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A Note on Laplacian Spectrum of Complementary Prisms

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Abstract. In this work, the Laplacian spectrum of Complementary Prism graph is considered. The complementary prism operation was introduced by Haynes et al. and denoted by $G\bar{G}$. Some upper and lower bounds obtained using majorization and operator definition of Laplacian. Beside Cardoso et al.'s results in literature about Laplacian spectrum of complementary prisms, an alternative proof about nonzero minimum and maximum Laplacian eigenvalue of complementary prism that contains disconnected components in the underlying graph G or $\bar{G}$ is provided. Also using this result, the lower and upper bound of nonzero minimum and maximum Laplacian eigenvalue of the complementary prism graph is emphasized.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *Laplacian matrix* of the graph G is the $n \times n$ matrix $L(G) = D(G) - A(G)$ where $D(G) = \text{diag}\{d_1, d_2, \ldots, d_n\}$ is the diagonal matrix of vertex degrees denoted by d_i for each $i \in V(G)$ and $A(G) = (a_{ij})$ is the $(0, 1)$-adjacency matrix of the graph G, that is, $a_{ij} = 1$ if i and j are adjacent vertices and $a_{ij} = 0$ otherwise.

$$L_{ij}(G) = \begin{cases} d_i & , i = j \\ -1 & , (i, j) \in E(G) \\ 0 & , otherwise \end{cases}$$

L can be viewed as an operator on the space of functions $f : V(G) \to \mathbb{R}$ satisfying

$$Lf(i) := d_i f(i) - \sum_{j, j\sim i} f(j). \tag{1.1}$$

$L(G)$ is real and symmetric matrix. Eigenvalues of L are real and non-negative, the smallest eigenvalue is equal to zero with constant eigenvector $\mathbf{1}$ and is a simple for connected graph. Multiplicity of zero eigenvalue of $L(G)$ is equal to number of connected components of graph. We denote λ_{min} and λ_{max} the smallest nonzero eigenvalue and largest eigenvalue, respectively. All eigenvectors of L orthogonal to each other. Thus, for an eigenvector $f = (f_i)$ corresponds

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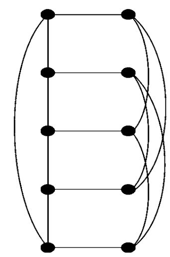

Figure 1. Example of Complementary Prisms of a graphs C_5 and $\overline{C_5}$

to a nonzero eigenvalue the sum $\sum_{i \in V(G)} f_i = 0$. In addition, smallest positive eigenvalue λ_{min} is called the algebraic connectivity of the graph [2]. Eigenvectors of $L(G)$ associated with algebraic connectivity is called Fiedler vectors.

Several research has been done for Laplacian spectrum of many graph operations which combines graphs with different ways. Adjacency, Laplacian, signless Laplacian spectra of product graphs and other operations have taken attention of many researchers. In this work, the Laplacian spectrum of Complementary prism graph is considered. The complementary prism operation was introduced by Haynes et al. [4] and denoted by $G\bar{G}$. In Section 2, some useful definitions and results are given. In Section 3, some upper and lower bounds are obtained using majorization and operator definition of Laplacian. In [1], it is provided a result about the Laplacian spectrum of complimentary prisms. Supporting result of Cardoso et al. about Laplacian spectrum of complementary prism graph, an alternative proof about nonzero minimum and maximum Laplacian eigenvalue of complementary prisms which has disconnected construction in underlying graph G or $\bar{G}$ is given.

2. Preliminaries

Definition 2.1 ([4]). Complementary prisms of a graph G, denoted as $G\bar{G}$. Let G be a graph and $\bar{G}$ be the complement of G is the graph with $V(G)$=$V(\bar{G})$ and $E(\bar{G})$=$E(K_n) \setminus E(G)$.

The complementary prism $G\overline{G}$ of G is the graph formed from the disjoint union $G \cup \bar{G}$ of G and $\bar{G}$ by adding the edges of a perfect matching between the corresponding vertices (same label) of G and $\bar{G}$.

$$L(G\bar{G}) = \begin{bmatrix} L(G) + I_n & -I_n \\ -I_n & L(\bar{G}) + I_n \end{bmatrix}$$

Example 2.2. Petersen Graph is a Complementary Prism graph shown in Figure 1.

Following theorem was proved by Cardoso et al. provide information about Laplacian spectrum of complementary prism graph.

Theorem 2.3 ([1]). *Let G be graph on n vertices with Laplacian eigenvalues* $\mu_1 \geqslant \mu_2 \geqslant ... \geqslant \mu_{n-1} \geqslant \mu_n = 0$. *For each* $i = 1, ..., n-1$, *if* $(\mu_i, \boldsymbol{u}_i)$ *is a* $\boldsymbol{L}$*-eigenpair of G, then*

$$\tau_{1,2}(\mu_i) = \frac{(n+2) \pm \sqrt{(n-2\mu_i)^2 + 4}}{2}$$

are $\boldsymbol{L}$*-eigenvalues of* $G\bar{G}$ *with associated eigenvectors* $\binom{\mathbf{u_i}}{(\mu_i - \tau_{1,2}(\mu_i))\mathbf{u_i}}$. *The others* $\boldsymbol{L}$*-eigenvalues of* $G\bar{G}$ *are 2 and 0 with associated eigenvectors* $\binom{\mathbf{j}}{-\mathbf{j}}$ *and* $\binom{\mathbf{j}}{\mathbf{j}}$*, respectively.*

Following useful majorization definition that will use to construct relationship between Laplacian spectra of G, $\bar{G}$ and $G\bar{G}$ in the next section.

Definition 2.4 ([3]). Suppose that $b = (b_1, \ldots, b_p)$ and $c = (c_1, \ldots, c_q)$ are sequences of non negative real numbers arranged in non-increasing order. We say that b *majorizes* c, denoted by $c \prec b$, if

$$\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} c_i, \quad 1 \leq k \leq \min\{p, q\}$$

and

$$\sum_{i=1}^{p} b_i = \sum_{i=1}^{q} c_i.$$

Theorem 2.5 (Fan 1954 [3]). *Let H and $\bar{H}$ be $n \times n$ Hermitian matrices having the form*

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix},$$

where $H_{11} : l \times l,\ H_{22} : m \times m,\ l + m = n$, and μ is sequence of spectrum of matrices in non increasing order. Then,

$$(\mu(H_{11}), \mu(H_{22})) = \mu(\bar{H}) \prec \mu(H).$$

3. Main Results

For the next result, the majorization concept will be used. For a symmetric matrix M, the notation $\mu(M) + 1$ means adding 1 to each element of spectrum of M.

Lemma 3.1. *Let $G\bar{G}$ be the complementary prism of G. Let μ is sequences of eigenvalues in non increasing order. Then,*

$$(\mu(L(G)) + 1, \mu(L(\bar{G})) + 1) \prec \mu(L(G\bar{G})).$$

Proof.

$$L(G\bar{G}) = \begin{bmatrix} L(G) + I_n & -I_n \\ -I_n & L(\bar{G}) + I_n \end{bmatrix}, \quad L' = \begin{bmatrix} L(G) + I_n & 0 \\ 0 & L(\bar{G}) + I_n \end{bmatrix}.$$

We can obtain the result applying Theorem 2.5 to L and L':

$$(\mu(L(G) + 1), \mu(L(\bar{G}) + 1)) = (\mu(L(G) + I_n), \mu(\bar{G}) + I_n) \prec \mu(L(G\bar{G})).$$ □

Remark 3.2. Let $G\bar{G}$ is complementary prism graph. It is known from Theorem 2.3, 2 is eigenvalue of $L(G\bar{G})$. Furthermore, if $f = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ is the corresponding eigenfunction, then

$$\sum_{i=1}^{n} x_i = -\sum_{i=1}^{n} y_i \neq 0.$$

Following lemma can be said from Theorem 2.3. It is also show relationshi between nonzero maximum and minimum eigenvalues using alternative way.

Lemma 3.3. *Let $G\bar{G}$ be the $2n$ order complementary prism of G.*

$$\lambda_{\min}(G\bar{G}) + \lambda_{\max}(G\bar{G}) = n + 2 \tag{3.1}$$

where $\lambda_1 = 0 < \lambda_{\min}(G\bar{G}) \leq \cdots \leq \lambda_{\max}(G\bar{G})$ is spectrum of $L(G\bar{G})$.

Proof. Let $f = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ be an eigenvector of $L(G\bar{G})$ where $\sum_{i \in V(G\bar{G})} f_i = 0$, corresponding to k^{th} $(2 \leq k \leq 2n)$ eigenvalue λ_k. We know from (1.1) that, for all $i \in V(G)$,

$$(d_i + 1)x_i - \sum_{\substack{j \sim i \\ i,j \in V(G)}} x_j - y_i = \lambda_k x_i \tag{3.2}$$

for all $\bar{i} \in V(\bar{G})$

$$(d_{\bar{i}} + 1)y_i - \sum_{\substack{j \sim i \\ i,j \in V(\bar{G})}} y_j - x_i = \lambda_k y_i$$

Let $f' = (-y_1, \dots, -y_n, x_1, \dots x_n)$ be an eigenfunction of $L(G\bar{G})$ corresponding to eigenvalue λ_m $(2 \leq m \leq 2n)$. Apply f' to (1.1)

$$(d_i + 1)y_i - \sum_{\substack{j\sim i \\ i,j\in G}} y_j + x_i = \lambda_m y_i$$

for all $i \in V(\bar{G})$

$$(d_{\bar{i}} + 1)x_i - \sum_{\substack{j\sim i \\ i,j\in V(\bar{G})}} x_j + y_i = \lambda_m x_i. \tag{3.3}$$

Adding equations (3.2) and (3.3),

$$(n + 1)(x_i + y_i) - (\sum_{j=1}^{n} x_j + \sum_{j=1}^{n} y_j - x_i - y_i) = (\lambda_k + \lambda_m)(x_i + y_i).$$

Using $\sum_{j\in G\bar{G}} f_i = \sum_{j=1}^{n} x_j + \sum_{j=1}^{n} y_j = 0$, therefore

$$(\lambda_k + \lambda_m) = n + 2. \tag{3.4}$$

Let$\lambda_k(G\bar{G})$ and $\lambda_m(G\bar{G})$ be in the spectrum of $L(G\bar{G})$. Assume that

$$\begin{aligned} \lambda_m(G\bar{G}) &= \lambda_{\min}(G\bar{G}), \\ \lambda_k(G\bar{G}) &\leq \lambda_{\max}(G\bar{G}). \end{aligned} \tag{3.5}$$

satisfying equation (3.4) such that

$$\lambda_k(G\bar{G}) + \lambda_{\min}(G\bar{G}) = n + 2. \tag{3.6}$$

So, we can write for another eigenvalues, $\lambda_t(G\bar{G})$ and $\lambda_n(G\bar{G})$ in the spectrum of $L(G\bar{G})$

$$\begin{aligned} \lambda_n(G\bar{G}) &= \lambda_{\max}(G\bar{G}), \\ \lambda_t(G\bar{G}) &\geq \lambda_{\min}(G\bar{G}). \end{aligned} \tag{3.7}$$

satisfying equation (3.4) such that

$$\lambda_t(G\bar{G}) + \lambda_{\max}(G\bar{G}) = n + 2. \tag{3.8}$$

Adding (3.5) and (3.7),

$$\lambda_{\min}(G\bar{G}) + \lambda_k(G\bar{G}) \leq \lambda_t(G\bar{G}) + \lambda_{\max}(G\bar{G})$$

for equality using equations (3.6) and (3.8),

$$\begin{aligned} \lambda_k(G\bar{G}) &= \lambda_{\max}(G\bar{G}), \\ \lambda_t(G\bar{G}) &= \lambda_{\min}(G\bar{G}). \end{aligned}$$

Hence, we obtain (3.1). □

Proposition 3.4. *Let $G\bar{G} = (G\bar{G})$ be a $2n$ order complementary prism of G. $\lambda_{\min}(G)$ and $\lambda_{\min}(\bar{G})$ are nonzero smallest eigenvalues of G and $\bar{G}$, respectively. For minimum eigenvalue of $L(G\bar{G})$*

$$\lambda_{\min}(G\bar{G}) \leq \min\{2, \frac{\lambda_{\min}(G) + \lambda_{\min}(\bar{G}) + 2}{2}\}.$$

Proof. Let $x = (x_1, \dots, x_n)$, $\|x\| = 1$, be an eigenvector of $L(G)$ corresponding to $\lambda_{\min}(G)$ and $y = (y_1, \dots, y_n)$, $\|y\| = 1$, is an eigenvector of $L(\bar{G})$ corresponding to $\lambda_{\min}(\bar{G})$. Using equation (1.1) for all $i \in V(G)$

$$d_i x_i - \sum_{(i,j)\in E(G)} x_j = \lambda_{\min}(G)x_i. \tag{3.9}$$

Similarly, for all $\bar{i} \in V(\bar{G})$

$$d_{\bar{i}} y_i - \sum_{i,j)\in E(\bar{G})} y_j = \lambda_{\min}(\bar{G})y_i. \tag{3.10}$$

Modify the each equations in (3.9) and (3.10) are multiplied by y_i, x_i respectively for all vertices in graph.

$$d_i x_i y_i - y_i \sum_{(i,j)\in E(G)} x_j = \lambda_{\min}(G)x_i y_i, \tag{3.11}$$

$$d_{\bar{i}} x_i y_i - x_i \sum_{(i,j)\in E(\bar{G})} y_j = \lambda_{\min}(\bar{G}) x_i y_i \tag{3.12}$$

equations can be obtained. Adding both side of equations (3.11) and (3.12),

$$(n-1)\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} x_i y_i = (\lambda_{\min}(G) + \lambda_{\min}(\bar{G}))\sum_{i=1}^{n} x_i y_i$$

is found. Therefore, $\lambda_{\min}(G) + \lambda_{\min}(\bar{G}) = n$ for $\sum_{i=1}^{n} x_i y_i \neq 0$. Assume that $\lambda_{\min}(G) + \lambda_{\min}(\bar{G}) < n$ then $\sum_{i=1}^{n} x_i y_i = 0$. Let $f = (f_i) \in \mathbb{R}^{2n}$ be a function as follows

$$f_i = \begin{cases} x_i/\sqrt{2} & , i \in G \\ y_i/\sqrt{2} & , i \in \bar{G} \end{cases} .$$

Hence,

$$\begin{aligned}
\lambda_{\min}(G\bar{G}) &\leq \frac{\sum_{(i,j)\in E(G\bar{G})}(f_i - f_j)^2}{\sum_i f_i^2} \\
&= \frac{\sum_{(i,j)\in E(G)}(f_i - f_j)^2 + \sum_{(i,j)\in E(\bar{G})}(f_i - f_j)^2}{\sum_i f_i^2} \\
&+ \frac{\sum_{(i,j)\in E(G,\bar{G})}(f_i - f_j)^2}{\sum_i f_i^2} \\
&= \frac{\lambda_{\min}(G)}{2} + \frac{\lambda_{\min}(\bar{G})}{2} + \sum_{(i,j)\in E(G,\bar{G})}(f_i - f_j)^2.
\end{aligned}$$

From the assumption, $\sum_{i=1}^{n} x_i y_i = 0$ then it can be get that $\sum_{(i,j)\in E(G,\bar{G})}(f_i - f_j)^2 = 1$. Thus,

$$\lambda_{\min}(G\bar{G}) \leq \frac{\lambda_{\min}(G) + \lambda_{\min}(\bar{G}) + 2}{2}$$

from Theorem 2.3, it is known that 2 is eigenvalue of $L(G\bar{G})$ matrix for $G\bar{G}$. Hence,

$$\lambda_{\min}(G\bar{G}) \leq \min\left\{2, \frac{\lambda_{\min}(G) + \lambda_{\min}(\bar{G}) + 2}{2}\right\}. \qquad \square$$

The following theorem that uses Rayhleigh quotient emphasizes the nonzero minimum and maximum Laplacian eigenvalues of complementary prism graph that contain disconnected components in the underlying graph G or $\bar{G}$.

Theorem 3.5. *Let $G\bar{G}$ be a $2n$ order complementary prism of G. If G or $\bar{G}$ is disconnected graph, then*

$$\lambda_{\min}(G\bar{G}) = \frac{(n+2) - \sqrt{n^2+4}}{2},$$

$$\lambda_{\max}(G\bar{G}) = \frac{(n+2) + \sqrt{n^2+4}}{2}$$

where $\lambda_{\min}(G\bar{G})$ and $\lambda_{\max}(G\bar{G})$ are minimum and maximum non zero eigenvalues of $L(G\bar{G})$, respectively.

Proof. Without loss of generality, G be a disconnected graph with k connected components denoted by G_j where $1 \leq j \leq k$. Let $f = f_j$ be an eigenfunction of $L(G\bar{G})$ corresponding to eigenvalue $\lambda \neq 2$

$$f_j = \begin{cases} x_j, & v \in V(G_j) \\ y_j, & v \in V(\bar{G}_j) \end{cases}$$

where $\sum_{v\in V(G)} f_j = \sum_{v\in V(\bar{G})} f_j = 0$. Using (1.1), for every $v \in V(G_j)$,

$$\begin{aligned}
d_i x_j - \sum_{E(G_j)} x_j - y_j &= \lambda x_j \\
x_j - y_j &= \lambda x_j \\
(1-\lambda)x_j &= y_j
\end{aligned} \tag{3.13}$$

for every vertex $\bar{v} \in V(\bar{G}_j)$

$$\begin{aligned} d_{\bar{i}} y_j - \sum_{(i,\bar{i}) \in E(\bar{G})} f_i - x_j &= \lambda y_j \\ d_{\bar{i}} y_j + d_i y_j - x_j &= \lambda y_j \\ (n+1) y_j - x_j &= \lambda y_j. \end{aligned} \tag{3.14}$$

Apply (3.13) to (3.14) provided that $x_j \neq 0$

$$\begin{aligned} (1-\lambda)(n+1-\lambda) x_j &= x_j \\ \lambda^2 - \lambda(n+2) + n &= 0. \end{aligned}$$

Hence,

$$\lambda = \frac{(n+2) + \sqrt{n^2+4}}{2}$$

or

$$\lambda = \frac{(n+2) - \sqrt{n^2+4}}{2}.$$

Let $g = (x_1, x_2, ..., x_n, y_1, ..., y_n)$ be an eigenfunction. We want to minimize Rayleigh quotient for any λ'

$$\lambda' = \frac{\sum\limits_{(i,j) \in E(G\bar{G})} (g_i - g_j)^2}{\sum_i g_i^2}$$

subject to

$$\sum_{i=1}^{n} x_i + y_i = 0$$

and

$$\sum_{i=1}^{n} x_i^2 + y_i^2 = 1.$$

Using Lagrange multipliers, define a Lagrangian function

$$F(x_1, ..., x_n, y_1, ..., y_n) = \sum_{(i,j) \in E(G\bar{G})} (g_i - g_j)^2 + \mu_1 \sum_{i=1}^{n} (x_i + y_i) + \mu_2 ((\sum_{i=1}^{n} x_i^2 + y_i^2) - 1).$$

For all x_i corresponding to $i \in \bar{G}_j$, for $1 \leq i \leq n$

$$\frac{\partial F}{\partial x_i} = 2(Lf)_i + \mu_1 + 2\mu_2 x_i = 0. \tag{3.15}$$

For all y_i corresponding to $i \in \bar{G}_j$, for $1 \leq i \leq n$

$$\frac{\partial F}{\partial y_i} = 2(Lf)_i + \mu_1 + 2\mu_2 y_i = 0. \tag{3.16}$$

Adding all equations (3.15) and (3.16) corresponding to all x_i and y_i, it is obtained that $\mu_1 = 0$. Also, substitute $\mu_1 = 0$ in equation (3.15);

$$\begin{aligned} 2(Lf)_i + 2\mu_2 x_i &= 0 \\ \lambda' x_i &= -\mu_2 x_i. \end{aligned}$$

Then μ_2 is equal to negative value of corresponding eigenvalue λ'.

$$\lambda' = \frac{\sum_{(i,j)\in E(G\bar{G})} (g_i - g_j)^2}{\sum_i g_i^2} \geq \sum_{(i,j)\in E(G_j\bar{G}_j)} (g_i - g_j)^2 + \sum_{(i,j)\in E(\bar{G}_j\bar{G}_k)} (g_i - g_j)^2. \tag{3.17}$$

All entries corresponding to a connected component G_j and $\bar{G}_j$ are constant in eigenfunction f. When the f is applied to Rayleigh quotient then summation is equal to zero on edges of G_j and $\bar{G}_j$. The remaining part different from zero is evaluated as the right side of the inequality (3.17). Let minimize

$$\sum_{(i,j)\in E(G_j\bar{G}_j)} (g_i - g_j)^2 + \sum_{(i,j)\in E(\bar{G}_j\bar{G}_k)} (g_i - g_j)^2$$

subject to

$$\sum_{i=1}^{n} x_i + y_i = 0$$

and

$$\sum_{i=1}^{n} x_i^2 + y_i^2 = 1.$$

where $g = (x_1, x_2, ..., x_n, y_1, ..., y_n)$. Similarly using Lagrange multipliers define following function

$$F'(x_1, ..., x_n, y_1, ..., y_n) = \sum_{(i,j)\in E(G_j\bar{G}_j)} (g_i - g_j)^2 + \sum_{(i,j)\in E(\bar{G}_j\bar{G}_k)} (g_i - g_j)^2 +$$

$$\mu_1 \sum_{i=1}^{n} (x_i + y_i) + \mu_2 ((\sum_{i=1}^{n} x_i^2 + y_i^2) - 1)$$

we can evaluate partial differential of F'. After adding all partial differential for each variable, it is obtained that $\mu_1 = 0$ and $x_i(1 + \mu_2) = y_i$ is the form of only critical point. Therefore, it is clear from (3.13) that eigenfunction f is minimize also the right side of inequality (3.17). Hence, $\lambda_{min} = \frac{(n+2)-\sqrt{n^2+4}}{2}$. Using Lemma 3.3, $\lambda_{\max}(G\bar{G}) = \frac{(n+2)+\sqrt{n^2+4}}{2}$. □

Theorem 3.6. *Let $\lambda_{\min}(G\bar{G})$ and $\lambda_{\max}(G\bar{G})$ are the smallest and largest nonzero eigenvalues of a complementary prism graph $G\bar{G}$. Then,*

$$\frac{(n+2) - \sqrt{n^2 + 4}}{2} \leq \lambda_{\min}(G\bar{G})$$

and

$$\lambda_{\max}(G\bar{G}) \leq \frac{(n+2) + \sqrt{n^2 + 4}}{2}$$

Proof. Assume that $H\bar{H}$ is a complementary prisms and H has two connected component such as G_1 and G_2. Also, $G_n\bar{G}_n$ is a complementary prism graph obtained by adding an edge between two connected component such that both G_n and $\bar{G}_n$ are connected graphs (see in Figure 2). Let $j \in V(H)$ that has same vertex degree in both $H\bar{H}$ and $G_n\bar{G}_n$ and $g = g_i$ is eigenfunction corresponding to $\lambda_{\min}(G_n\bar{G}_n)$. Using form of the eigenvector $f = f_i$ corresponding to $\lambda_{\min}(H\bar{H})$

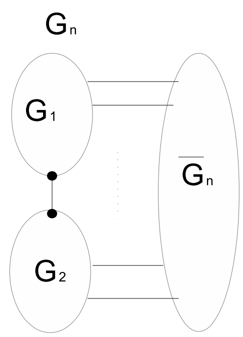

FIGURE 2. $G_n\bar{G}_n$ is obtained by $H\bar{H}$ where H has two disconnected component

defined in proof of Theorem 3.5, $\lambda_{\min}(H\bar{H})$ can be expressed from (1.1):

$$\begin{aligned}\lambda_{min}(H\bar{H}) &= \frac{f_j - f_{\bar{j}}}{f_j}\\ &= \frac{d_j f_j - \sum\limits_{(i,j)\in E(H)} f_i - f_{\bar{j}}}{f_j}\\ &\leq \frac{(d_j-1)g_j - \sum\limits_{(i,j)\in E(G_n)} g_i + g_j - g_{\bar{j}}}{g_j}\\ &= \frac{d_j g_j - \sum\limits_{(i,j)\in E(G_n)} g_i - g_{\bar{j}}}{g_j}\\ &= \lambda_{\min}(G_n\bar{G}_n).\end{aligned}$$

Thus, using Theorem 3.5, lower bound for nonzero smallest eigenvalue of any complementary prism $G\bar{G}$ can be written as

$$\frac{(n+2)-\sqrt{n^2+4}}{2} \leq \lambda_{\min}(G\bar{G}).$$

Moreover, using Lemma 3.3, upper bound for largest eigenvalue of any complementary prism can be expressed;

$$\lambda_{\max}(G\bar{G}) \leq \frac{(n+2)+\sqrt{n^2+4}}{2}.$$ □

Now using Theorem 3.1, we can define the bound for $\lambda_{\max}(G\bar{G})$ and $\lambda_{\min}(G\bar{G})$.

Proposition 3.7. *Let $G\bar{G}$ is $2n$ order graph. Then, for nonzero maximum eigenvalues of $L(G\bar{G})$*

$$\lambda_{\max}(G\bar{G}) \geq \max\{\lambda_{\max}(G), \lambda_{\max}(\bar{G})\} + 1.$$

Proof. Let $b = (\lambda_{\max}(G\bar{G}), \ldots, \lambda_{\min}(G\bar{G}), 0)$ is sequence of spectrum of $L(G\bar{G})$ in nonincreasing order and

$$c = (\max\{\lambda_{\max}(G), \lambda_{\max}(\bar{G})\} + 1, \ldots, \min\{\lambda_{\min}(G), \lambda_{\min}(\bar{G})\} + 1, 1, 1)$$

is sequence of spectrum of $\lambda(L(G) + I_n)$ and $\lambda(L(\bar{G}) + I_n)$ in nonincreasing order. We know from Theorem 3.1 that b majorizes c. Therefore, using Definition 2.4,

$$\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} c_i, \quad 1 \leq k \leq 2n \tag{3.18}$$

and

$$\sum_{i=1}^{2n} b_i = \sum_{i=1}^{2n} c_i. \tag{3.19}$$

If $k = 1$, then $\lambda_{\max}(G\bar{G}) \geq \max\{\lambda_{\max}(G), \lambda_{\max}(\bar{G})\} + 1$. □

Remark 3.8. From (3.18) and (3.19) we have

$$\sum_{i=1}^{2n-2} b_i \geq \sum_{i=1}^{2n-2} c_i \quad \text{and} \quad \sum_{i=1}^{2n} b_i = \sum_{i=1}^{2n} c_i.$$

Hence, $\lambda_{\min}(G\bar{G}) \leq 2$. We know from Theorem 2.3 that 2 is always eigenvalue of Laplacian of complementary prisms. Therefore, we can also obtain that 2 is an upper bound for nonzero minimum eigenvalue by majorization.

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