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Numerical Solution of One Boundary Value Problem Using Finite Difference Method

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Abstract. Many problem of physics and engineering are modelled by boundary value problems for ordinary or partial differential equations. Usually, it is impossible to find the exact solution of the boundary value problems, so we have to apply various numerical methods. There are different numerical methods (for example, the Explicit Euler method, the Runge-Kutta method, the Improved Euler method, Finite difference method and finite element method) for determining the approximate solutions of initial and boundary-value problems. One of them is the finite difference method, which is the simplest scheme. This method can be applied to higher of ordinary differential equations, provided it is possible to write an explicit expression for the highest order derivative and the system has a complete set of initial conditions. In this study, we are interested in the finite difference method for new type boundary value problems. We describe the numerical solutions of some two-point boundary value problems by using finite difference method. This method are based upon the approximations that allow to replace the differential equations by algebraic system of equations and the unknowns solutions are related to grid points. In this article, we have presented a finite difference method for solving second order boundary value problems for ordinary differential equations with an internal singularity. This method tested on several model problems for the numerical solution.

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1. Introduction

Sturm-Liouville type differential equations are used to understand different kind of processes in natural sciences. To understand the nature of such processes, we need to solve these type equations using some analytical methods which satisfy the certain boundary conditions. In general, it is impossible to solve these boundary-value problems analytically. So we prefer numerical techniques to solve this problems. A literature regarding the numerical solution of the Sturm-Liouville problem is given . [1, 4–6] Note that, finite difference methods are numerical methods for approximating the solution to various type differential equations using finite difference equations to approximate derivatives. The idea is to replace ordinary or partial derivatives appearing in the boundary-value problem by finite differences that approximate

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them. Note that the finite difference methods deal without interior singular point and corresponding transmission conditions [2, 9]. It is our main goal here to develop finite difference method to deal with an additional transmission conditions at the interior singular point.

2. Analysis of the Method

The finite difference method is based on converting the differential equation into system of algebraic equations by dividing the range into finite intervals and by replacing derivative expressions at the interior nodal points with appropriate differences. [7, 8] Let us consider the boundary-value problem for Sturm-Liouville equations.

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \tag{2.1}$$

subject to end-point boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta \tag{2.2}$$

where p, q and f are all smooth functions and α, β are real constants. In order to discretize problem (2.1), (2.2) the definition range $[a, b]$ is divided into N equal ranges $[x_0, x_1], [x_1, x_2], ..., [x_{N-1}, x_N]$.
where

$$a = x_0 < x_1 < ... < x_N = b, \quad x_i = a + ih, \quad h = \frac{b-a}{N}.$$

The derivative expressions in the boundary value problem are replaced by finite difference expressions as

$$y'(x) \approx \frac{y(x+h) - y(x-h)}{2h} \tag{2.3}$$

and

$$y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}. \tag{2.4}$$

Denoting y_i as the approximation of the value of the unknown function y(x) at node x_i and substituting (2.3) and (2.4) in the boundary value problem (2.1)-(2.2) we have the numerical solution $y_0, y_1, ..., y_N$ of the following linear system of algebraic equations

$$\left(1 - \frac{1}{2}hp_i\right)y_{i-1} + \left(-2 + h^2 q_i\right)y_i + \left(1 + \frac{1}{2}hp_i\right)y_{i+1} = h^2 f(x_i) \ \ 1 \le i \le N-1 \quad , \quad i = 1, 2, 3, ..., N-1$$

where

$$y_0 = \alpha, \quad y_N = \beta.$$

Note that the finite difference equation at each nodal point $x = x_i$ involves solution values at three nodal point x_{i-1} , x_i and x_{i+1}. This system of equations can be written in the matrix and vector form

$$My = B \tag{2.5}$$

with

$$M = \begin{pmatrix} -2+h^2q_1 & 1+\frac{1}{2}hp_1 & 0 & 0 & \cdots & 0 \\ 1-\frac{1}{2}hp_2 & -2+h^2q_2 & 1+\frac{1}{2}hp_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1-\frac{1}{2}hp_{N-2} & -2+h^2q_{N-2} & 1+\frac{1}{2}hp_{N-2} & 0 \\ 0 & 0 & \cdots & 0 & 1-\frac{1}{2}hp_{N-1} & -2+h^2q_{N-1} \end{pmatrix}_{N-1\times N-1},$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \end{pmatrix} \quad and \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{N-1} \end{pmatrix}$$

where

$$b_i = \begin{cases} h^2 f_1 - \left(1 - \frac{1}{2}hp_1\right), & i = 1 \\ h^2 f_i, & i = 2, 3, ..., N-2 \\ h^2 f_{N-1} - \left(1 + \frac{1}{2}hp_{N-1}\right)\beta, & i = N-1 \end{cases}.$$

The linear system of algebraic equations (2.5) is tridiagonal and can be solved efficiently by the Crout or Cholesky algoritm [3].

3. Development of The Finite Difference Method for Solving Transmission Problems

Let us consider the following boundary value problem on the disjoint intervals $[-1, 0)$ and $(0, 1]$ consisting of the linear differential equation

$$y'' + y' + e^{-2x}y = 0, \qquad x \in [-1, 0) \cup (0, 1] \tag{3.1}$$

together with boundary conditions at the end-points $x = -1, 1$ given by

$$y(-1) = 0, \qquad y(1) = 1 \tag{3.2}$$

and with additional transmission conditions at the interior point of singularity $x = 0$, given by

$$y(-0) = y(+0), \quad y'(-0) = 2y'(+0).$$

At first we shall investigate this problem without transmission conditions. We can show that the exact solution of the BVP (3.1)-(3.2) is

$$y = -csc(\frac{1}{e} - e)sin(e - e^{-x}).$$

For simplicity consider the uniform cartesian grid $x_i = -1 + ih, \quad i = 0, 1, \quad \dots, 10$ for $N = 10$, i.e $h = \frac{1-(-1)}{10} = 0,2$ where in particular $x_0 = -1, \quad x_{10} = 1$. By using the central finite difference method (FDM) at a typical grid point x_i, we obtain

$$(2 - h)y_{i-1} + (-4 + 2h^2e^{-2x_i})y_i + (2 + h)y_{i+1} = 0 \tag{3.3}$$

for $i = 1, 2, \dots, 9$. Consequently, the finite difference solution $y_i \approx y(x_i)$ is defined as the solution of the linear algebraic system of equations (3.3) In a tridiagonal matrix-vector form, this linear algebraic system of equations can be written as

$$\begin{pmatrix} -4+2h^2e^{-2x_1} & 2+h & 0 & \cdots & 0 & 0 & 0 \\ 2-h & -4+2h^2e^{-2x_2} & 2+h & \cdots & 0 & 0 & 0 \\ 0 & 2-h & -4+2h^2e^{-2x_3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & -4+2h^2e^{-2x_7} & 2+h & 0 \\ 0 & 0 & 0 & \cdots & 2-h & -4+2h^2e^{-2x_8} & 2+h \\ 0 & 0 & 0 & \cdots & 0 & 2-h & -4+2h^2e^{-2x_9} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_7 \\ y_8 \\ y_9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -2-h \end{pmatrix}.$$

The solution of this system of algebraic equations can be found by using MATLAB-Octave. An error analysis of the obtained results are given in the following table

x	FDM Solution	Exact Solutions	EROR
-1	0	0	0
-0.8	0.68969	0.66514	0.024552
-0.6	1.1298	1.0981	0.031699
-0.4	1.3534	1.3236	0.029877
-0.2	1.4269	1.4023	0.024641
0	1.4096	1.3908	0.018768
0.2	1.3442	1.3308	0.013394
0.4	1.2579	1.249	0.008868
0.6	1.1668	1.1616	0.0052016
0.8	1.0794	1.0771	0.0022897
1	1	1	0

In table the exact solution is compared with the numerical solutions of FDM for the BVP (3.1)-(3.2) Now we shall investigate the BVP (3.1)-(3.2) under additional transmission conditions at the interior singular point $x = 0$, given by

$$y(-0) = y(+0), \quad y'(-0) = 2y'(+0).$$

We can find that the exact solution of this problem is has the following form

$$y = \begin{cases} \frac{-12sin(e)}{sin(2-\frac{1}{e}-e)+5sin(\frac{1}{e}-e)}cos(e^{-x}) + \frac{12cos(e)}{sin(2-\frac{1}{e}-e)+5sin(\frac{1}{e}-e)}sin(e^{-x}) , & x \in [-1, 0) \\ [sec(\frac{1}{e}) - \frac{5cos(e)-cos(2-e)}{sin(2-\frac{1}{e}-e)+5sin(\frac{1}{e}-e)}tan(\frac{1}{e})]cos(e^{-x}) + \frac{5cos(e)-cos(2-e)}{sin(2-\frac{1}{e}-e)+5sin(\frac{1}{e}-e)}sin(e^{-x}), & x \in (0, 1] \end{cases}.$$

If we select $N = 19$ and apply the transmission conditions then we have two additional algebraic equations.

$$y_9 - y_{10} = 0 \tag{3.4}$$

and

$$y_8 - y_9 - 2y_{10} + 2y_{11} = 0. \tag{3.5}$$

The solution of the algebraic system of equations (3.3), (3.4), (3.5) can be found by using MATLAB/Octave.

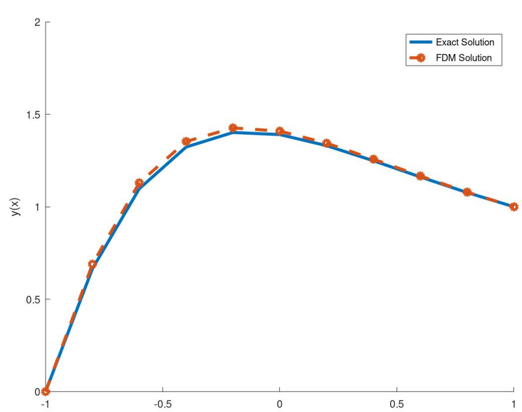

Figure 1. Graph of the exact solution and the FDM solution

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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