $tr(h^2)$.*

Proof. Let us consider a concircular $\varphi-$recurrent nearly cosymplectic manifold. By virtue of (2.1) and (7.2), we have

$$-(\nabla_W\widetilde{C})(X,Y)Z+\eta((\nabla_W\widetilde{C})(X,Y)Z)\xi=A(W)\widetilde{C}(X,Y)Z. \tag{7.3}$$

Contracting (7.3) with U, we obtain

$$-g((\nabla_W\widetilde{C})(X,Y)Z,U)+\eta((\nabla_W\widetilde{C})(X,Y)Z)\eta(U)=A(W)g(\widetilde{C}(X,Y)Z,U). \tag{7.4}$$

Let e_i $(i=1,2,...,2n+1)$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (7.4) and taking summation over i, $1\leq i\leq 2n+1$, we get

$$(\nabla_W S)(Y,Z)=-A(W)\{S(Y,Z)-\frac{r}{2n+1}g(Y,Z)\}. \tag{7.5}$$

Replacing Z by ξ in (7.5) and using (2.1) and (2.7), for a constant r, we have

$$(\nabla_W S)(Y,\xi)=A(W)\eta(Y)\{tr(h^2)+\frac{r}{2n+1}\}. \tag{7.6}$$

Using (2.7) and (2.4) in (2.12), we obtain

$$(\nabla_W S)(Y,\xi)=-[S(Y,hW)+tr(h^2)g(Y,hW)]. \tag{7.7}$$

In view of (7.6) and (7.7), we have

$$S(Y,hW)=-\{tr(h^2)+\frac{r}{2n+1}\}A(W)\eta(Y)-tr(h^2)g(Y,hW). \tag{7.8}$$

Taking $Y=\xi$, and using (2.5) and (2.8) a characteristic vector field in (7.8), we get

$$A(W)\{tr(h^2)+\frac{r}{2n+1}\}=0. \tag{7.9}$$

Using (2.11) in (7.9), for non-vanishing A, we have

$$\lambda=\frac{tr(h^2)(2n+1)}{n}.$$

So, we have desired result. □

References

[1] Z. Olszak, *Nearly Sasakian manifolds*, Tensor, N.S., **33**(1979), 26.
[2] Z. Olszak, *Five-dimensional nearly Sasakian manifolds*, Tensor, N.S., **34**(1980), 273-276.
[3] H. Endo, *On the curvature tensor of nearly cosymplectic manifolds of constant φ-sectional curvature*. An. Ştiint. Univ. Al. I. Cuza Iaşi. Mat. **51**(2)(2005), 439-454 .
[4] A. De Nicola, G. Dileo, I. Yudin, *On nearly Sasakian and nearly cosymplectic manifolds*, Ann. Mat., (2017), https://doi.org/10.1007/s10231-017-0671-2.
[5] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Math., **203**, Birkhauser Boston 2002.
[6] D.E. Blair, D.K. Showers, *Almost Contact Manifolds with Killing Structures Tensors II.*, J. Dier. Geom., **9**(1974), 577-582.
[7] D.E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. **509**, *Springer-Verlag*, Berlin, (1976).
[8] S. K. Chaubey, *On generalized φ-recurrent trans-Sasakian manifolds*, (to appear).
[9] P. Libermann, *Sur les automorphismes innit·esimaux des structures symplectiques et de atructures de contact*, Coll. G·eom. Di. Globale, (1959), 3759.
[10] R.S. Hamilton, *The Ricci flow on surfaces, Mathematical and General relativity (Santa Cruz, CA, 1986)*, American Math. Soc. Contemp. Math. **71** (1988), 237-262.
[11] H.G. Nagaraja, C.R. Premalatha, *Ricci solitons in Kenmotsu manifolds*, J. Math. Anal., **3** (2) (2012), 18-24.
[12] R. Sharma, *Certain results on K-contact and (κ,μ)-contact manifolds*, J. Geom., **89** (2008), 138-147.
[13] M.M. Tripathi, *Ricci solitons in contact metric manifolds*, arXiv:0801, 4222v1, [math DG] (2008).
[14] U.C. De, *On φ-recurrent Kenmotsu manifolds*, Turk J. Math., **33** (2009), 17-25.
[15] J. P. Jaiswal, R. H. Ojha, *On generalized φ-recurrent LP-Sasakian manifolds*, Kyungpook Math. J., **49**(2009), 779-788.
[16] A. Basari, C. Murathan, *On generalized φ-recurrent Kenmotsu manifolds*, Fen Derg. **3**(1)(2008), 91-97.
[17] D. A. Patil, D. G. Prakasha, C. S. Bagewadi, *On generalized φ-recurrent Sasakian manifolds*, Bull. of Math. Anal. and Appl., **1** (3)(2009), 42-48.
[18] J. P. Jaiswal, R. H. Ojha, *On generalized φ-recurrent LP-Sasakian manifolds*, Kyungpook Math. J., **49**(2009), 779-788.
[19] U. C. De, N. Guha, *On generalized recurrent manifolds*, J. Nat. Acad. Math. India, **9**(1991), 85-92.
[20] B. Prasad, *A pseudo projective curvature tensor on Riemannian manifold*, Bull. Cal. Math. Soc., **94** (2002), 163-169.
[21] K.Yano, *Concircular geometry-I. Concircular transformations*, Proc. Japan Acad., **16** (1940), 195-200.