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SPECTRUM AND SYMMETRIES OF THE IMPULSIVE DIFFERENCE EQUATIONS

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Abstract. This paper deals with the spectral analysis and symmetries of the second order difference equations with impulse. We determine a transfer matrix and this allows us to investigate the locations of eigenvalues and spectral singularites of the difference operator generated in $\ell_2(\mathbb{Z})$.

1. Introduction

As is well known, impulsive equations appear as a natural description of the observed evolution phenomena in several real world problems [1–5]. The books on the subject of impulsive equations by Samoilenko [1] summarize and organize the theories and applications of impulsive equations and have a great contribution to the theory. In recent years, impulsive points have been a subject of both theoretical and experimental research and we observe an increasing interest about this area. Theory of impulsive differential equations has been motivated by a number of applied problems (control theory, population dynamics, chemotherapeutic treatment in medicine and some physics problems). Besides, impulsive difference equations are basic tools to do investigations in numerical analysis having applications in economics, social sciences, biology, engineering, etc... For the mathematical theory of impulsive difference equations, we refer to [6–8]. The spectral analysis of self–adjoint (Hermitian) and non–selfadjoint (non–Hermitian) difference operators were investigated in detail by several authors [9–12]. But the theory of impulsive

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difference equations is a new and important branch of operator theory which is interesting and useful.

In spectral theory, the concept of a spectral singularity of a second–order linear differential operator has been known since the pioneering work of Naimark [13] and recently studied by Guseinov [14]. It was proved that the spectral singularities are the spectral points of continuous spectrum and they spoil the completeness of the eigenfunctions of certain non-Hermitian operators. Hermitian operators have no spectral singularities and they have real spectrums. But the reality of the spectrum of an operator doesn't necessarily mean that it is Hermitian. A class of non-Hermitian operators which are called $\mathcal{P}$,$\mathcal{T}$, and $\mathcal{PT}$–symmetric, have real spectrum [15, 16]. A kind of such operators was studied in the recent years and the physical meaning and potential applications of spectral singularities were understood quite recently [14, 17]. In [17], Mostafazadeh considers the second–order time–independent Schrödinger equation

$$-\psi''(x) = k^2\psi(x), \quad x \in \mathbb{R}\setminus\{0\} \tag{1.1}$$

with a general point interaction

$$_+(0) = B\Psi_-(0), \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C} \tag{1.2}$$

at a single point $x = 0$, where

$$\Psi_\pm(x) := \begin{pmatrix} \psi_\pm(x) \\ \psi'_\pm(x) \end{pmatrix}, \quad \pm x \geq 0$$

introduces the two-component wave function and ψ_-, ψ_+ define respectively the restrictions of the solution of (1.1). His study becomes for us a tool to form the impulsive difference equation with general point interaction at a single point $n = 0$. The aim of present article is to explore the eigenvalues and spectral singularities depending on the choice of the coupling constants a, b, c, d for impulsive difference point interaction.

2. Spectral Singularities and Eigenvalues of the Impulsive Discrete Operator

We consider the second–order difference equation

$$y_{n-1} + y_{n+1} = \lambda y_n, \quad n \in \mathbb{Z}\setminus\{-1, 0, 1\} \tag{2.1}$$

with impulsive condition

$$\begin{pmatrix} y_1 \\ \Delta y_1 \end{pmatrix} = \mathbf{B}\begin{pmatrix} y_{-1} \\ \nabla y_{-1} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.2}$$

where a, b, c, d are complex numbers and $\lambda := z + \frac{1}{z}$ is a spectral parameter. Also, $\triangle$ denotes the forward difference operator and $\bigtriangledown$ denotes the backward difference operator, i.e.

$$\triangle y_n := y_{n+1} - y_n,$$
$$\bigtriangledown y_n := y_n - y_{n-1}.$$

$\{y_n{}^-\}$ and $\{y_n{}^+\}$ be respectively the restrictions of the solution $\{y_n\}$ to the sets of negative and positive integer numbers, i.e.

$$y_n^-(z) := y_n(z), \quad n \in \mathbb{Z}^-,$$
$$y_n^+(z) := y_n(z), \quad n \in \mathbb{Z}^+.$$

Clearly, $n = 0$ is the interaction (impulsive) point and matrix $\mathbf{B}$ is used to continue the solution from negative integer numbers to the positive integer numbers.

Now, let $z \in \mathbb{C} \backslash \{-1, 0, 1\}$ and take into account this interaction for (2.1). By the help of linearly independent solutions of (2.1), we can write the general solution as

$$y_n^-(z) = A_- z^n + B_- z^{-n}, \quad n \in \mathbb{Z}^-,$$
$$y_n^+(z) = A_+ z^n + B_+ z^{-n}, \quad n \in \mathbb{Z}^+,$$

where $A_\pm$ and $B_\pm$ are constant coefficients. If we introduce the two-component wave function

$$\Psi(y_n^-) := \begin{pmatrix} y_n^- \\ \nabla y_n^- \end{pmatrix}, \quad n \in \mathbb{Z}^-,$$

$$\Psi(y_n^+) := \begin{pmatrix} y_n^+ \\ \triangle y_n^+ \end{pmatrix}, \quad n \in \mathbb{Z}^+,$$

we can express the point interaction (2.2) by imposing the matching condition

$$\Psi(y_1) = \mathbf{B}\Psi(y_{-1}), \quad \mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}.$$

Using the point interaction given by (2.2), we obtain

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = M \begin{pmatrix} A_- \\ B_- \end{pmatrix}, \quad M = (M_{ij}); \quad i, j = 1, 2 \tag{2.3}$$

where

$$M := N_1^{-1} \mathbf{B} N_2$$

such that

$$N_1 := \begin{pmatrix} z & z^{-1} \\ z^2 - z & z^{-2} - z^{-1} \end{pmatrix}, \quad N_2 := \begin{pmatrix} z^{-1} & z \\ z^{-1} - z^{-2} & z - z^2 \end{pmatrix}.$$

Combining (2.1)-(2.3), we find

$$M_{11}(z) = \frac{1}{z(1-z^2)}\left[-bz^{-2} + (a+2b+d)\,z^{-1} - (a+b+c+d)\right],$$

$$M_{12}(z) = \frac{z}{1-z^2}\left[(a+b)z^{-1} + (b+d)\,z - (a+2b+c+d)\right],$$

$$M_{21}(z) = \frac{z}{1-z^2}\left[-(b+d)\,z^{-1} - (a+b)z + (a+2b+c+d)\right],$$

$$M_{22}(z) = \frac{z^3}{1-z^2}\left[bz^2 - (a+2b+d)\,z + (a+b+c+d)\right].$$

Thus, we obtain

$$\begin{aligned} M_{11}(z^{-1}) &= M_{22}(z), \\ M_{12}(z^{-1}) &= M_{21}(z). \end{aligned}$$

Now, consider the left- and right-going scattering solutions of (2.1)-(2.2) that we denote by y_n^l and y_n^r, respectively. They are expressed as

$$y_n^l(z) = \begin{cases} A_+^+ z^n + B_+^+ z^{-n}, & n \to +\infty \\ A_-^+ z^n + B_-^+ z^{-n}, & n \to -\infty, \end{cases}$$

$$y_n^r(z) = \begin{cases} A_+^- z^n + B_+^- z^{-n}, & n \to +\infty \\ A_-^- z^n + B_-^- z^{-n}, & n \to -\infty, \end{cases}$$

where $A_\pm^\pm$ and $B_\pm^\pm$ are complex coefficients. Let L denote the operator generated by (2.1)-(2.2) in $\ell_2(\mathbb{Z})$. By using the definition of the Wronskian of the Jost solutions of the operator L acting in $\ell_2(\mathbb{Z})$, we can give the following theorem [18]:

Theorem 2.1. *The following asymptotics hold:*

$$W[y_n^l, y_n^r](z) = M_{22}\left(\frac{1}{z} - z\right), \quad n \to +\infty, \tag{2.4}$$

$$W[y_n^l, y_n^r](z) = \frac{M_{22}}{\det M}\left(\frac{1}{z} - z\right), \quad n \to -\infty. \tag{2.5}$$

Proof. The left- and right-going scattering solutions are defined in terms of their asymptotic behaviors

$$\begin{aligned} y_n^l(z) &\to z^n, \quad n \to +\infty, \\ y_n^r(z) &\to z^{-n}, \quad n \to -\infty \end{aligned}$$

and so we obtain

$$A_+^+ = B_-^- = 1, \quad A_-^- = B_+^+ = 0. \tag{2.6}$$

Next, for the left-going scattering solution using the expression (2.3), we have

$$\begin{pmatrix} A_+^+ \\ B_+^+ \end{pmatrix} = M \begin{pmatrix} A_-^+ \\ B_-^+ \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} A_-^+ \\ B_-^+ \end{pmatrix} = \frac{1}{\det M} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So we have

$$A_-^+ = \frac{M_{22}}{\det M}, \quad B_-^+ = -\frac{M_{21}}{\det M} \quad . \tag{2.7}$$

Similarly, for the right-going scattering solution, we get

$$A_+^- = M_{12}, \quad B_+^- = M_{22}. \tag{2.8}$$

Now, we can express the left- and right-going scattering solutions of (2.1)-(2.2) as

$$\begin{aligned} y_n^l(z) &= \begin{cases} z^n, & n \to +\infty \\ \dfrac{M_{22}}{\det M} z^n - \dfrac{M_{21}}{\det M} z^{-n}, & n \to -\infty \end{cases} \\ y_n^r(z) &= \begin{cases} M_{12} z^n + M_{22} z^{-n}, & n \to +\infty \\ z^{-n}, & n \to -\infty. \end{cases} \end{aligned} \tag{2.9}$$

Since the Wronskian of linearly independent solutions is independent of n, (2.9) can be used to perform the Wronskian of the Jost solutions for $n \to +\infty$ and for $n \to -\infty$.

(i) For $n \to +\infty$, we get

$$\begin{aligned} W\left[y_n^l, y_n^r\right](z) &= z^n \left(M_{12} z^{n+1} + M_{22} z^{-n-1}\right) - z^{n+1}\left(M_{12} z^n + M_{22} z^{-n}\right) \\ &= M_{22}\left(\frac{1}{z} - z\right). \end{aligned}$$

(ii) For $n \to -\infty$, we see

$$\begin{aligned} W\left[y_n^l, y_n^r\right](z) &= \left(\frac{M_{22}}{\det M} z^n - \frac{M_{21}}{\det M} z^{-n}\right) z^{-n-1} - \left(\frac{M_{22}}{\det M} z^{n+1} - \frac{M_{21}}{\det M} z^{-n-1}\right) z^{-n} \\ &= \frac{M_{22}}{\det M}\left(\frac{1}{z} - z\right). \end{aligned}$$

Therefore, the proof is completed. □

The transfer matrix for a piecewise continuous scattering potentials has unit determinant [17]. For the point interaction (2.2), we consider

$$\det M = \det \mathbf{B} = ad - bc = 1. \tag{2.10}$$

The interactions violating this condition are called *anomalous point interactions* [17].

Using Theorem 2.1 and (2.10), we have the following:

Corollary 2.2. *The necessary and sufficient condition to investigate the eigenvalues and spectral singularities of the operator L is to investigate the zeros of the function M_{22}.*

Since the spectral singularities and eigenvalues of L correspond to λ values for which $M_{22}(z) = 0$, we need to examine the zeros of the function M_{22}, i.e.,

$$bz^2 - (a + 2b + d)\, z + (a + b + c + d) = 0. \tag{2.11}$$

$\sigma_d(L)$ and $\sigma_{ss}(L)$ will denote the eigenvalues and spectral singularities of L, respectively. Therefore, by the definitions of eigenvalues and spectral singularities of an operator, we can write [11, 13],

$$\sigma_{ss}(L) = \left\{\lambda : \lambda = z + \frac{1}{z},\ z \in D_0,\ M_{22}(z) = 0\right\} \tag{2.12}$$

and

$$\sigma_d(L) = \left\{\lambda : \lambda = z + \frac{1}{z},\ z \in D_1,\ M_{22}(z) = 0\right\}, \tag{2.13}$$

where $D_1 := \{z : 0 < |z| < 1\}$ and $D_0 := \{z : |z| = 1\}$.

In order to examine the zeros of (2.11), we consider the following cases.

Case 1. $b \neq 0$: *In this case,* (2.11) *gives*

$$z_{1,2} = 1 + \mu \pm \sqrt{\mu^2 - \nu},$$

where

$$\mu := \frac{a + d}{2b}, \quad \nu := \frac{c}{b}$$

and therefore, a spectral singularity appears whenever

$$\left|1 + \mu \pm \sqrt{\mu^2 - \nu}\right| = 1$$

and an eigenvalue exists if

$$0 < \left|1 + \mu \pm \sqrt{\mu^2 - \nu}\right| < 1.$$

Case 2. $b = 0$ *and* $a + d = tr\mathbf{B} \neq 0$*: In this case,* (2.11) *gives*

$$z = 1 + \frac{c}{a + d}$$

and thus, a spectral singularity appears whenever

$$|a + c + d| = |a + d|$$

and an eigenvalue exists if

$$0 < |a + c + d| < |a + d| .$$

Case 3. $b = 0$ *and* $a + d = tr\mathbf{B} = 0$*: Then the condition of the existence of a spectral singularity or an eigenvalue, namely* $M_{22} = 0$*, implies that* $c = 0$*. In this case,* $\mathbf{B} = a\sigma_3$ *and* $M = a\sigma_1$*, where*

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, M *is independent of* λ*,* M_{22} *vanishes identically, and the interaction is anomalous for* $a \neq \pm i$*.*

3. $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{PT}$-Symmetries

In this section, we examine the consequences of imposing $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{PT}$–symmetries on the point interaction (2.2) and their spectral singularities, eigenvalues.

3.1. $\mathcal{P}$–Symmetry.

Definition 3.1. Let $\mathcal{P}$ be the parity (reflection) operator acting in the space of complex sequences $y = \{y_n\}$, $y_n : \mathbb{Z} \to \mathbb{C}$. Then for all $n \in \mathbb{Z}$, we have

$$\mathcal{P}y_n := y_{-n}.$$

Definition 3.2. The point interaction (2.2) is $\mathcal{P}$ invariant (or has $\mathcal{P}$–symmetry) if

$$\mathcal{P}\begin{pmatrix} y_1 \\ \triangle y_1 \end{pmatrix} = \mathbf{B}\mathcal{P}\begin{pmatrix} y_{-1} \\ \nabla y_{-1} \end{pmatrix}, \tag{3.1}$$

where the action of $\mathcal{P}$ on a two-component wave function is defined componentwise.

Hence, it is easy to verify that

$$\mathcal{P} \triangle y_n = - \nabla y_{-n}$$
$$\mathcal{P} \nabla y_n = - \triangle y_{-n}.$$

Theorem 3.3. *The point interaction* (2.2) *has* $\mathcal{P}$*–symmetry if and only if*

$$\sigma_3 = \mathbf{B}\sigma_3\mathbf{B}. \tag{3.2}$$

Proof. Assume that the point interaction (2.2) has $\mathcal{P}$–symmetry. Since (2.2) can be expressed as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ - \triangle y_1 \end{pmatrix} = \mathbf{B}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_{-1} \\ - \nabla y_{-1} \end{pmatrix},$$

we get the following by Definition 3.1 :

$$\sigma_3\mathcal{P}\begin{pmatrix} y_{-1} \\ \nabla y_{-1} \end{pmatrix} = \mathbf{B}\sigma_3\mathcal{P}\begin{pmatrix} y_1 \\ \triangle y_1 \end{pmatrix}.$$

Using (3.1), it is easy to write

$$\sigma_3 \mathcal{P} \begin{pmatrix} y_{-1} \\ \triangledown y_{-1} \end{pmatrix} = \mathbf{B}\sigma_3 \mathbf{B} \mathcal{P} \begin{pmatrix} y_{-1} \\ \triangledown y_{-1} \end{pmatrix}.$$

This gives a consequence that

$$\sigma_3 = \mathbf{B}\sigma_3\mathbf{B}.$$

Now, let (3.2) holds. In exactly the same way, we can prove that the interaction (2.2) has $\mathcal{P}$–symmetry. □

In terms of the entries of $\mathbf{B}$, we can give that the point interaction (2.2) has $\mathcal{P}$–symmetry as the following theorem:

Theorem 3.4. *The point interaction* (2.2) *has $\mathcal{P}$-symmetry if and only if*

$$ad - bc = 1, \quad a = d, \quad a, b, c, d \in \mathbb{C} \qquad .$$

Proof. Assume that (2.2) has $\mathcal{P}$–symmetry, we obtain

$$\begin{pmatrix} \mathcal{P} y_1 \\ \mathcal{P} \vartriangle y_1 \end{pmatrix} = \mathbf{B} \begin{pmatrix} \mathcal{P} y_{-1} \\ \mathcal{P} \triangledown y_{-1} \end{pmatrix}$$

from (3.1). Then it verifies that

$$\begin{pmatrix} y_{-1} \\ - \triangledown y_{-1} \end{pmatrix} = \mathbf{B} \begin{pmatrix} y_1 \\ - \vartriangle y_1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_{-1} \\ \triangledown y_{-1} \end{pmatrix} = \mathbf{B} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ \vartriangle y_1 \end{pmatrix}.$$

Therefore, we write

$$\begin{pmatrix} y_{-1} \\ \triangledown y_{-1} \end{pmatrix} = \widetilde{\mathbf{B}} \begin{pmatrix} y_1 \\ \vartriangle y_1 \end{pmatrix},$$

where

$$\widetilde{\mathbf{B}} := \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Since $\widetilde{\mathbf{B}}^{-1} = \mathbf{B}$, the coefficients of the matrix $\mathbf{B}$ satisfy

$$a = \frac{d}{ad - bc}, \quad b = \frac{b}{ad - bc}, \quad c = \frac{c}{ad - bc}, \quad d = \frac{a}{ad - bc} \qquad .$$

This completes the proof. □

Theorem 3.5. *If the point interaction* (2.2) *has $\mathcal{P}$–symmetry, then we can summarize the conditions for the existence of spectral singularities and eigenvalues as follows:*

(i) *If* $b \neq 0$, *then there exists a spectral singularity whenever*

$$|a + b \pm 1| = |b|$$

and an eigenvalue exists whenever

$$0 < |a + b \pm 1| < |b| \, .$$

(ii) *If* $b = 0$, *then there exists a spectral singularity whenever*

$$|c \pm 2| = 2$$

and an eigenvalue exists whenever

$$0 < |c \pm 2| < 2.$$

3.2. $\mathcal{T}$–**Symmetry.**

Definition 3.6. Let $\mathcal{T}$ be the time–reversal operator acting on the space of complex sequences $y = \{y_n\}$, $y_n : \mathbb{Z} \to \mathbb{C}$. Then for all $n \in \mathbb{Z}$, we have

$$\mathcal{T} y_n := y_n^*,$$

where "$*$" denotes the complex conjugate of y.

Definition 3.7. The point interaction (2.2) is the time–reversal invariant (or has $\mathcal{T}$–symmetry) if

$$\mathcal{T} \begin{pmatrix} y_1 \\ \triangle y_1 \end{pmatrix} = \mathbf{B} \mathcal{T} \begin{pmatrix} y_{-1} \\ \nabla y_{-1} \end{pmatrix}, \tag{3.3}$$

where the action of $\mathcal{T}$ on a two-component wave function is defined componentwise.

Theorem 3.8. *The point interaction* (2.2) *has* $\mathcal{T}$*–symmetry if and only if* $\mathbf{B}$ *is real matrix.*

Proof. Taking account of the definition of $\mathcal{T}$–symmetry for the point interaction (2.2), it is easy to see that this relation is equivalent to the requirement that $\mathbf{B}$ is a real matrix, i.e. a, b, c, d must be real. □

3.3. $\mathcal{PT}$–**Symmetry.**

Definition 3.9. The point interaction (2.2) is $\mathcal{PT}$ invariant (or has $\mathcal{PT}$–symmetry) if

$$\mathcal{PT} \begin{pmatrix} y_1 \\ \triangle y_1 \end{pmatrix} = \mathbf{B} \mathcal{PT} \begin{pmatrix} y_{-1} \\ \nabla y_{-1} \end{pmatrix}. \tag{3.4}$$

Theorem 3.10. *The point interaction* (2.2) *has* $\mathcal{PT}$*–symmetry if and only if*

$$\sigma_3 = \mathbf{B}^* \sigma_3 \mathbf{B} \tag{3.5}$$

holds.

Proof. Let the point interaction (2.2) has $\mathcal{PT}$–symmetry. By taking complex conjugate from (2.2), we write

$$\begin{pmatrix} y_1^* \\ \triangle y_1^* \end{pmatrix} = \mathbf{B}^* \begin{pmatrix} y_{-1}^* \\ \nabla y_{-1}^* \end{pmatrix}$$

and thus, we get

$$\begin{pmatrix} \mathcal{T} y_1 \\ \mathcal{T} \triangle y_1 \end{pmatrix} = \mathbf{B}^* \begin{pmatrix} \mathcal{T} y_{-1} \\ \mathcal{T} \nabla y_{-1} \end{pmatrix}.$$

It follows from that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{T} y_1 \\ -\triangle \mathcal{T} y_1 \end{pmatrix} = \mathbf{B}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{T} y_{-1} \\ -\nabla \mathcal{T} y_{-1} \end{pmatrix}.$$

Hence, we obtain

$$\sigma_3 \begin{pmatrix} \mathcal{PT} y_{-1} \\ \mathcal{PT} \nabla y_{-1} \end{pmatrix} = \mathbf{B}^* \sigma_3 \begin{pmatrix} \mathcal{PT} y_1 \\ \mathcal{PT} \triangle y_1 \end{pmatrix}$$

so that

$$\sigma_3 \mathcal{PT} \begin{pmatrix} y_{-1} \\ \nabla y_{-1} \end{pmatrix} = \mathbf{B}^* \sigma_3 \mathbf{B} \mathcal{PT} \begin{pmatrix} y_1 \\ \triangle y_1 \end{pmatrix}.$$

Using (3.4), it implies that

$$\sigma_3 = \mathbf{B}^* \sigma_3 \mathbf{B}.$$

Now, let (3.5) holds. We prove that the point interaction is $\mathcal{PT}$–invariant in a similar way. Therefore, this completes the proof. □

Expressing this relation in terms of the entries of $\mathbf{B}$ and solving the resulting equations yield the following theorem:

Theorem 3.11. *Let a, b, c and d be the complex numbers as follows*

$$a = r_a e^{i\alpha}, \quad b = r_b e^{i\beta}, \quad c = r_c e^{i\theta}, \quad d = r_d e^{i\delta}.$$

The point interaction (2.2) has $\mathcal{PT}$–symmetry if and only if

$$\mathbf{B} = e^{i\frac{\alpha+\delta}{2}} \begin{pmatrix} \sqrt{1+\epsilon_2 r_b r_c}\, e^{i\frac{\alpha-\delta}{2}} & \epsilon_1 \epsilon_2 r_b \\ \epsilon_1 r_c & \sqrt{1+\epsilon_2 r_b r_c}\, e^{-i\frac{\alpha-\delta}{2}} \end{pmatrix},$$

where $\alpha + \delta = 2\pi k$, $k \in \mathbb{Z}$ and

$$\epsilon_1 := (-1)^m, \quad m \in \mathbb{Z}, \quad \epsilon_2 := \begin{cases} \epsilon_1, & 0 \le r_b r_c \le 1 \\ 1, & r_b r_c > 1 \end{cases}.$$

Proof. Let the point interaction (2.2) has $\mathcal{PT}$–symmetry. Using (3.5), we have the following relation

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Last equality implies that

$$a^*a = 1 + b^*c, \quad a^*b - b^*d = 0, \quad c^*a - d^*c = 0, \quad d^*d = 1 + c^*b$$

and it is clear from that

$$|a|^2 = |d|^2 = 1 + c^*b, \quad a^*b = b^*d, \quad c^*a = d^*c.$$

These relations give us the following cases:

$$\begin{array}{lll}
I. & |a|^2 = |d|^2 & \Rightarrow \; r_a = r_d \\[2mm]
II. & a^*b = b^*d & \Rightarrow \; r_a r_b e^{(\beta-\alpha)i} = r_b r_d e^{(\delta-\beta)i} \\
 & & \Rightarrow \; \beta = \dfrac{\alpha+\delta}{2} + n\pi, \quad n \in \mathbb{Z} \\[2mm]
III. & d^*c = c^*a & \Rightarrow \; r_c r_d e^{(\theta-\delta)i} = r_a r_c e^{(\alpha-\theta)i} \\
 & & \Rightarrow \; \theta = \dfrac{\alpha+\delta}{2} + m\pi, \quad m \in \mathbb{Z} \\[2mm]
IV. & |a|^2 = |d|^2 = 1 + c^*b & \Rightarrow \; r_a^2 = r_d^2 = 1 + r_b r_c e^{(n-m)\pi i}, \quad m, n \in \mathbb{Z} \\
 & & \Rightarrow \; r_a^2 = r_d^2 = 1 + r_b r_c (-1)^{n-m}, \quad m, n \in \mathbb{Z}
\end{array}$$

and so, we get

$$\begin{cases} a = \sqrt{1 + (-1)^{n-m} r_b r_c}\, e^{i\alpha} \\ b = (-1)^n r_b e^{i\frac{\alpha+\delta}{2}} \\ c = (-1)^m r_c e^{i\frac{\alpha+\delta}{2}} \\ d = \sqrt{1 + (-1)^{n-m} r_b r_c}\, e^{i\delta} \end{cases},$$

where $m, n \in \mathbb{Z}$. Since $r_b, r_c \in [0, +\infty)$, we can write

$$1 + (-1)^{n-m} r_b r_c \geq 0$$

and so

$$1 + (-1)^{n-m} r_b r_c \geq 0 \quad \Rightarrow \quad (-1)^{n-m} \geq -\frac{1}{r_b r_c} \; .$$

Next, we investigate this relation corresponds to the $(n-m)$ integer.

(**I**) Let $(n-m)$ be an even number.

(**I.a**) Let $n = 2n_1$ and $m = 2m_1$, where $n_1, m_1 \in \mathbb{Z}$. Since

$$1 + (-1)^{n-m} r_b r_c = 1 + r_b r_c,$$

we obtain

$$\begin{cases} a = \sqrt{1 + r_b r_c} e^{i\alpha} \\ b = r_b e^{i\frac{\alpha+\delta}{2}} \\ c = r_c e^{i\frac{\alpha+\delta}{2}} \\ d = \sqrt{1 + r_b r_c} e^{i\delta}. \end{cases}$$

(**I.b**) Let $n = 2n_2 + 1$ and $m = 2m_2 + 1$ where $n_1, m_1 \in \mathbb{Z}$. In this case,

$$1 + (-1)^{n-m} r_b r_c = 1 + r_b r_c$$

and we get similarly

$$\begin{cases} a = \sqrt{1 + r_b r_c} e^{i\alpha} \\ b = -r_b e^{i\frac{\alpha+\delta}{2}} \\ c = -r_c e^{i\frac{\alpha+\delta}{2}} \\ d = \sqrt{1 + r_b r_c} e^{i\delta}. \end{cases}$$

(**II**) Let $(n - m)$ be an odd number.

(**II.a**) Let $n = 2n_1$ and $m = 2m_1 + 1$ where $n_1, m_1 \in \mathbb{Z}$.

$$1 + (-1)^{n-m} r_b r_c = 1 - r_b r_c.$$

This implies that

$$0 \le r_b r_c \le 1$$

and we obtain

$$\begin{cases} a = \sqrt{1 - r_b r_c} e^{i\alpha} \\ b = r_b e^{i\frac{\alpha+\delta}{2}} \\ c = -r_c e^{i\frac{\alpha+\delta}{2}} \\ d = \sqrt{1 - r_b r_c} e^{i\delta}. \end{cases}$$

(**II.b**) Let $n = 2n_2 + 1$ and $m = 2m_2$ where $n_2, m_2 \in \mathbb{Z}$. We find

$$1 + (-1)^{n-m} r_b r_c = 1 - r_b r_c$$

so that

$$0 \le r_b r_c \le 1.$$

Therefore, we obtain

$$\begin{cases} a = \sqrt{1 - r_b r_c} e^{i\alpha} \\ b = -r_b e^{i\frac{\alpha+\delta}{2}} \\ c = r_c e^{i\frac{\alpha+\delta}{2}} \\ d = \sqrt{1 - r_b r_c} e^{i\delta}. \end{cases}$$

Consequently, we also give the both cases with a single expression such that

$$\begin{cases} a = \sqrt{1+\epsilon_2 r_b r_c}e^{i\alpha} \\ b = \epsilon_1\epsilon_2 r_b e^{i\frac{\alpha+\delta}{2}} \\ c = \epsilon_1 r_c e^{i\frac{\alpha+\delta}{2}} \\ d = \sqrt{1+\epsilon_2 r_b r_c}e^{i\delta} \end{cases} \tag{3.6}$$

where

$$\epsilon_1 := (-1)^m, \quad \epsilon_2 := \begin{cases} \epsilon_1, & 0 \le r_b r_c \le 1 \\ 1, & r_b r_c > 1 \end{cases}$$

and thus, the transmission matrix $\mathbf{B}$ is given as

$$\mathbf{B} = e^{i\frac{\alpha+\delta}{2}} \begin{pmatrix} \sqrt{1+\epsilon_2 r_b r_c}e^{i\frac{\alpha-\delta}{2}} & \epsilon_1\epsilon_2 r_b \\ \epsilon_1 r_c & \sqrt{1+\epsilon_2 r_b r_c}e^{-i\frac{\alpha-\delta}{2}} \end{pmatrix}, \tag{3.7}$$

where $\alpha+\delta = 2\pi k$, $k \in \mathbb{Z}$. □

Theorem 3.12. *If the point interaction* (2.2) *has* $\mathcal{PT}$*–symmetry, then we can summarize the conditions for the existence of spectral singularities and eigenvalues as follows:*

(i) *Suppose that* $b \neq 0$*. Then, we have spectral singularities if*

$$a+c+d=0 \qquad \textit{or} \qquad 2a+4b+c+2d=0$$

and eigenvalues if

$$1 < |a+b|^2 < 1+|b|^2 \qquad \textit{or} \qquad |a+b| \neq 1 \qquad \textit{and} \qquad |a+2b| < 1 < |a|\ .$$

(ii) *Suppose that* $b=0$ *and* $a+d = tr\mathbf{B} \neq 0$*. Then, we have a spectral singularity if*

$$|a+c+d| = |a+d|$$

and an eigenvalue if

$$0 < |a+c+d| < |a+d|\,.$$

4. Conclusions

In this paper, we study the spectral analysis of an impulsive difference operator in $\ell_2(\mathbb{Z})$ generated by a second-order difference equation with an impulsive interior point. In literature, there exist a lot of paper investigating the properties of eigenvalues and spectral singularities of difference operators (see [9–11]), but this study differs from the others with some aspects. First of all, none of the difference equations considered in these studies have a discontinuous point. We handle the difference equation with one discontinuity given at $n = 0$. Furthermore, unlike the well known methods, we determine a transfer matrix which enables us to find the locations of eigenvalues and spectral singularities by the help of the zeros of a quadratic polynomial. The rest of paper is devoted to a detailed analysis of certain symmetries that the impulsive condition possesses. This paper is the first one that points up the effects of a single interaction point of an impulsive difference boundary value problem. In further research, one can study the general form of Equation (2.1) with more interaction points.

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