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Coincidence Point Theorems on b-Metric Spaces via C_F-Simulation Functions

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Abstract
In this paper, we investigate the existence and uniqueness of the coincidence points with the C_F-simulation function for two nonlinear operators on the b-metric space. Our results improve and generalize some of the results available in the literature.

Keywords: b-metric space, Fixed point, Simulation function
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1. Introduction

From a theoretical standpoint, there are many different ways to solve the problems encountered in mathematics and related sciences. In the recent years, the most remarkable theory is the fixed point theory which is used in many areas. The most known theory is the Banach contraction principle [1] and this theory has numerous applications in important areas (see [2], [3]).

Recently, Khojasteh et al. [4] introduced the concept of simulation function. Then, they introduced the non-linear Z-contraction of the simulation class of functions. The well known Banach contraction principle ensures the existence and uniqueness of fixed point of a contraction on a complete metric space. After this interesting principle, several authors generalized this principle by introducing the various contractions on metric spaces (see [4],[5]).

Now, we give some concepts and results from the literature used throughout the study.

Definition 1.1. *[6] Let X be a non-empty set and let $d : X \times X \longrightarrow [0,\infty)$ be a function satisfying the following conditions:*

(i) $d(x,y) = 0 \Longleftrightarrow x = y$, *for all* $x,y \in X$,

(ii) $d(x,y) = d(y,x)$, *for all* $x,y \in X$,

(iii) $d(x,y) \leq s[d(x,y)+d(y,z)]$, *for some real* $s \geq 1$, *for all* $x,y,z \in X$.

Then, d is called a b-metric on X and (X,d) is called a b-metric space.

Lemma 1.2. *[7] Let (X,d) be a metric space and $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence in (X,d), then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$. For all $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$, $d(x_{m_k}, x_{n_k-1}) < \varepsilon$ and

(1) $\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k+1}) = \varepsilon$,

(2) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \varepsilon$,

(3) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon$,

(4) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \varepsilon$.

Lemma 1.3. *[8] Let (X,d) be a b-metric space for some real $s \geq 1$ and $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a b-Cauchy sequence in (X,d), then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$, $d(x_{m_k}, x_{n_k-1}) < \varepsilon$ and

(1) $\varepsilon \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq s\varepsilon$,

(2) $\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq s^2\varepsilon$,

(3) $\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq s^2\varepsilon$,

(4) $\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq s^3\varepsilon$.

Definition 1.4. *[9] Let X be a nonempty set and $T, g : X \longrightarrow X$ be mappings.*

(1) A point $x \in X$ is called a fixed point of the mapping T if $Tx = x$.

(2) A point $x \in X$ is called a coincidence point of the mappings T and g if $Tx = gx$.

(3) A point $x \in X$ is called a common fixed point of the mappings T and g if $Tx = gx = x$.

Definition 1.5. *[9] Let $T, g : X \longrightarrow X$ be mappings on a b-metric space (X,d). If*

$$\lim_{n\to\infty} d(Tgx_n, gTx_n) = 0,$$

for all $\{x_n\} \subseteq X$ such that the $\{gx_n\}$ and $\{Tx_n\}$ sequences are convergent and have the same limit points, then T and g are called compatible.

Remark 1.6. *[10] If T and g commuting (that is, $Tgx = gTx$ for all $x \in X$), then T and g are compatible.*

Definition 1.7. *[4] Let $T, g : X \longrightarrow X$ be functions and $\{x_n\} \subseteq X$. The sequence $\{x_n\}$ is a Picard-Jungck sequence with a pair of (T,g) if $gx_{n+1} = Tx$, for each $n \geq 0$*

Definition 1.8. *[11] Let $F : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ be a continuous function and satisfy the following conditions:*

(a) $F(s,t) \leq s$;

(b) $F(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$; *for all* $s, t \in [0,\infty)$.

Then, F is called a C-class function.

We denote C-class functions as $\mathtt{C}$.

Definition 1.9. *[4] Let $F : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ be a function. There exists a $C_F \geq 0$ such that* *(a)* $F(s,t) > C_F \Rightarrow s > t$;

(b) $F(t,t) \leq C_F$, $\forall s,t \in [0,\infty)$.

Then, F has property C_F.

Definition 1.10. *[5] Let $\zeta : [0,\infty)\times[0,\infty) \longrightarrow \mathbb{R}$ be a function satisfying the following conditions:*

(ζa) $\zeta(0,0) = 0$;

(ζb) $\zeta(t,s) < F(t,s)$*, for all* $s,t > 0$*; the function* $F : [0,\infty)\times[0,\infty) \longrightarrow \mathbb{R}$ *is the element of* $\mathcal{C}$ *with property* C_F.

(ζc) If $\{t_n\}$, $\{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ *and* $t_n < s_n$*, then* $\limsup_{n\to\infty} \zeta(t_n,s_n) < C_F$.

Then, it is called a C_F-simulation function.

We denote the class of all C_F-simulation functions as Z_F.

Definition 1.11. *Let (X,d) be a b-metric space for some real $s \geq 1$ and $f,g : X \longrightarrow X$ be mappings. ζ is an element of Z_F such that*

$$\zeta(s^4 d(Tx,Ty),(gx,gy)) \geq C_F, \tag{1.1}$$

for all $x,y \in X$ with $gx \neq gy$. Then, T is called a $(Z_{F,b},g)$-contraction.

2. Main results

In this section, we introduce our main results.

Remark 2.1.

(1) By axiom (ζ_b), it is clear that a simulation function must verify $\zeta(r,s^4 r) < C_F$ for all $r > 0$.

(2) Furthermore, ζ is the elements of Z_F such that

$$d(Tx,Ty) \leq s^4 d(Tx,Ty) < d(gx,gy), \tag{2.1}$$

for all $x,y \in X$ with $gx \neq gy$. T is a $(Z_{F,b},g)$-contraction.

To prove, assume that $gx \neq gy$. Then, $d(gx,gy) > 0$. If $Tx = Ty$, then $0 = d(Tx,Ty) = s^4 d(Tx,Ty) < d(gx,gy)$. On the contrary case, if $Tx \neq Ty$, then $0 < d(Tx,Ty)$, by property (ζ_b) and (1.1), we have that

$$C_F \leq \zeta(s^4 d(Tx,Ty), d(gx,gy)) < F(d(gx,gy), s^4 d(Tx,Ty))$$

so (2.1) holds. In other words $d(Tx,Ty) \leq s^4 d(Tx,Ty) < d(gx,gy)$ is obtained.

Lemma 2.2. *If T is a $(Z_{F,b},g)$-contraction in a b-metric space (X,d) and $x,y \in X$ are coincidence points of T and g, then $Tx = gx = gy = Ty$. In particular, the following conditions hold.*

(1) If T (or g) is injective within the entire set of coincidence points of T and g, then T and g have a single coincidence point at most.

(2) If T and g have a common fixed point, it is unique.

Proof. To prove, assume that $gx \neq gy$. Then, $d(gx,gy) > 0$. Using (1.1) the following is obtained

$$C_F \leq \zeta(s^4 d(Tx,Ty), d(gx,gy)) = \zeta(s^4 d(gx,gy), d(gx,gy)).$$

Due to the item (1) of Remark 2.1, contradiction is obtained. In this case, our assumption is incorrect. Therefore, if x and y are coincidence points of T and g, then $Tx = gx = gy = Ty$. The proof is completed. $\square$

Theorem 2.3. *Let T be a $(Z_{F,b},g)$-contraction in b-metrik space (X,d). Suppose that there is a Picard-Jungck sequence $\{x_n\}$ of (T,g). In addition, at least one of the following conditions holds.*

(a) $(g(X),d)$ *(or $(T(X),d)$) is complete.*

(b) (X,d) *is complete, T and g are b-continuous and compatible.*

(c) (X,d) *is complete, T and g are b-continuous and commuting.*

T and g have at least one coincidence point. Furthermore, either the sequence $\{gx_n\}$ contains a coincidence point of T and g, or at least one of the following conditions holds.

In case (a), the sequence $\{gx_n\}$ converges to $u \in g(X)$ and any point of $v \in X$ is a coincident point of T and g such that $gv = u$.

In cases (b) and (c), the sequence $\{gx_n\}$ is convergent to a coincidence point of T and g.

In addition, if $x,y \in X$ are the coincidence points of T and g, then $Tx = gx = gy = Ty$. If T (or g) is injective within the entire set of coincidence points of T and g, then T and g have a single coincidence point at most.

Proof. The proof is completed if $\{x_n\}$ contains a coincidence point of T and g. Suppose that $\{x_n\}$ does not contain any coincidence points of T and g, for all $n \geq 0$; that is,

$$gx_n \neq Tx_n = gx_{n+1}.$$

In this case, we have

$$d(gx_n, gx_{n+1}) > 0 \tag{2.2}$$

for all $n \geq 0$.

Now, the evidence will be examined in three cases.

Step 1. Using (ζb) and (1.1), $s \geq 1$ and for all $n \geq 0$,

$$\begin{aligned} C_F &\leq \zeta(s^4(d(Tx_n, Tx_{n+1})), d(gx_n, gx_{n+1})) \\ &= \zeta(s^4 d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1})) \\ &< F(d(gx_n, gx_{n+1}), s^4 d(gx_{n+1}, gx_{n+2})), \end{aligned} \tag{2.3}$$

for all $n \leq 0$, $0 < d(gx_{n+1}, gx_{n+2}) \leq s^4 d(gx_{n+1}, gx_{n+2}) < d(gx_n, gx_{n+1})$. Similarly, we can prove that $d(gx_{n+2}, gx_{n+3}) < d(gx_{n+1}, gx_{n+2})$. Therefore, $\{d(gx_n, gx_{n+1})\}$ is sub-zero, non-increasing and convergent.

Let $r > 0$ and $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = r$. Using axiom ($\zeta$c) to the sequences $\{t_n = d(gx_{n+1}, gx_{n+2})\}$ and $\{s_n = d(gx_n, gx_{n+1})\}$ with $t_n < s_n$,

$$C_F \leq \limsup_{n\to\infty} \zeta(s^4 d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1}) = \limsup_{n\to\infty} \zeta(s^4 t_n, s_n) < C_F.$$

Due to the with (2.3),

$$C_F \leq \limsup_{n\to\infty} \zeta(s^4 d(gx_{n+1}, gx_{n+2}), d(gx_n, gx_{n+1})),$$

for all $n \geq 0$, a contradiction is obtained. In this case, our assumption is incorrect. Therefore, we have $r = 0$; that is,

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0,$$

holds.

Step 2. Suppose that the sequence $\{gx_n\}$ is not a b-Cauchy sequence in (X,d). Then, there exits an $\varepsilon > 0$ and sequences of positive integers $\left\{gx_{n(k)}\right\}$ and $\left\{gx_{m(k)}\right\}$ with $n(k) > m(k) \geq k$ such that $d(gx_{m(k)}, gx_{n(k)}) > \varepsilon$, $d(gx_{m(k)}, gx_{n(k)-1}) < \varepsilon$. T, using (ζb) axiom and $(Z_{F,b}, g)$ contraction,we have

$$\begin{aligned} C_F &\leq \zeta(s^4(d(Tx_{m(k)}, Tx_{n(k)})), d(gx_{m(k)}, gx_{n(k)})) \\ &= \zeta(s^4 d(gx_{m(k)+1}, gx_{n(k)+1}), d(gx_{m(k)}, gx_{n(k)})) \\ &< F(d(gx_{m(k)}, gx_{n(k)}), s^4 d(gx_{m(k)+1}, gx_{n(k)+1})). \end{aligned}$$

It now consists of two different situations.

Case (i): $s = 1$.

In this case, (X,d) is a metric space. By Lemma 1.2 there exits $\varepsilon > 0$ and sequence of positive integers $\{gx_{n(k)}\}$ and $\{gx_{m(k)}\}$ such that $n(k) > m(k) \geq k$ with $d(gx_{m(k)}, gx_{n(k)}) > \varepsilon$, $d(gx_{m(k)}, gx_{n(k-1)}) < \varepsilon$ and satisfying (1)-(4) of Lemma 1.2 and using (ζc), $\{t_n = d(gx_{m(k)+1}, gx_{n(k)+1})\}$ and $\{s_n = d(gx_{m(k)}, gx_{n(k)})\}$, we have

$$\begin{aligned}
C_F &\leq \limsup_{n\to\infty} \zeta(d(gx_{m(k)+1}, gx_{n(k)+1}), d(gx_{m(k)}, gx_{n(k)})) \\
&< F(d(gx_{m(k)}, gx_{n(k)}), d(gx_{m(k)+1}, gx_{n(k)+1})) \\
&< C_F
\end{aligned}$$

which is a contradiction.

Case (ii): $s > 1$.

In this case, (X,d) is a b-metric space. By Lemma 1.3 there exist $\varepsilon > 0$ and sequences of positive integers $\{gx_{n(k)}\}$ and $\{gx_{m(k)}\}$ such that $n(k) > m(k) \geq k$ with $d(gx_{m(k)}, gx_{n(k)}) > \varepsilon$, $d(gx_{m(k)}, gx_{n(k-1)}) < \varepsilon$ and satisfying (1)-(4) of Lemma 1.3, we have

$$\begin{aligned}
C_F &\leq \limsup_{n\to\infty} \zeta(s^4 d(gx_{m(k)+1}, gx_{n(k)+1}), d(gx_{m(k)}, gx_{n(k)})) \\
&< F(d(gx_{m(k)}, gx_{n(k)}), s^4 d(gx_{m(k)+1}, gx_{n(k)+1})) \\
&< C_F
\end{aligned}$$

which is a contradiction.

Consequently, by (i) and (ii), we have $\{gx_n\}$, is a b-Cauchy sequence in (X,d).

Step 3. By assumptions (a), (b), (c), we will prove that T and g have a coincidence point.

Case (a): Suppose that $(g(X))$ (or $(T(X),d)$) is complete. We also found that the sequence $\{gx_n\}$ is a b-Cauchy sequence. In case for all $n \geq 0$, $gx_{n+1} = Tx_n \in T(X) \subseteq g(X)$, taking into account these, $u \in g(X)$, that is,

$$\lim_{n\to\infty} d(gx_n, u) = 0.$$

Since $Tx_n = gx_{n+1}$, for all n we have,

$$\lim_{n\to\infty} d(Tx_n, u) = 0. \tag{2.4}$$

Let $v \in X$ be any point such that $gv = u$. Suppose that v is not a coincidence point of T and g, then $gv = u \neq Tv$. In this case, we have $\delta = d(Tv, gv) > 0$. Using (2.4), $n_0 \in N$ be such that $d(gx_n, gv) < \delta$ for all $n \geq n_0$. This means that $d(gx_n, gv) < \delta = d(Tv, gv)$, for all $n \geq n_0$.

In particular, $gx_n \neq Tv$ for all $n \geq n_0$, then

$$d(Tx_n, Tv) = d(gx_{n+1}, gv) > 0, \text{ for all } n \geq n_0. \tag{2.5}$$

On the other hand, if $gx_n = gv$ for all $n \geq n_1$, it contradicts the condition (2.2) for $\exists\, n_1 \in N$.

Therefore, the sequence $\{gx_n\}$ has a subsequence $\{gx_{\delta(n)}\}$ with

$$gx_{\delta(n)} \neq gv. \tag{2.6}$$

Now, let $n_2 \in N$ such that $\delta(n_2) \geq n_0$. Therefore, for all $n \geq n_2$, by (2.5) and (2.6), $d(gx_{\delta(n)}, gv) > 0$ and $d(Tx_{\delta(n)}, Tv) > 0$. Using($\zeta$b),

$$\begin{aligned}
C_F &\leq \zeta(s^4 d(Tx_{\delta(n)}, Tv), d(gx_{\delta(n)}, gv))) \\
&< F(d(gx_{\delta(n)}, gv), s^4 d(Tx_{\delta(n)}, Tv))
\end{aligned}$$

this means that;

$$0 \leq d(Tx_{\delta(n)}, Tv) \leq s^4 d(Tx_{\delta(n)}, Tv) < d(gx_{\delta(n)}, gv) = d(gx_{\delta(n)}, u).$$

By $\lim_{n\to\infty} d(gx_{\delta(n)},u)=0$, $\lim_{n\to\infty} d(Tx_{\delta(n)},Tv)=0$. However, $\{Tx_{\delta(n)}\}=\{gx_{\delta(n)+1}\}$ is a supsequence of $\{gx_n\}$ and converges to gv. Due to the uniqueness of the limit, we have $gv=Tv$. This contradicts our assumption. Then, $u=gv=Tv$. In other words, v is a coincidence point of T and g.

Case (b): Suppose that (X,d) is complete. T and g are continuous and compatible. In this case the sequence $\{gx_n\}$ is $\{gx_n\}\longrightarrow u\in X$, since (X,d) is a b-Cauchy sequence on the complete b-metric space. Since T is continuous, $\{ggx_n\}\longrightarrow gu$. Since g is continuous, $\{Tgx_n\}\longrightarrow Tu$. Moreover, T and g are compatible, $\{Tx_n=gx_{n+1}\}$ and $\{gx_n\}$ have the same limit points, we deduce that

$$d(Tu,gu)=\lim_{n\to\infty} d(Tgx_n,ggx_{n+1})=\lim_{n\to\infty} d(Tgx_n,gTx_n)=0.$$

Therefore, u is a coincidence point of T and g.

Case (c): Suppose that (X,d) is complete and T and g are continuous and commuting. In this case, if T and g are commuting, then T and g will be compatible which is the same with case (b). □

Example 2.4. *Let $X=[0,1]$ and $d:X\times X\longrightarrow[0,\infty)$ be defined as*

$$d(x,y)=\begin{cases}0, & x=y,\\ (x-y)^2, & x\neq y,\end{cases}$$

Then, d is a b-metric with coefficient $s=2$ but it is not a metric. Consider the mappings $T,g:X\longrightarrow X$ defined by $Tx=x+3$ and $gx=5x+1$ for all $x\in X$. In order to solve the non-linear equation

$$x+3=5x+1$$

Theorem 2.3 can be applied using the simulation function $\zeta(t,s)=s-t$.

$$\begin{aligned}\zeta(s^4d(Tx,Ty),d(gx,gy)) &= d(gx,gy)-s^4d(Tx,Ty)\\ &=(5x+1-5y-1)^2-2^4.(x+3-y-3)^2\\ &=25(x-y)^2-16(x-y)^2\\ &=9(x-y)^2\\ &\geq 0.\end{aligned}$$

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