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A Note on a Banach Algebra

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Abstract. In this paper, we discuss and investigate Segal algebra using the Hardy -Littlewood maximal operator in amalgam spaces.

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1. Main Results

Let G be a locally compact abelian group with Haar measure μ. An amalgam space $(L^p, \ell^q)(G)$ $(1 \leq p, q \leq \infty)$ is a Banach space of measurable (equivalence classes of) functions on G which belong locally to L^p and globally to ℓ^q. The first systematic study of amalgams on the real line was undertaken by [3]. In 1979, Stewart [5] extended the definition of Holland to locally compact abelian groups using the Structure Theorem for locally compact groups. A Banach function space (shortly BF-space) on G is a Banach space $(B, \|.\|_B)$ of measurable functions which is continuously embedded into $L^1_{loc}(G)$, i.e. for any compact subset $K \subset G$ there exists some constant $C_K > 0$ such that $\|f\chi_K\|_{L^1} \leq C_K \|f\|_B$ for all $f \in B$.

We denote by $L^p_{loc}(G)$ $(1 \leq p \leq \infty)$ the space of (equivalence classes of) functions on G such that f restricted to any compact subset E of G belongs to $L^p(G)$. Let $1 \leq p, q < \infty$. The amalgam of L^p and ℓ^q on the real line is the normed space

$$(L^p, \ell^q) = \left\{ f \in L^p_{loc}(\mathbb{R}) : \|f\|_{pq} < \infty \right\},$$

where

$$\|f\|_{pq} = \left[\sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p \, dx \right]^{q/p} \right]^{1/q}. \tag{1.1}$$

We make the appropriate changes for p, q infinite. Now we show that the norm $\|.\|_{pq}$ makes (L^p, ℓ^q) into a Banach space [3].

Theorem 1.1. *Let $J_k = [k, k+1)$, $k \in \mathbb{Z}$ and $1 \leq p, q < \infty$. The space (L^p, ℓ^q) is a Banach space with respect to the norm $\|.\|_{pq}$.*

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Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in (L^p, ℓ^q). Then given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m \geq N$, then

$$\|f_n - f_m\|_{pq} = \left(\sum_{k\in\mathbb{Z}} \|f_n - f_m\|_{p,J_k}^q \right)^{1/q} < \varepsilon. \tag{1.2}$$

Hence, for any fixed k,

$$\|f_n - f_m\|_{p,J_k} < \varepsilon \quad (n, m \geq N).$$

Thus $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^p(J_k)$ for $k \in \mathbb{Z}$. Define $f = \sum_{k\in\mathbb{Z}} f^k \chi_{J_k}$, where $f^k \in L^p(J_k)$. Now we show that $f \in (L^p, \ell^q)$. Using Fatou's Lemma (applied to the right-hand series viewed as integral over the integers)

$$\begin{aligned} \|f\|_{pq}^q &= \sum_{k\in\mathbb{Z}} \left\|f^k\right\|_{p,J_k}^q = \sum_{k\in\mathbb{Z}} \lim_{n\to\infty} \|f_n\|_{p,J_k}^q \\ &\leq \liminf_{n\to\infty} \|f_n\|_{pq}^q . \end{aligned} \tag{1.3}$$

The last quantity is finite since $\{f_n\}_{n=1}^{\infty}$, being Cauchy, is bounded in norm. Hence, the left side of (1.3) is finite, that is, $f \in (L^p, \ell^q)$. To show that $\{f_n\}_{n=1}^{\infty}$ convergences in (L^p, ℓ^q) to f we have, for $m \geq N$

$$\|f_m - f\|_{p,J_k}^q = \lim_{n\to\infty} \|f_m - f_n\|_{p,J_k}^q ,$$

$$\begin{aligned} \|f_m - f\|_{pq}^q &= \sum_{k\in\mathbb{Z}} \lim_{n\to\infty} \|f_m - f_n\|_{p,J_k}^q \\ &\leq \liminf_{n\to\infty} \sum_{k\in\mathbb{Z}} \|f_m - f_n\|_{p,J_k}^q \\ &< \varepsilon^q \end{aligned}$$

by (1.2). Thus the Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ converges to f, which is what we wished to show. □

The following definition of $(L^p, \ell^q)(G)$ is due to Stewart [5]. By the Structure Theorem ([2], Theorem 24.30), $G = \mathbb{R}^a \times G_1$, where a is a nonnegative integer and G_1 is a locally compact abelian group which contains an open compact subgroup H. Let $I = [0, 1)^a \times H$ and $J = \mathbb{Z}^a \times T$, where T is a transversal of H in G_1, i.e. $G_1 = \bigcup_{t\in T} (t + H)$ is a coset decomposition of G_1. For $\alpha \in J$ we define $I_\alpha = \alpha + I$, and therefore G is equal to the disjoint union of relatively compact sets I_α. We normalize μ so that $\mu(I) = \mu(I_\alpha) = 1$ for all α. Let $1 \leq p, q \leq \infty$. The amalgam space $(L^p, \ell^q)(G) = (L^p, \ell^q)$ is a Banach space

$$\left\{ f \in L_{loc}^p(G) : \|f\|_{pq} < \infty \right\},$$

where

$$\begin{aligned} \|f\|_{pq} &= \left[\sum_{\alpha\in J} \|f\|_{L^p(I_\alpha)}^q \right]^{1/q} \quad \text{if } 1 \leq p, q < \infty, \\ \|f\|_{\infty q} &= \left[\sum_{\alpha\in J} \sup_{x\in I_\alpha} |f(x)|^q \right]^{1/q} \quad \text{if } p = \infty,\ 1 \leq q < \infty, \\ \|f\|_{p\infty} &= \sup_{\alpha\in J} \|f\|_{L^p(I_\alpha)} \quad \text{if } 1 \leq p < \infty, q = \infty. \end{aligned} \tag{1.4}$$

If $G = \mathbb{R}$, then we have $J = \mathbb{Z}$, $I_\alpha = [\alpha, \alpha + 1)$ and (1.4) becomes (1.1).

The amalgam spaces (L^p, ℓ^q) satisfy the following relations and inequalities [5]:

$$\begin{gathered} (L^p, \ell^{q_1}) \subset (L^p, \ell^{q_2}) \quad q_1 \leq q_2 \\ (L^{p_1}, \ell^q) \subset (L^{p_2}, \ell^q) \quad p_1 \geq p_2 \\ (L^p, \ell^p) = L^p \\ (L^p, \ell^q) \subset L^p \cap L^q, \quad p \geq q \\ L^p \cup L^q \subset (L^p, \ell^q), \quad p \leq q \\ \|f\|_{pq_2} \leq \|f\|_{pq_1}, \quad q_1 \leq q_2 \\ \|f\|_{p_1 q} \leq \|f\|_{p_2 q}, \quad p_1 \leq p_2. \end{gathered}$$

Note that C_c is included in all amalgam spaces.

Theorem 1.2. *If p, q, r, s are exponents such that $1/p + 1/r - 1 = 1/m \le 1$ and $1/q + 1/s - 1 = 1/n \le 1$, then*

$$(L^p, \ell^q) * (L^r, \ell^s) \subset (L^m, \ell^n).$$

Moreover, if $f \in (L^p, \ell^q)$ and $g \in (L^r, \ell^s)$, then

$$\begin{aligned} \|f * g\|_{mn} &\le 2^a \|f\|_{pq} \|g\|_{rs} \text{ if } m \ne 1 \\ \|f * g\|_{1n} &\le 2^{2a} \|f\|_{1q} \|g\|_{1s}. \end{aligned} \tag{1.5}$$

By (1.5) the inequality

$$\|f * g\|_{pq} \le C \|f\|_{pq} \|g\|_{11} = C \|f\|_{pq} \|g\|_1$$

satisfies for all $f \in (L^p, \ell^q)$ and $g \in \left(L^1, \ell^1\right) = L^1$, where $C \ge 1$, i.e. the amalgam space (L^p, ℓ^q) is a Banach L^1-module with respect to convolution ([4], p. 60). Also it is easy to see that the amalgam space $\left(L^p, \ell^1\right)$ is a Banach algebra under convolution $p \ge 1$. In fact using Young's inequality for amalgams (1.5), we have

$$\|f * g\|_{p1} \le C \|f\|_1 \|g\|_{p1} \le C \|f\|_{p1} \|g\|_{p1}.$$

Thus $|||f|||_{p1} = C \|f\|_{p1}$ defines a norm in $\left(L^p, \ell^1\right)$ under which $\left(L^p, \ell^1\right)$ is a Banach algebra. Recall that $\left(L^p, \ell^1\right) \subset L^1$.

The translation operator T_y is given by $T_y f(x) = f(x - y)$ for $x \in G$. $(B, \|.\|_B)$ is called strongly translation invariant if one has $T_y f \in B$ and $\left\|T_y f\right\|_B = \|f\|_B$ for all $f \in B$ and $y \in G$.

Theorem 1.3 ([4], Theorem 3.11)**.** *Let $1 \le p, q < \infty$. If for each $y \in G$ and $f \in (L^p, \ell^q)$, then $\left\|T_y f\right\|_{pq} \le 2^a \|f\|_{pq}$, i.e. the amalgam space (L^p, ℓ^q) is translation invariant.*

Theorem 1.4 ([4], Theorem 3.14)**.** *Let $1 \le p, q < \infty$. Then the mapping $y \to T_y$ is continuous from G into (L^p, ℓ^q).*

A subalgebra $S(G)$ of $L^1(G)$ is called a Segal algebra if:

(**S − 1**) $S(G)$ is dense in $L^1(G)$ and if $f \in S(G)$ then $T_y f \in S(G)$;

(**S − 2**) $S(G)$ is a Banach algebra under some norm $\|.\|_{S(G)}$ which also satisfies $\|f\|_{S(G)} = \left\|T_y f\right\|_{S(G)}$ for all $f \in S(G)$, $y \in G$;

(**S − 3**) if $f \in S(G)$, then for every $\varepsilon > 0$ there exists a neighborhood U of the identity element of G such that $\left\|T_y f - f\right\|_{S(G)} < \varepsilon$ for all $y \in U$.

Now we use the fact that (L^p, ℓ^q) has an equivalent translation-invariant norm $\|.\|^\sharp_{pq}$.

Theorem 1.5 (([4], Theorem 1.21), [1])**.** *A function f belongs to (L^p, ℓ^q), $1 \le p, q \le \infty$, iff the function $f^\sharp$ on G defined by $f^\sharp(x) = \|f\|_{L^p(x+E)}$belongs to $L^q(G)$. If $\|f\|^\sharp_{pq} = \left\|f^\sharp\right\|_q$, then $2^{-a}\|f\|_{pq} \le \|f\|^\sharp_{pq} \le 2^a \|f\|_{pq}$,where E is open precompact neighborhood of 0 and*

$$\|f\|^\sharp_{pq} = \left[\int_G \|f\|^q_{L^p(x+E)}\, dx\right]^{1/q}.$$

Hence the norms $\|.\|_{pq}$ and $\|.\|^\sharp_{pq}$ are equivalent.

Corollary 1.6. *If $f \in (L^p, \ell^q)$ and $f^\sharp(x) = \|f\|_{L^p(x+E)}$, then it is obtained that*

$$\left(T_y f\right)^\sharp(x) = \left\|T_y f\right\|_{L^p(x+E)} = \|f\|_{L^p(x+y+E)} = f^\sharp(x + y) = T_{-y} f^\sharp(x)$$

and

$$\left\|T_y f\right\|^\sharp_{pq} = \left\|\left(T_y f\right)^\sharp\right\|_q = \left\|T_{-y} f^\sharp\right\|_q = \left\|f^\sharp\right\|_q = \|f\|^\sharp_{pq}.$$

So the space (L^p, ℓ^q) is strongly translation invariant with respect to $\|.\|^\sharp_{pq}$.

Theorem 1.7 ([4], Theorem 4.16)**.** *Let $1 \le p < \infty$. Then the amalgam space $\left(L^p, \ell^1\right)$ becomes a Segal algebra with respect to $\|.\|^\sharp_{pq}$.*

Definition 1.8. For $x \in \mathbb{R}^n$ and $r > 0$ we denote an open ball with center x and radius r by $B(x, r)$. For $f \in L^1_{loc}(\mathbb{R}^n)$, we denote the (centered) Hardy-Littlewood maximal operator Mf of f by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

where the supremum is taken over all balls $B(x, r)$.

Let $1 \leq p, q < \infty$. We define the space

$$A^{p,q}(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) : Mf \in (L^p, \ell^q) \right\},$$

and the norm $\|.\|$ given by

$$\|f\| = \|f\|_1 + \|Mf\|_{pq}$$

for $f \in A^{p,q}(\mathbb{R}^n)$ due to the fact that Mf is a sublinear operator.

Theorem 1.9. *The space $(A^{p,q}(\mathbb{R}^n), \|.\|)$ is a Banach algebra according to convolution.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $A^{p,q}(\mathbb{R}^n)$. Clearly $(f_n)_{n\in IN}$ and $(Mf_n)_{n\in IN}$ are Cauchy sequences in L^1 and (L^p, ℓ^q), respectively. Since L^1 is Banach space, then there exist $f \in L^1$ such that $\|f_n - f\|_1 \to 0$. For $\varepsilon > 0$ there exists a positive integer N such that $\|M(f_n - f_m)\|_{pq} < \varepsilon$ whenever $n, m \geq N$. Without loss of generality we may assume f_n converges to f almost everywhere.

We have, $m \geq N$,

$$\begin{aligned} \|M(f - f_m)\|_{pq}^q &= \sum_{k\in\mathbb{Z}} \|M(f - f_m)\|_{p,J_k}^q \\ &= \sum_{k\in\mathbb{Z}} \lim_{n\to\infty} \|M(f_n - f_m)\|_{p,J_k}^q \\ &\leq \liminf_{n\to\infty} \sum_{k\in\mathbb{Z}} \|M(f_n - f_m)\|_{p,J_k}^q \\ &= \liminf_{n\to\infty} \|M(f_n - f_m)\|_{pq}^q < \varepsilon^q. \end{aligned}$$

Therefore $f \in A^{p,q}(\mathbb{R}^n)$ and $\|f_m - f\| \to 0$ as $m \to \infty$. This asserts that $A^{p,q}(\mathbb{R}^n)$ is a Banach space.

Now let $f, g \in A^{p,q}(\mathbb{R}^n)$ be given. Then we write $f, g \in L^1$. Since L^1 is a Banach algebra under convolution, then $f * g \in L^1$ and the inequality

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

is satisfied. It is well known that the maximal operator is bounded in (L^p, ℓ^q). Hence using

$$\|M(f * g)\|_{pq} \leq C \|f * g\|_{pq}$$

and

$$\|f * g\|_{pq} \leq C \|f\|_{pq} \|g\|_1, \tag{1.6}$$

where $C \geq 1$, we have

$$\begin{aligned} \|f * g\| &= \|f * g\|_1 + \|M(f * g)\|_{pq} \\ &\leq \|f\|_1 \|g\|_1 + C \|f * g\|_{pq} \\ &\leq \|f\|_1 \|g\|_1 + C \|f\|_{pq} \|g\|_1 \\ &\leq C \|f\| \|g\|. \end{aligned} \tag{1.7}$$

Therefore $f * g \in (L^p, \ell^q)$. If we define the norm $\|f\|^* = C(q, r) \|f\|$ on $A^{p,q}(\mathbb{R}^n)$, then $A^{p,q}(\mathbb{R}^n)$ is a Banach algebra by (1.6) and (1.7). □

Recall that,

$$\begin{aligned} M\left(T_y f\right) &= T_y M f, \\ C_c(\mathbb{R}^n) &\subset L^1(\mathbb{R}^n) \cap (L^p, \ell^q) \subset A^{p,q}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \end{aligned}$$

for $1 < p, q < \infty$.

Theorem 1.10. *The space* $\left(A^{p,q}(\mathbb{R}^n), \|.\|_{q,r}^{p1}\right)$ *is a Segal algebra.*

Proof. Since $C_c(\mathbb{R}^n) \subset A^{p,q}(\mathbb{R}^n)$ and $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, then $A^{p,q}(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$. So $A^{p,q}(\mathbb{R}^n)$ is a Segal Algebra by Theorem 1.4, Corollary 1.6 and Theorem 1.9. □

Conflicts of Interest

The author declare that there are no conflicts of interest regarding the publication of this article.

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