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Equiconvergence with Fourier Series for Non-Classical Sturm-Liouville Problems

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Abstract. Sturm-Liouville type boundary-value problems arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of chemistry, aerodynamics, electrodynamics of complex medium or polymer rheology. For example the vibration of a homogeneous loaded strings, the earth's free oscillations, the interaction of atomic particles, sound, surface waves, heat transfer in a rod with heat capacity concentrated at the ends, electromagnetic waves and gravitational waves can be solved using the Sturmian theory. A large class of physical problems require the investigation of the Sturm-Liouville type problems with discontinuities. Examples are vibration problems under various loads such as a vibrating string with a tip mass or heat conduction through a liquid solid interface.

In this study we shall investigate some properties of the eigenfunctions of one discontinuous Sturm-Liouville Problem. We shall prove some preliminary results related to the basic solutions, Green's function, resolvent operator and selfadjointness of the considered problem. Particularly we shall present a new approach for constructing the Green's function which is not standard one generally found in textbooks. The obtained results are implemented to the investigation of the basis properties of the system of eigenfunctions in modified Hilbert spaces. Finally, we shall show that the eigenfunction expansion series regarding the convergence behaves in the same way as an ordinary Fourier series.

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1. Introduction

The Sturmian theory is one of the most actual and extensively developing field in spectra theory of differential operators. As a rule , the separation of variables method was applied on the two-order partial differential equation to obtain a Sturm-Liouville problem for each independent variable. This method is a cornerstone in the study of partial differential equations, appearing in physics and engineering. Some physical problems require the investigation of the Sturm-Liouville type problems with the discontinuous coefficients and corresponding transmission conditions.

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Basically boundary-value problems with continuous coefficients and without interior singularities have been studied. There is a quite substantial literature on such type problems (see, for example, [1,5–7,9,12–19]). In recent years, there has been growing interest in boundary- value problems with discontinuous coefficients(see, for example, [2–4,8–11,19] and references cited therein). In this study we shall investigate the folowing discontinuous Sturm-Liouville problem which consists of Sturm-Liouville equation,

$$\Theta y := -y''(x) + q(x)y(x) = \lambda y(x) \tag{1.1}$$

to hold on two disjoint intervals (a, c) and (c, b) , where discontinuity in y and y' at the interior point of discontinuity $x = c$ are prescribed by so-called transmission conditions, given by

$$t_c(y) := \cos\gamma y(c-) + \sin\gamma y(c+) = 0,\ 0 < \gamma < \pi \tag{1.2}$$

$$t_c(y') = \cos\delta y'(c-) + \sin\delta y'(c+) = 0,\ 0 < \delta < \pi \tag{1.3}$$

together with the end-point boundary conditions, given by

$$\theta_a(y) =:= \cos\alpha y(a) + \sin\alpha y'(a) = 0,\ \ 0 < \alpha < \pi, \tag{1.4}$$

$$\theta_b(y) =:= \cos\beta y(b) + \sin\beta y'(b) = 0,\ 0 < \beta < \pi, \tag{1.5}$$

where the coefficient $q(x)$ is real-valued, continuous on $[a, c)\cup \in (c, b]$ and has a finite limits $q(c\mp) = \lim_{x\to c\mp} q(x)$. Note that, this type of eigenvalue problems arise in varied physical transfer problems. For example, in electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem.

2. Some Preliminary Results

At first, we shall prove some result related to the basic solutions, Green's function, resolvent operator and selfadjointness of the considered problem which is not standard one generally found in textbooks. For this we shall introduce a new inner product in the Hilbert space $L_2(a, c) \oplus L_2(c, b)$ and define a linear operator £ in this space such a way that the considered problem can be interpreted as the seladjoint problem. Throughout in below we shall assume that, $\cot\alpha \neq 0, \cot\beta \neq 0,\ \cot\gamma > 0$. In the Hilbert Space $H = L_2(a, c) \oplus L_2(c, b)$ let us define a new inner product by

$$< f, g >_H \ \ := \ \ \cos\gamma \int_a^{c-} f(x)\overline{g(x)}dx + \frac{1}{\cot\delta}\int_{c+}^{b} f(x)\overline{g(x)}dx$$

for $f, g \in L_2(a, c) \oplus L_2(c, b)$.
Let us define a linear operator $\Lambda : H \to H$ by action low

$$\Lambda f = -f'' + q(x)f$$

on the domain $D(\Lambda)$ consisting of all $f \in H$ satisfying the conditions
D_1) $f\ f'\ inAC_{loc}(a, c) \oplus AC_{loc}(c, b)$ and there are finite left- and right-sides limits $f(c\mp)$ and $f'(c\mp)$.
D_2) The functions

$$f^-(x) := \begin{cases} f(x), & \text{for } x \in [a, c) \\ f(c-0), & \text{for } x = c \end{cases} \quad , \quad f^+(x) := \begin{cases} f(x), & \text{for } x \in (c, b] \\ f(c+0), & \text{for } x = c \end{cases}$$

are absolutely continuous in $[a, c]$ and $[c, b]$ respectively.
D_3) $\Theta f^- \in L_2[a, c], \Theta f^+ \in L_2[c, b]$
D_4) $\theta_a(y) = \theta_b(y) = t_c(y) = t_c(y') = 0$
Obviously, the problem (1.1) – (1.5) can be written in the operator equation form

$$\Lambda f = \lambda f,\ \ f \in D(\Lambda)$$

in the Hilbert space H.

Theorem 2.1. *The operator Λ is densely defined(i.e. $D(\Lambda)$ is dense in H) and symmetric in the Hilbert space H.*

Corollary 2.2. *All eigenvalues of the operator Λ are real.*

Remark 2.3. With no restriction of generality we can assume that all eigenfunctions of the problem (1.1) – (1.5) are real-valued.

Theorem 2.4. *The set of eigenvalues of the operator Λ coincide with the set of eigenvalues of the problem (1.1) − (1.5). operator Λ is densely defined(i.e. $D(\Lambda)$ is dense in H) and symmetric in the Hilbert space H.*

With a view to obtain the resolvent operator $R(\lambda,\Lambda) := (\lambda-\Lambda)^{-1}$ we shall solve the operator

$$(\lambda-\Lambda)y = f$$

for $f \in H$. This equation is equivalent to the two interval differential equation

$$y'' + (\lambda - q(x))y = f(x), \quad x \in (a,b) \cup (b,c) \tag{2.1}$$

together with boundary-transmission conditions (1.2), (1.3), (1.4), (1.5). Let us $u(x,\lambda)$ and $\upsilon(x,\lambda)$ be linear independent solutions of the equation (1.1) and let $\omega(\lambda) = W(u(x,\lambda)\upsilon(x,\lambda))$ is the wronskian.

Theorem 2.5. *Let λ be any complex number such that $\omega(\lambda) \neq 0$. Then the inhomogeneous boundary-value- transmission problem (2.1), (1.2), (1.3), (1.4), (1.5) has a unique solution $y = y(x,\lambda)$ given by*

$$y(x,\lambda) \quad = \quad \cos\gamma\langle G(x,.;\lambda),\overline{f(.)}\rangle_{L_2(a,c)} + \frac{1}{\cot\delta}\langle G(x,.;\lambda),\overline{f(.)}\rangle_{L_2(c,b)}$$

where

$$G(x,z;\lambda) = \begin{cases} \frac{\upsilon(z,\lambda)u(x,\lambda)}{\omega(\lambda)} & for\ a \le z \le x \le b, \quad x,z \neq c \\ \frac{u(z,\lambda)\upsilon(x,\lambda)}{\omega(\lambda)} & for\ a \le x \le z \le b, \quad x,z \neq c \end{cases}$$

is the corresponding Green's function.

Theorem 2.6. *All eigenvalues of the operator Λ are simple.*

Theorem 2.7. *The linear differential operator Λ is self adjoint in the Hilbert space H.*

3. Main Results

Theorem 3.1. *The zeros of $\omega(\lambda)$ coincide with the eigenvalues of the problem (1.1) − (1.5).*

If $\lambda_n,\ n = 1,2,3...$ are the eigenvalues of the operator Λ and $u_n(x,\lambda_n) := y(x,\lambda_n)$ are the corresponding system of eigenfunction, then

$$\cos\gamma\int_a^{c-0} u_n(x)u_m(x)dx + \frac{1}{\cot\delta}\int_{c+0} bu_n(x)u_m(x)dx = 0 \quad for\ \ n \neq m$$

It is easy to see that there are $\kappa_n \neq 0$ $n = 0,1,2,....$ such that

$$u(x,\lambda_n) = \kappa_n\vartheta(x,\lambda_n),$$

Then we have

$$\omega'(\lambda_n) = ||u_n||_H^2\kappa_n.$$

Let

$$\phi_n(x) := \frac{1}{||u_n||_H}u_n(x)$$

we can define that

$$res_{\lambda=\lambda_n}y(x,\lambda) =< f,\Phi_n >_H \phi_n(x) \tag{3.1}$$

From (3.1) it follows easily that

$$res_{\lambda=\lambda_n}R(\lambda,\Lambda) =< f,\phi_n >_H \phi_n(x)$$

Hence

$$\begin{aligned} res_{\lambda=\lambda_n} < R(\lambda,\Lambda)f,f > &= < res_{\lambda=\lambda_n}R(\lambda,\Lambda)f,f >=< f,\phi_n >< \phi_n,f > \\ &= |\langle f,\phi_n\rangle|^2 \end{aligned}$$

Theorem 3.2. *For $f \in H$ we have the following modified Parseval's equality*

$$\|f\|_H^2 = \sum_{n=0}^{\infty}(\cos\gamma \int_a^{c-0} f(x)\phi_n(x)dx + \frac{1}{\cot\delta}\int_{c+0}^{b} f(x)\phi_n(x)dx)^2$$

Theorem 3.3. *Assume that the following conditions be satisfied:*
(i) the function $f(x)$ is continuously differentiable in each of intervals $[a,c)$ and $(c,b]$ with finite one-hand limits $f(c\pm), f'(c\pm)$
ii)the functions

$$f_-(x) = \begin{cases} f(x), & for\ a \le x < c \\ f(c-), & for\ x = c \end{cases}, \quad f_-(x)'(x) = \begin{cases} f'(x), & for\ a \le x < c \\ f'(c-), & for\ x = c \end{cases}$$

are absolutely continuous on $[a,c]$ and the functions

$$f_+(x) = \begin{cases} f(x), & for\ c < x \le b \\ f(c+), & for\ x = c \end{cases}, \quad f_+(x)'(x) = \begin{cases} f'(x), & for\ c < x \le b \\ f'(c+), & for\ x = c \end{cases}$$

are absolutely continuous on $[c,b]$
iii)$-f'' + q(x)f(x) \in H$
iv)the function $f(x)$ satisfied the both transmission conditions (1.2), (1.3) and (1.4). Then

$$f(x) = \sum_{n=0}^{\infty}(\cos\gamma \int_a^{c-0} f(x)\phi_n(x)dx + \frac{1}{\cot\delta}\int_{c+0}^{b} f(x)\phi_n(x)dx)\phi_n(x)$$

in the sense of uniform convergence on whole $[a.c) \cup (c,b]$.

Below for sake of simplicity we consider only the case where $\cot\gamma\cot\delta = 1$. Then we can find that the next asymptotic formula for the eigenvalues $\lambda_n = S_n^2$ is true;

$$S_n = \frac{(n-1)\pi}{b-a} + O(\frac{1}{n})$$

Theorem 3.4. *Let $f \in H$. Then the series*

$$\sum_{n=0}^{\infty} < f, \phi_n >_H \phi_n(x)$$

regarding the convergence behaves in the same way as an ordinary Fourier series in each of the separate intervals (a,c) and (c,b). In particular it converges to $\frac{1}{2}(f(x+0)+f(x-0))$ if $f(x)$ is of bounded variation in the neighbourhood of $x \in (a,c) \cup (c,b)$.

References

[1] Aliyev, Z.S., Dun'yamalieva, A.A., *Basis properties of root functions of the Sturm-Liouville problem with a spectral parameter in the boundary conditions*, Doklady Mathematics, **88/1**(2013), 441–445. 1

[2] Allahverdiev, P.B., Bairamov, E., Uğurlu,, E., *Eigenparameter dependent Sturm-Liouville problems in boundary conditions with transmission conditions*, J. Math. Anal. Appl., **401**(2013), 388–396. 1

[3] Ao, J., Sun, J., *Matrix representations of Sturm-Liouville problems with coupled eigenparameter-dependent boundary conditions*, Applied Mathematics and Computation, **244**(2014), 142–148. 1

[4] Bairamov, E., Uğurlu,, E., *Krein's Theorems for a Dissipative Boundary Value Transmission Problem*, Complex Anal. Oper. Theory, **7**(2013), 831–842. 1

[5] Binding, P.A., Browne, P.J., Patrick J., *Oscillation theory for indefinite Sturm-Liouville problems with eigenparameter dependent boundary conditions*, Proc. Roy. Soc. Edinburg Sect., **127/6**(1997), 1123–1136. 1

[6] Buterin, S.A., *On half inverse problem for differential pencils with the spectral parameter in boundary conditions*, Tamkang J. Math., **42/3**(2011), 355–364. 1

[7] Fulton, C.T., *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proc. Roy. Soc. Edin., **77A**(1977), 293–308. 1

[8] Kandemir, M., Mukhtarov, O.Sh, Yakubov, Y.Y., *Irregular boundary value problems with discontinuous coefficients and the eigenvalue parameter*, Mediterr, J. Math., **6**(2009), 317–338. 1

[9] Li, K., Sun, J., Hao, X., Bao, Q., *Spectral analysis for discontinuous non-self-adjoint singular Dirac operators with eigenparameter dependent boundary condition*, J. Math. Anal. Appl., **453**(2017), 304–316. 1

[10] Mukhtarov, O.Sh., Olğar, H., Aydemir, K., *Resolvent operator and spectrum of new type boundary value problems*, Filomat, **29:7**(2015), 1671–1680. 1

[11] Olğar, H., Mukhtarov, O.Sh., *Weak eigenfunctions of two-interval Sturm-Liouville problems together with interaction conditions*, Journal of Mathematical Physics, **58**(2017), 042201. 1

[12] Panakhov E.S., Sat, M., *Reconstruction of potential function for Sturm-Liouville operator with coulomb potential*, Boundary Value Problems, **49**(2013), pp.2013. 1

[13] Schneider, A., *A note on eigenvalue problems with eigenvalue parameter in the boundary condition*, Math. Z., **136**(1794), 163–167. 1

[14] Sherstyuk, A.I., Problems of Theoretical Physics, Gos. Univ., Leningrad Vol. 3., 1988. 1

[15] Shkalikov, A.A., *Boundary value problems for ordinary differential equations with a parameter in boundary condition*, Trudy Sem. Imeny I.G. Petrowsgo, **9**(1983), 190–229. 1

[16] Walter, J., *Regular eigenvalue problems with eigenvalue parameter in the boundary conditions*, Math. Z., **133**(1973), 301–312. 1

[17] Wang A., Zettl, A., *Self-adjoint Sturm-Liouville problems with discontinuous boundary conditions*, Methods And Applications of Analysis, **22**(**1**)(2015), 037–066. 1

[18] Yakubov, S., Yakubov, Y., *Abel basis of root functions of regular boundary value problems*, Math. Nachr., **19**7(1999), 157–187. 1

[19] Zhang, X., Sun, J., *The determinants of fourth order dissipative operators with transmission conditions*, J. Math. Anal. Appl., **410/1**(2014), 55–69. 1