



AN OVERVIEW ON TOPOLOGY OPTIMIZATION METHODS EMPLOYED IN STRUCTURAL ENGINEERING

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Abstract

Any mechanical performance measure of a structure is strongly related with its topology. Size and shape optimization cannot give the best structural performance, since these methods cannot change the structure's topology. Hence, topology optimization should be employed to obtain the best performance. In this paper, a review of topology optimization is provided. At first, the general topology optimization problem is defined. Then, modern topology optimization methods are presented and discussed.

Keywords: Topology Optimization, Structural Optimization, Optimum Structural Design, Review

YAPISAL MÜHENDİSLİKTE KULLANILAN TOPOLOJİ ENİYİLEMESİ YÖNTEMLERİ ÜZERİNE GENEL BİR BAKIŞ

Özet

Bir yapının gösterdiği mekanik performans, o yapının topolojisi ile çok yakından alakalıdır. Boyut ve şekil eniyilemeleri sonucunda, yapının topolojisinde bir değişiklik olmadığı için, en iyi performans elde edilemez. Netice itibarıyla, en iyi performansın elde edilebilmesi için topoloji eniyilemesinden faydalanılması gerekmektedir. Bu çalışmada, topoloji eniyilemesi yöntemleri hakkında bir derleme sunulmuştur. İlk olarak, genel topoloji eniyilemesi problemi tanıtılmış, ardından modern topoloji eniyilemesi yöntemleri tartışılmıştır.

Anahtar Kelimeler: Topoloji Eniyilemesi, Yapısal Eniyileme, Eniyilenmiş Tasarım, Derleme

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1. INTRODUCTION

Structural optimization is a subspace of the general optimization field which concerns with obtaining optimized structures to achieve certain needs while satisfying some constraints. Three types of structural optimization problems are encountered in the field: size, shape and topology optimization. In size optimization, the geometry of the structure is known, but certain sizes (dimensions) are the design variables to be determined. In shape optimization, the topology is known, but the boundary curves/surfaces of the structure either parameterized or discretized design variables are to be found. Whereas in topology optimization, usually the whole design domain is discretized, then the boundary curves/surfaces as well as number of holes and the location of the holes inside the design domain can be altered.

Topology optimization [1, 2] is basically the determination of optimum material distribution in a design space which minimizes (or maximizes) an objective function while satisfying some constraints. The objective function can be compliance minimization (i.e., stiffness maximization) for static problems and, fundamental frequency or frequency gap maximization for dynamic problems [3]. A topology optimization problem is solved in a certain design space in which applied loads and boundary conditions are specified. In this design space, a volume fraction value, which indicates the ratio of solid volume to total design space volume, is also defined. Volume fraction determines how much of the design space will include solid material. In order to conduct topology optimization, a finite element model is needed. For two dimensional structures, a finite element model can be constructed by employing large number of square elements, which can be considered as pixels. Then, the topology optimization problem can be summarized as determining which pixels will include material and which pixels will be void, for the optimized design.

Topology optimization is the most comprehensive method to be employed in structural design, since it involves simultaneous size and shape optimization, as well. Because of this promise, it is widely studied in the structural engineering field [4-6]. In this study, a brief review on topology optimization fundamentals and modern topology optimization methods utilized in optimum structural design is provided.

2. TOPOLOGY OPTIMIZATION

2.1. Description of Topology

Mathematically, topology is concerned with the deformable objects, and all distortions are considered as transformations or reversibly unique mappings [7]. The topological transformations or topological mappings are the transformations which do not change the neighborhood relations, in other words, topological transformations cause topologically equivalent domains, therefore, topological property of a domain is invariant for all topological mappings [7].

Topological domains consist of all subsets of \mathbb{R}^3 . The degree of connection of domains determines its topology class. Domains belonging to the same topology class are considered as topologically equivalent [7] (see Figure 1a). Naming of topology classes can be done as follows: n -fold connected domains require $(n-1)$ cuts from one boundary to another, to convert them into a simply connected domain [7] (see Figure 1b).

According to the above definitions, size and shape optimizations cause topological mappings that result in topologically equivalent structures. However, topology optimization transforms the structure from one topology class to another.

2.2. Topology Optimization Problem

Although exact analytical solutions of topology optimization problems reveal basic characteristics of optimal designs, they are only capable of solving problems having simple load and support conditions [8]. Therefore, for more realistic problems, it is necessary to use a discretized design domain. Most of the studies in the literature use finite element formulation with a fixed mesh as the discretized model [9]. Here, the general form of topology optimization problem is provided for this model as:

$$\begin{aligned} & \text{minimize: } H(\mathbf{u}(\mathbf{x}), \mathbf{x}) \\ & \text{subject to: } G_j(\mathbf{u}(\mathbf{x}), \mathbf{x}) \geq 0 \quad \text{for } j = 1, 2, \dots, P \\ & \quad \quad \quad x_e = 0 \text{ or } 1 \quad \text{for } e = 1, 2, \dots, N \end{aligned} \quad (1)$$

where \mathbf{x} is the design variable vector, \mathbf{u} is the state field, H is the objective function, G_j is the j th constraint, P is the total number of constraints, x_e is the e th structural member that constitutes the design variable vector \mathbf{x} , N is the total number of design variables (structural members). One of

the mostly used constraints above is the so called volume (or mass) constraint [9].

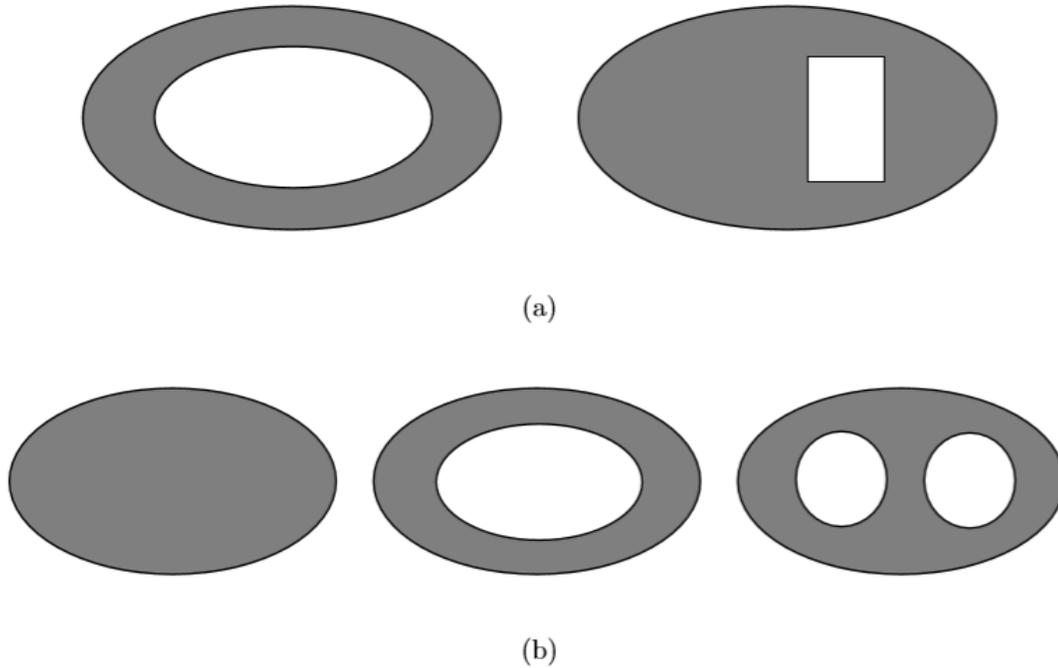


Figure 1. (a) Topologically equivalent domains. (b) Topological classes of simply, two-fold, three-fold connected domains.

The formulation given in Equation (1) most of the time represents a nonlinear topology optimization problem. Here, it should be emphasized that, design variables can only take a value of 0 or 1. In general, topology optimization problems lack solutions (i.e., ill-posed) with this current form [7, 9]. If design variables are defined as continuous, the problem relaxes and efficient gradient-based optimization algorithms can be utilized [9]. Then, the continuous topology optimization problem can be written as:

$$\begin{aligned} & \text{minimize: } H(\mathbf{u}(\mathbf{x}), \mathbf{x}) \\ & \text{subject to: } G_j(\mathbf{u}(\mathbf{x}), \mathbf{x}) \geq 0 \quad \text{for } j = 1, 2, \dots, P \\ & \quad \quad \quad 0 \leq x_e \leq 1 \quad \text{for } e = 1, 2, \dots, N \end{aligned} \quad (2)$$

The formulation in Equation (2) is extensively used in the modern topology optimization literature [9]. For instance a static topology optimization problem, where the objective function is minimization of the compliance (or equivalently maximization of the stiffness) of the structure, can be written as:

$$\begin{aligned} \text{minimize: } & H(\mathbf{x}) = \mathbf{u}(\mathbf{x})^t \mathbf{K} \mathbf{u}(\mathbf{x}) \\ \text{subject to: } & \mathbf{K} \mathbf{u} - \mathbf{f} = \mathbf{0} \\ & aV_0 - V(\mathbf{x}) \geq 0 \\ & 0 \leq x_e \leq 1 \quad \text{for } e = 1, 2, \dots, N \end{aligned} \quad (3)$$

where \mathbf{u} is the displacement field, \mathbf{K} is the global stiffness matrix, \mathbf{f} is the load vector, $V(\mathbf{x})$ is the total solid material volume, V_0 is the total design space volume, a is the volume fraction value which is the ratio between the total solid material volume and the total design space volume. In the problem given in Equation (3), mass constraint is employed via the volume fraction term. Moreover, static equilibrium equation must be satisfied in all of the iteration steps. A sample compliance minimization (i.e., stiffness maximization) topology optimization problem [10] description and its solution is provided in Figure 2. In this problem the volume fraction, a , is selected as 0.5 (i.e., half void-half solid design). The finite element discretization is 50×150 , therefore design space consists of 7500 elements (i.e., $V_0 = 7500$). The beam dimensions are $1 \text{ m} \times 3 \text{ m}$. Applied vertically downward force's (F) magnitude is 1 kN and an isotropic material with $E = 210 \text{ GPa}$ and $\nu = 0.3$ is utilized. In Figure 2, at left, the design space, applied load and the boundary conditions are seen. Whereas, at right, the topology optimized design is presented.

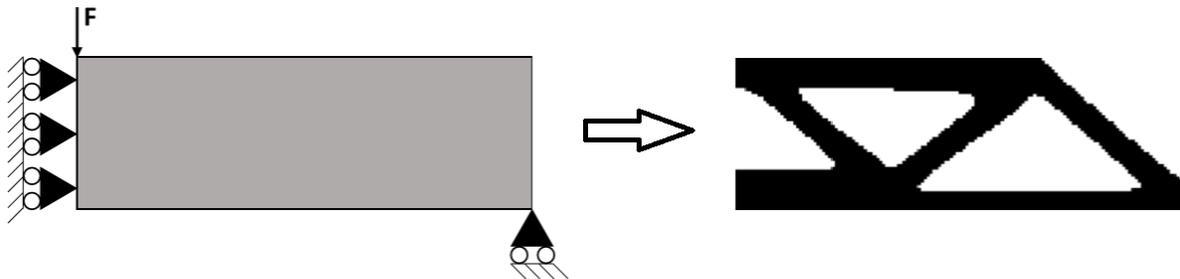


Figure 2. A compliance minimization topology optimization problem [10].

On the other hand, for dynamic problems, objective function can be n th natural frequency maximization or frequency gap maximization. A topology optimization problem in which n th natural frequency is maximized with a mass constraint can be defined as:

$$\text{maximize: } \omega_n^2(\mathbf{x}) = \frac{\mathbf{u}_n^t \mathbf{K} \mathbf{u}_n}{\mathbf{u}_n^t \mathbf{M} \mathbf{u}_n}$$

$$\text{subject to: } \mathbf{K}\mathbf{u}_n - \omega_n^2 \mathbf{M}\mathbf{u}_n = \mathbf{0} \quad (4)$$

$$aV_0 - V(\mathbf{x}) \geq 0$$

$$0 \leq x_e \leq 1 \quad \text{for } e = 1, 2, \dots, N$$

where ω_n is the n th natural frequency, \mathbf{u}_n is the n th mode shape vector and \mathbf{M} is the global mass matrix of the structure. Similarly a frequency gap maximization topology optimization problem in which the frequency gap between n th and $n+1$ th natural frequencies are maximized for a given mass constraint can be defined as:

$$\text{maximize: } \frac{\omega_{n+1}^2(\mathbf{x})}{\omega_n^2(\mathbf{x})}$$

$$\text{subject to: } \mathbf{K}\mathbf{u}_n - \omega_n^2 \mathbf{M}\mathbf{u}_n = \mathbf{0} \quad (5)$$

$$\mathbf{K}\mathbf{u}_{n+1} - \omega_{n+1}^2 \mathbf{M}\mathbf{u}_{n+1} = \mathbf{0}$$

$$aV_0 - V(\mathbf{x}) \geq 0$$

$$0 \leq x_e \leq 1 \quad \text{for } e = 1, 2, \dots, N$$

Sample natural frequency maximization and frequency gap maximization topology optimization problems are provided in Figure 3 and Figure 4, respectively.

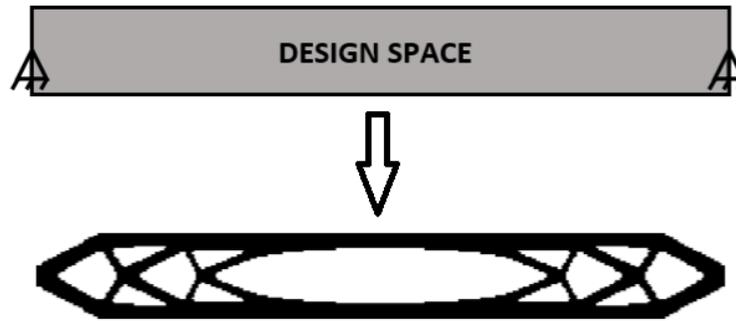


Figure 3. A fundamental frequency maximization topology optimization problem [10].

In Figure 3, a fundamental frequency maximization topology optimization problem [10] with 0.5 volume fraction value (a) is solved. The finite element discretization is 40×320 , therefore the total design space consists of 12800 elements (i.e., $V_o = 12800$). The beam dimensions are $1 \text{ m} \times 8 \text{ m}$. An isotropic material with $E = 210 \text{ GPa}$, $\nu = 0.3$ and $\rho = 7800 \text{ kg/m}^3$ is employed. At top of the Figure 3, the design space is provided and at bottom, the topology optimized design is

presented.

On the other hand, in Figure 4, a frequency gap maximization topology optimization problem [11] with 0.5 volume fraction value is given. The objective function is to maximize the gap between the second and the first natural frequencies of a compliant inertial amplification mechanism [12]. The finite element discretization is 50×100 , and mechanism has 50 mm height (vertical) and 100 mm length (horizontal). An isotropic material with $E = 210$ GPa, $\nu = 0.3$ and $\rho = 7800$ kg/m³ is used. At left of the Figure 4, the design space is given, whereas at right, the topology optimized design is presented.

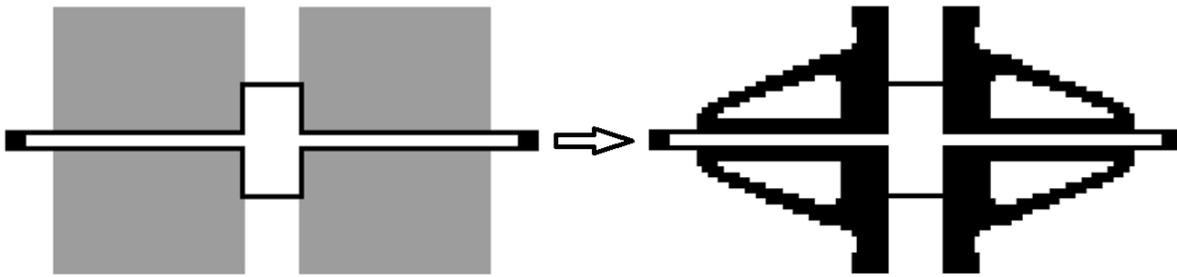


Figure 4. A frequency gap maximization topology optimization problem [11] of a compliant inertial amplification mechanism [12].

2.3 Early Historical Progress Achieved in Structural Topology Optimization

Topology optimization studies can be traced back more than a hundred years ago [13]. In Michell's pioneering work [13], optimal layout of truss structures which minimizes the weight is studied for a single load condition while only stress constraints are being employed. This type of structures are known as Michell trusses to honor their founder. Further discussions on Michell trusses are provided in [14]. However, the subject was untouched until 1950s [15]. Between 1950s and 1980s, various analytical and numerical studies on layout optimization are done (for detailed discussions see [8, 15-17]). In that period, conducted works are referred as layout optimization, since grid-like structures are studied. In topology optimization, the ratio of solid structure's volume to total design space volume is referred as the volume fraction. Therefore, layout optimization deals with low volume fraction designs like truss topology optimization [8, 18].



During 1980s, the necessity of studying high volume fraction design spaces emerged. In those years, it was discovered that, for the problems with high volume fractions, it is necessary to consider the microstructure of the system (see [15] for historical progress). The so-called microstructure approach is considered under the topic of generalized shape optimization [8, 18]. More clearly, generalized shape optimization is the topology optimization performed for high volume fraction design spaces and it is the most general problem formulation that also covers layout optimization. In this study, “generalized shape optimization” term is omitted. Instead, “topology optimization” is used. So, when the term “topology optimization” is seen in the text, it should be understood that the problem deals with the most general case, i.e., problems with high volume fractions.

The introduction of microstructure approach to topology optimization field leads the modern topology optimization techniques to emerge. In the next section, the basic categories of these techniques are discussed in detail.

3. BASIC CATEGORIES OF MODERN TOPOLOGY OPTIMIZATION METHODS

For almost 30 years, numerous topology optimization methods emerged. In a recent study [9], the close relationship among these methods are pointed out. However, a coarse classification of the existing methods is still possible. To that end, in this section, modern topology optimization methods are categorized and the popular ones are introduced within each subsection. One can refer to [7, 9, 18, 19], to extend the idea about the subject.

Here, it should be noted that, it is a tradition to develop and test the topology optimization algorithms on compliance minimization problems with a mass constraint. Unless specified, throughout this chapter, the methods presented are also for that kind of topology optimization problems, as well.

3.1. Homogenization Methods

The discrete nature of topology optimization problem formulation given in Equation (1) leads lack of existence of solutions even for the simple problems such as minimum compliance case [8]. One way of making the problem formulation relaxed and continuous as given in Equation (2)



is to introduce composite microstructures to the design domain [20]. To that end, mathematical theory of homogenization (see [21] for a brief review) is used to obtain a homogenized equivalent material model which replaces the composite, in turn, making the optimization problem relaxed and design variables continuous. The usage of homogenization theory is the reason why these techniques are called homogenization methods in the literature [7, 22-24].

In these methods, design domain is considered as being consisted of composite microstructures. These composite microstructures are composed of periodically repeating porous unit cells. These unit cells can be a hole-in-cell type or a layered type [7]. Other unit cell composite types can be found in the literature as well [20]. It is assumed that there exists infinitely many periodically distributed small unit cells within the microstructure. This assumption leads the continuous variation of material density throughout the microstructure [7]. This periodic microstructure's effective macroscopic properties depend on its unit cell geometry, and homogenization theory is utilized to calculate them at this point [7]. Following this procedure, a well-posed topology optimization problem formulation is obtained, as desired.

There exists mainly two homogenization methods which are classified according to the unit cell type used in the microstructure [18]. One of them uses optimal unit cell structures which are previously found for a certain type of problem. However, that situation brings a disadvantage, since advanced mathematical treatment to obtain optimal unit cells is needed for other problem types [18]. The other method utilizes non optimal unit cells [20]. This type of unit cell consists of a rectangular void inside and isotropic material on the edges. For this type of structure, rectangular void lengths (a and b) and unit cell orientation angle with respect to global coordinate system (θ) are the design variables. Design variables a and b are continuous and they can vary between 0 and 1. By homogenization, mechanical properties of the microstructure which is composed of infinitely many unit cells can be defined in terms of a , b and θ . In other words, when homogenization method is employed, a , b and θ variables actually define the configuration of the microstructure. In discretized domain, a microstructure corresponds to an individual finite element. Simple microstructure definition of this type is used in the first homogenization method topology optimization study in the literature [22] and caused the modern topology optimization field to emerge.

3.2. Density Methods

Density methods [25-27] are introduced after the homogenization methods. In these methods, for an individual finite element (i.e., microstructure) there exists only one design variable which is the element density (x_e). A microstructure's material property contribution to the whole structure depends on this density design variable which can take any value between 0 and 1. With this formulation, various well-known optimization algorithms (see [28]) can be utilized to solve the topology optimization problem. Moreover, density methods involve penalization parameters, which penalize intermediate densities to obtain manufacturable solid-void designs with clear material boundaries instead of the perforated ones acquired with homogenization methods. Various density methods can be found in the literature [9, 19]. Here, the most popular of them, "Solid Isotropic Microstructure with Penalization" (SIMP) method [25, 26] is addressed.

The SIMP method [25, 26] is a density method in which elastic properties of a microstructure is proportional to an exponential multiple of an element density (x_e) function. Exponential multiplier of an element density function (penalization number, p) causes the continuous and differentiable variation of the microstructure property of an individual finite element, therefore the topology optimization problem is converted into a continuous problem (like Equation 2). This allows one to use the gradient based mathematical programming methods. Moreover, in the presence of a volume constraint, the use of penalization number (p) values larger than 1 (i.e., $p > 1$) penalizes microstructures with intermediate densities [7]. When $p > 1$, the microstructures with intermediate densities contribute to structure's elasticity less than the structure's mass. In other words, they do not provide enough stiffness for the structure compared to their mass. So it becomes uneconomical for the structure to hold them. Consequently microstructures (finite elements) with intermediate densities are penalized and they are forced to vanish through the optimization process. As a result, a void-solid (0-1) type of a final design is obtained. On the other hand, the common numerical instabilities in topology optimization problems, such as local minima problem, checkerboard pattern formation and more importantly, mesh dependency issues [29], can also occur in the SIMP method with its current formulation, as well.

In Section 2.2, the need to transform the original problem in Equation (1) to Equation (2) is mentioned. One way to successfully achieve this is the homogenization approach (see Section

3.1). On the other hand, the relaxation attained by the SIMP method with its current form is not satisfactory enough to make the problem well-posed [7]. The nonexistence of solutions forces the optimization to converge to different results for different finite element mesh discretizations. This phenomenon is known as the mesh dependency of solutions in the literature [29]. To close the design space in an appropriate sense, i.e., to make the optimization problem well-posed, problem regularization is needed [7]. Thus, lack of convergence related with mesh dependency can be discarded. There exist many problem regularization techniques in the literature [30-33].

3.3 Discrete Methods

Usage of continuous density design variables in the topology optimization problems results in porous or grey regions in the optimized structure if proper penalization or problem regularization is not applied. Discrete methods concern with Equation (1) instead of Equation (2) in an attempt to obtain directly 0-1 (i.e., void-solid) designs. For this discrete problem, a natural choice is to use non-gradient optimization approaches such as Genetic Algorithms, Simulated Annealing, Particle Swarms, Ant Colonies, etc. However, these methods are inefficient for topology optimization problems in which large number of design variables are involved [34]. They can only solve problems with coarse finite element meshes [34]. Moreover, these algorithms also need some type of problem regularization or filtering to obtain physically reliable designs.

There exists another type of approach so-called evolutionary methods [35]. Inspired from the natural evolutionary process, evolutionary structural optimization (ESO) method [36] is suggested. In the ESO method, the optimum is sought by removing some of the inefficient material from the design domain in every iteration step. The determination of inefficient material depends on the type of objective function and constraints. However, as the material can only be removed (not added) during optimization, sometimes non-optimized designs are obtained [15]. To cope with this problem, “bi-directional evolutionary structural optimization” (BESO) method [37] is suggested. In the BESO method, material is being added in the vicinity of overstressed elements while in the inefficient areas, material is being removed.

Despite its conceptual simplicity, ESO/BESO methods are subjected to criticism [38-40]. These include, intuitive nature of the procedure, obtaining non optimal designs at the end of the process and difficulties encountered with multiple constraints [39]. As a consequence of these, a new

BESO algorithm is suggested recently [41, 42]. In the new BESO algorithm, design variable sensitivities are utilized with penalization in order to sort the elements with respect to their efficiency. One further novelty in the method is, none of the elements are completely removed from the design space, instead, inefficient elements get a predetermined minimum value. By this, it is claimed that, void elements can become solid in any iteration step of the algorithm. It is argued in the literature that, the new BESO algorithm can be categorized as a discrete update version of the standard SIMP scheme [9].

3.4. Boundary Variation Methods

Boundary variation methods are the recent development in the topology optimization field. They emerged from the need to obtain crisp and smooth edged structures as a result of optimization procedure. These methods have their roots in shape optimization techniques, however, they are fundamentally different, because they allow formation and disappearance of void regions inside the design domain in addition to allowing structural boundary movements [19]. There exist numerous boundary variation methods in the literature, but here, only the most popular level set methods are considered. For other alternative approaches, refer to [9, 19].

In the level set approach [43-45], the level set function determines where the boundaries, void and solid sections take place. Material boundaries exist where the level set function takes a certain constant value (usually it is 0). Where the level set function values are below this constant there exist void regions, whereas solid regions take place where the level set function values are greater than that constant. Changes in structure's topology are achieved by level set function updating, and mostly it is performed by solving the augmented Hamilton-Jacobi equation with diffusive and reactive terms [9]. In this manner, the original Hamilton-Jacobi equation is utilized for shape updating, reactive term is used for nucleation of new holes and diffusive term is employed to smooth the level set field [9].

Level set methods are mainly composed of level set function parameterization, mechanical model and optimization parts [46]. Firstly, design variables are parameterized via level set function. Then, geometry described by the level set function is mapped onto the mechanical structural model. Finally, design update is performed [46]. Depending on the level set method type used, in each of these three parts, regularization techniques can be employed to regularize ill-posedness of

the topology optimization problem [46]. Many level set methods utilize the regularization techniques used in density methods. Moreover, similar material interpolation schemes with density methods are utilized in the geometry mapping part of many level set methods, as well. A comprehensive review on the level set methods can be found in [46].

4. CONCLUSION

In this paper, a brief review on topology optimization methods used in structural engineering is presented. Topology optimization problem is defined and early historical progress in the field is addressed. The modern structural topology optimization methods are classified into four main categories as: homogenization methods, density methods, discrete methods and boundary variation methods. Each of these methods is discussed considering major points. Finally, the list of the pioneering works for the basic categories of the modern topology optimization methods is provided in Table 1.

Table 1. The list of the pioneering works for modern topology optimization methods.

The Pioneering Work	Method	Year
Reference [22]	Homogenization	1988
Reference [25]	Density	1989
Reference [36]	Discrete	1993
Reference [44]	Boundary Variation	2003

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