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Compact Operators in the Class $\left(bv_k^\theta, bv\right)$

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Abstract: The space bv of bounded variation sequence plays an important role in the summability. More recently this space has been generalized to the space bv_k^θ and the class $\left(bv_k^\theta, bv\right)$ of infinite matrices has been characterized by Hazar and Sarıgöl [2]. In the present paper, for $1 < k < \infty$, we give necessary and sufficient conditions for a matrix in the same class to be compact, where θ is a sequence of positive numbers.

Keywords: Matrix transformations, Sequence spaces, bv_k^θ spaces.

1 Introduction

Let ω be the set of all complex sequences, ℓ_k and c be the set of k-absolutely convergent series and convergent sequences. In [2] , the space bv_k^θ has been defined by

$$bv_k^\theta = \left\{ x = (x_k) \in w : \sum_{n=0}^{\infty} \theta_n^{k-1} \left|\triangle x_n\right|^k < \infty, \ x_{-1} = 0 \right\},$$

which is a BK space for $1 \leq k < \infty$, where (θ_n) is a sequence of nonnegative terms and $\triangle x_n = x_n - x_{n-1}$ for all n.

Also, in the special case $\theta_n = 1$ for all n, it is reduced to bv^k, studied by Malkowsky, Rakočević and Živković [1] , and $bv_1^\theta = bv$.

Let U and V be subspaces of w and $A = (a_{nv})$ be an arbitrary infinite matrix of complex numbers. By $A(x) = (A_n(x))$, we denote the A-transform of the sequence $x = (x_v)$, i.e.,

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

provided that the series are convergent for $v, n \geq 0$. Then, A defines a matrix transformation from U into V, denoted by $A \in (U, V)$, if the sequence $Ax = (A_n(x)) \in V$ for all sequence $x \in U$.

Lemma 1.1 ([6]) . Let $1 < k < \infty$ and $1/k + 1/k^* = 1$. Then, $A \in (\ell_k, \ell)$ if and only if

$$\|A\|'_{(\ell_k,\ell)} = \left\{ \sum_{\nu=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}| \right)^{k^*} \right\}^{1/k^*} < \infty$$

and there exists $1 \leq \xi \leq 4$ such that $\|A\|'_{(\ell_k,\ell)} = \xi \|A\|_{(\ell_k,\ell)}$

If S and H are subsets of a metric space (X, d) and $\varepsilon > 0$, then S is called an ε-net of H , if, for every $h \in H$, there exists an $s \in S$ such that $d(h, s) < \varepsilon$; if S is finite, then the ε-net S of H is called a finite ε-net of H. By M_X, we denote the collection of all bounded subsets of X. If $Q \in M_X$, then the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q) = \inf \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\}.$$

The function $\chi : M_X \to [0, \infty)$ is called the Hausdorff measure of noncompactness [5].

If X and Y are normed spaces, $\mathcal{B}(X, Y)$ states the set of all bounded linear operators from X to Y and is also a normed space according to the norm $\|L\| = \sup_{x \in S_X} \|L(x)\|$, where S_X is a unit sphere in X, $i.e.$, $S_X = \{x \in X : \|x\| = 1\}$. Further, a lineer operator $L : X \to Y$ is said to be compact if the sequence $(L(x_n))$ has convergent subsequence in Y for every bounded sequence $x = (x_n) \in X$. By $\mathcal{C}(X, Y)$ we denote the set of such operators.

The following results are need to compute Hausdorff measure of noncompactness.

Lemma 1.2 ([4]). Let X and Y be Banach spaces, $L \in \mathcal{B}(X,Y)$. Then, Hausdorff measure of noncompactness of L, denoted by $\|L\|_{\chi}$, is defined by

$$\|L\|_{\chi} = \chi\left(L\left(S_X\right)\right),$$

and

$$L \in \mathcal{C}(X,Y) \text{ iff } \|L\|_{\chi} = 0.$$

Lemma 1.3 ([5]). Let Q be a bounded subset of the normed space X where $X = \ell_k$ for $1 \leq k < \infty$.If $P_r : X \to X$ is the operator defined by $P_r(x) = (x_0, x_1, ..., x_r, 0, ...)$ for all $x \in X$, then

$$\chi(Q) = \lim_{r\to\infty} \sup_{x\in Q} \|(I - P_r)(x)\|,$$

where I is the identity operator on X.

Lemma 1.4 ([4]). Let X be normed sequence space, χ_T and χ denote Hausdorff measures of noncompactness on M_{X_T} and M_X, the collections of all bounded sets in X_T and X, respectively. Then,

$$\chi_T(Q) = \chi(T(Q)) \text{ for all } Q \in M_{X_T},$$

where T is an infinite triangle matrix.

2 Compact operators on the space bv_k^θ

More recently the class $\left(bv_k^\theta, bv\right)$, $1 < k < \infty$, has been characterized by Hazar and Sarıgöl [2] in the following form. In the present paper, by computing Hausdorff measure of noncompactness, we characterize compact operators in the same class.

Theorem 2.1. Let $A = (a_{nv})$ be an infinite matrix of complex numbers for all $n, v \geq 0$ and $1 < k < \infty$. Then, $A \in \left(bv_k^\theta, bv\right)$ if and only if

$$\lim_{n\to\infty} \sum_{j=\nu}^{\infty} a_{nj} \text{ exists for each } v \tag{2.1}$$

$$\sup_m \sum_{\nu=0}^{m} \left| \theta_\nu^{-1/k^*} \sum_{j=\nu}^{m} a_{nj} \right|^{k^*} < \infty \text{ for each } n \tag{2.2}$$

$$\sum_{\nu=0}^{\infty} \left(\sum_{n=0}^{\infty} \left| \theta_\nu^{1/k^*} \sum_{j=\nu}^{\infty} \left(a_{nj} - a_{n-1,j}\right) \right| \right)^{k^*} < \infty. \tag{2.3}$$

Also, for special case $\theta_v = 1$, it is reduced to the following result of [1].

Corollary 2.2. Let $A = (a_{nv})$ be an infinite matrix of complex numbers for all $n, v \geq 0$ and $1 < k < \infty$. Then, $A \in \left(bv^k, bv\right)$ if and only if (2.1) holds,

$$\sup_m \sum_{\nu=0}^{m} \left| \sum_{j=\nu}^{m} a_{nj} \right|^{k^*} < \infty \text{ for each } n,$$

$$\sum_{\nu=0}^{\infty} \left(\sum_{n=0}^{\infty} \left| \sum_{j=\nu}^{\infty} \left(a_{nj} - a_{n-1,j}\right) \right| \right)^{k^*} < \infty.$$

Now we give the following theorem.

Theorem 2.3. Let $1 < k < \infty$ and $\theta = (\theta_n)$ be a sequence of positive numbers. If $A \in \left(bv_k^\theta, bv\right)$, then there exists $1 \leq \xi \leq 4$ such that

$$\|A\|_\chi = \frac{1}{\xi}\lim_{r\to\infty}\left\{\sum_{n=r+1}^{\infty}\left(\sum_{v=0}^{\infty}|d_{nv}|\right)^{k^*}\right\}^{1/k^*}, \tag{2.4}$$

and $A \in \mathcal{C}\left(bv_k^\theta, bv\right)$ if and only if

$$\lim_{r\to\infty}\sum_{n=r+1}^{\infty}\left(\sum_{v=0}^{\infty}|d_{nv}|\right)^{k^*} = 0 \tag{2.5}$$

where

$$d_{nj} = \theta_j^{-1/k^*}\sum_{v=j}^{\infty}\left(a_{nv} - a_{n-1,v}\right)$$

Proof. Define $T_1 : bv_k^\theta \to \ell_k$ and $T_2 : bv \to \ell$ by $T_1(x) = \theta_v^{1/k^*}(x_v - x_{v-1})$ and $T_2(x) = x_v - x_{v-1},\ x_{-1} = 0$. Then, it clear that T_1 and T_2 are isomorhism preseving norms, $i.e.$, $\|x\|_{bv_k^\theta} = \|T_1(x)\|_{\ell_k}$ and $\|x\|_{bv} = \|T_2(x)\|_\ell$. So, bv_k^θ and bv are isometrically isomorhic to ℓ_k and ℓ, respectively, $i.e.$, $bv_k^\theta \simeq \ell_k$ and $bv \simeq \ell$. Now let $T_1(x) = y$ for $x \in bv_k^\theta$. Then, $x = T_1^{-1}(y) \in S_{bv_k^\theta}$ if and only if $y \in S_{\ell_k}$, where $S_X = \left\{x \in X : \|x\|_X = 1\right\}$. Also, it is seen easily (see [3]) that $T_2AT_1^{-1} = D$ and $A \in \left(bv_k^\theta, bv\right)$ iff $D \in (\ell_k, \ell)$. Further, by Lemma 1.1, there exists $1 \leq \xi \leq 4$ such that

$$\begin{aligned}
\|A\|_{\left(bv_k^\theta, bv\right)} &= \sup_{x\neq\theta}\frac{\|A(x)\|_{bv}}{\|x\|_{bv_k^\theta}} = \sup_{x\neq\theta}\frac{\left\|T_2^{-1}DT_1(x)\right\|_{bv}}{\|x\|_{bv_k^\theta}} \\
&= \sup_{x\neq\theta}\frac{\|D(y)\|_\ell}{\|y\|_{\ell_k}} = \|D\|_{(\ell_k,\ell)} \\
&= \frac{1}{\xi}\|D\|'_{(\ell_k,\ell)}
\end{aligned}$$

and so, by Lemmas 1.2, 1.3 and 1.4, we have

$$\begin{aligned}
\|A\|_\chi &= \chi\left(AS_{bv_k^\theta}\right) = \chi(T_2AS_{bv_k^\theta}) \\
&= \chi(DT_1S_{bv_k^\theta}) = \lim_{r\to\infty}\sup_{y\in S_{\ell_k}}\|(I-P_r)\,D(y)\|_\ell \\
&= \lim_{r\to\infty}\sup_{y\in S_{\ell_k}}\left\|D^{(r)}(y)\right\| = \lim_{r\to\infty}\left\|D^{(r)}\right\|_{(\ell_k,\ell)} \\
&= \frac{1}{\xi}\lim_{r\to\infty}\left\{\sum_{n=r+1}^{\infty}\left(\sum_{v=0}^{\infty}|d_{nv}|\right)^{k^*}\right\}^{1/k^*}
\end{aligned}$$

where $P_r : \ell \to \ell$ is defined by $P_r(y) = (y_0, y_1, ..., y_r, 0, ...)$, and

$$d_{nv}^{(r)} = \begin{cases} 0, & 0 \leq n \leq r \\ d_{nv}, & n > r \end{cases}$$

So the proof is completed by Lemma 1.2.

In the special case $\theta_n = 1$, the following result is immediate.

Corollary 2.4. Let $1 < k < \infty$. If $A \in \left(bv^k, bv\right)$, then there exists $1 \leq \xi \leq 4$ such that

$$\|A\|_\chi = \frac{1}{\xi}\lim_{r\to\infty}\left\{\sum_{n=r+1}^{\infty}\left(\sum_{v=0}^{\infty}|d_{nv}|\right)^{k^*}\right\}^{1/k^*}$$

and

$$A \in \mathcal{C}\left(bv_k, bv\right) \text{ iff } \lim_{r\to\infty}\sum_{n=r+1}^{\infty}\left(\sum_{v=0}^{\infty}|d_{nv}|\right)^{k^*} = 0$$

where

$$d_{nj} = \sum_{v=j}^{\infty}\left(a_{nv} - a_{n-1,v}\right)$$

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