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W–LINE CONGRUENCES

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Abstract. By utilizing the Darboux frames, along with a regular surface whose parametric curves are lines of curvature, we analyzed the normal line congruence which preserves the asymptotic curves between its focal surfaces. This allows deriving systems of partial differential equations through which the problem of determining the director surface and the corresponding normal line congruence could be solved. Moreover, a necessary and sufficient condition that the focal surfaces of the normal line congruence are degenerates into curves is derived. As a result the middle focal surface of the normal line congruence is presented as a new surface interrogation tool.

1. Introduction

In Euclidean 3-space, a two-parameter set of lines is called a line congruence. For instance, the normal vector field of a surface constitute such a line congruence but this is not the general situation. Hence, the line congruence of normals forms a special class; which is called normal line congruence. The lines of a line congruence meet a given plane in such a way that through a point of the plane one line, or at most a finite member, pass. Similar results hold if a surface is taken instead of a plane; this surface is called the reference surface or director surface of the line congruence. The lines of the line congruence which pass through a curve on the surface form a one-parameter set of lines i.e. a ruled surface (parameter ruled surface). It is known that on each generator of line congruence, there are two special points, called the focal points. This terminology is justified by the fact that a line congruence can be considered as the set of lines tangents two surfaces, the focal surfaces of the line congruence. Therefore there are two surfaces such that the generating lines of the line congruence are tangents to these surfaces.

There are several different ways that the representation of the line geometry. One of them is the dual vector system; a point on a dual unit sphere corresponds to a straight line in the 3-dimensional Euclidean system. So, the one parameter motion of this point corresponds to a ruled surface, while its two real parameter

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motion corresponds to a line congruence. Nowadays, the differential geometry of the line congruence and the focal surfaces have been widely applied in design and manufacturing, (e.g. Computer Aided Geometric Design/Computer Aided Manufacturing) of products and many other areas such as motion analysis and simulation of rigid bodies via dual number and dual vector systems and model-based object recognition systems [10-13].

This work is organized in the following way: In sec. 2, we present a brief introduction to the basic definitions of the representation of the Darboux frame on a regular surface whose parametric curves are lines of curvature and the normal line congruence. Sec. 3 is dedicated to the main results; we form systems of partial differential equations related to the following properties: the representation preserves, asymptotic curves, and the element area between the focal surfaces. Meanwhile, a necessary and sufficient condition that the focal surfaces of the normal line congruence are degenerates into curves has been derived. Especially, we have been paid pay attention to the director surface to be minimal surface and Weingarten-surface since the focal surfaces have special geometrical properties. Finally, the generalization middle focal surface is presented as a new surface interrogation tool.

2. Line Congruence in Euclidean 3-Space E^3

In the following, we will present some facts about classical results of differential line geometry in order to introduce the notations which will be used through the next sections. These and more recent descriptions about line congruences can be found in the works [1-4,6,8].

Let the vector function $\mathbf{r} = \mathbf{r}(u_1, u_2)$ represent a regular non-spherical and-non developable surface M in Euclidean 3-Space E^3, i.e. $\mathbf{r} : U \subset \mathbb{R}^2 \to E^3$ be a regular parametrized surface and g_{ij} and h_{ij} are the coefficients of the first and second fundamental forms of the surface M, and suppose that the u_1-and u_2 curves of this parametrization are lines of curvature, i.e., the elements g_{12} and h_{12} vanish identically ($g_{12} = h_{12} = 0$). Consider now the unit vectors $\mathbf{e}_1 = \mathbf{e}_1(u_1, u_2)$, $\mathbf{e}_2 = \mathbf{e}_2(u_1, u_2)$, are the tangents of the parametric curves u_2=const., u_1 =const., and the unit vector $\mathbf{e}_3 = \mathbf{e}_3(u_1, u_2)$ of the normal to the surface M at any regular point, then we have:

$$\mathbf{e}_1 = \frac{1}{\sqrt{g_{11}}}\frac{\partial \mathbf{r}}{\partial u_1},\ \mathbf{e}_2 = \frac{1}{\sqrt{g_{22}}}\frac{\partial \mathbf{r}}{\partial u_2},\ \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2, \tag{2.1}$$

which are invariants vector functions on the surface. Using that u_1-and u_2 curves are curvature lines on the surface, we can calculate $ds = \sqrt{g_{11}}du_1$-and $d\overline{s} = \sqrt{g_{22}}du_2$, the arc length parameters of the curves u_2=const., u_1 =const., respectively. The moving frame $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ on the surface M at every regular point is then called the Darboux frame. Hence, by means of the derivatives with respect to the arc length parameter of the curves u_2=const. with tangent $\mathbf{e}_1$ on M, the derivative formula with respect to the Darboux frame, may be stated as [1]:

$$\frac{\partial}{\partial s}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}=\begin{pmatrix}0 & q & k\\ -q & 0 & 0\\ -k & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}, \tag{2.2}$$

where $k=\frac{h_{11}}{g_{11}}=<\frac{\partial \mathbf{e}_1}{\partial s},\mathbf{e}_3>$, and $q=\frac{-(g_{11})_{u_2}}{2g_{11}\sqrt{g_{22}}}=<\frac{\partial \mathbf{e}_1}{\partial s},\mathbf{e}_2>$ are the normal and geodesic curvatures of the curves u_2=const., respectively. Similarly, the derivative formula of the Darboux frame of the curves u_1=const., with tangent $\mathbf{e}_2$ on M is:

$$\frac{\partial}{\partial \overline{s}}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}=\begin{pmatrix}0 & \overline{q} & 0\\ -\overline{q} & 0 & \overline{k}\\ 0 & -\overline{k} & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}, \tag{2.3}$$

also $\overline{k}=\frac{h_{22}}{g_{22}}=<\frac{\partial \mathbf{e}_2}{\partial \overline{s}},\mathbf{e}_3>$, and $\overline{q}=\frac{(g_{22})_{u_1}}{2g_{22}\sqrt{g_{11}}}=-<\frac{\partial \mathbf{e}_2}{\partial \overline{s}},\mathbf{e}_1>$ have the same meaning as in (2.2), for the curves u_1=const. on the surface M. We shall denote $\partial/\partial s$ and $\partial/\partial\overline{s}$ by the suffixes 1 and 2.

Since k, $\overline{k}$, and q, $\overline{q}$ are the invariant quantities of curvature on M, these invariants and their derivatives must fulfill the Gauss and Mainardi-Codazzi equations [1]:

$$\left.\begin{array}{c}-q^2+q_2-k\overline{k}=\overline{q}_1+\overline{q}^2,\\ q(\overline{k}-k)+k_2=0,\\ \overline{q}(\overline{k}-k)+\overline{k}_1=0.\end{array}\right\} \tag{2.4}$$

As stated earlier, given a set of unit vectors $\mathbf{e}_3=\mathbf{e}_3(u_1,u_2)$, the normal line congruence in E^3 is defined in the parameter form:

$$C_N:\mathbf{y}(u_1,u_2,\mu)=\mathbf{r}(u_1,u_2)+\mu\mathbf{e}_3(u_1,u_2),\ \mu\in\mathbb{R}, \tag{2.5}$$

where $\mathbf{r}=\mathbf{r}(u_1,u_2)$ is its director surface and $\mathbf{e}_3=\mathbf{e}_3(u_1,u_2)$ is the unit vector field along the direction of the generating lines of the congruence.

3. Main Results

It is known that the consecutive normals along a line of curvature on M : $\mathbf{r}=\mathbf{r}(u_1,u_2)$ intersect, the points of intersection being the corresponding center of curvature. The locus of the centers of curvature for all points of the surface M is called the surface of centers or centro-surface of M. In general it consists of two sheets, conjugated to the two families of lines of curvature and called focal surfaces of M. The parametric representations of the focal surfaces of C are given by [6,7,14]:

$$\left.\begin{array}{ll}F: & \mathbf{x}(u_1,u_2)=\mathbf{r}(u_1,u_2)+\rho\mathbf{e}_3(u_1,u_2),\ \rho=\frac{1}{k}\neq 0,\\ \overline{F}: & \overline{\mathbf{x}}(u_1,u_2)=\mathbf{r}(u_1,u_2)+\overline{\rho}\mathbf{e}_3(u_1,u_2),\ \overline{\rho}=\frac{1}{\overline{k}}\neq 0.\end{array}\right\} \tag{3.1}$$

Let g^i_{jk}, and h^i_{jk} $(i=1,2)$ are the coefficients of the first and second fundamental forms of the focal surfaces $\mathbf{x}=\mathbf{x}(u_1,u_2)$, and $\overline{\mathbf{x}}=\overline{\mathbf{x}}(u_1,u_2)$, respectively, one can obtain:

$$\left.\begin{array}{l} g^1_{11} = \rho_1^2, \quad g^1_{12} = \rho_1\,\rho_2, \quad g^1_{22} = (1-\frac{\rho}{\overline{\rho}})^2 + \rho_2^2, \\ g^2_{11} = (1-\frac{\overline{\rho}}{\rho})^2 + \overline{\rho}_1^2, \quad g^2_{12} = \overline{\rho}_1\,\overline{\rho}_2, \quad g^2_{22} = \overline{\rho}_2^2, \end{array}\right\} \tag{3.2}$$

and

$$\left.\begin{array}{l} h^1_{11} = -\frac{\rho_1}{\rho}\;, \quad h^1_{12} = 0, \quad h^1_{22} = \frac{\rho\,\overline{\rho}_1}{\overline{\rho}^2}, \\ h^2_{11} = \frac{\overline{\rho}\rho_2}{\rho^2}\;, \quad h^2_{12} = 0, \quad h^2_{22} = -\frac{\overline{\rho}_2}{\overline{\rho}}. \end{array}\right\} \tag{3.3}$$

Hence, the parametric curves on the focal surfaces, which correspond to the lines of curvature on the director surface, are conjugate, but not (generally) lines of curvature. The expression for the Gaussian curvatures of the focal surfaces F, and $\overline{F}$, at the corresponding points, are:

$$\left.\begin{array}{l} K_x = -\frac{\overline{\rho}_1}{\rho_1(\,\overline{\rho}-\rho)^2}, \\ \overline{K}_{\overline{\mathbf{x}}} = -\frac{\rho_2}{\overline{\rho}_2(\,\overline{\rho}-\rho)^2}. \end{array}\right\} \tag{3.4}$$

Moreover, the Mainardi-Codazzi equations may be given as in the following form

$$\left.\begin{array}{l} \frac{\partial}{\partial \overline{s}}[\ln \frac{\sqrt{g_{11}}}{\rho} - \int \frac{d\rho}{\overline{\rho}-\rho}] = 0, \\ \frac{\partial}{\partial s}[\ln \frac{\sqrt{g_{22}}}{f(\rho)} - \int \frac{d\overline{\rho}}{\rho-\overline{\rho}}] = 0. \end{array}\right\} \tag{3.5}$$

The integration of equations (3.5) is reducible to

$$\sqrt{g_{11}} = \rho a(s) e^{\int \frac{d\rho}{\overline{\rho}-\rho}}, \ \sqrt{g_{11}} = b(\overline{s}) e^{\int \frac{d\overline{\rho}}{\rho-\overline{\rho}}}. \tag{3.6}$$

Without changing the parametric curves, we may assume that $a(s) = b(\overline{s}) = 1$, then we get:

$$g_{11} = \rho^2 e^{2\int \frac{d\rho}{\overline{\rho}-\rho}}, \ g_{12} = 0, \quad g_{22} = \overline{\rho}^2 e^{2\int \frac{d\overline{\rho}}{\rho-\overline{\rho}}}. \tag{3.7}$$

Therefore

$$h_{11} = \rho e^{-2\int \frac{d\rho}{\overline{\rho}-\rho}}, \quad h_{12} = 0, \qquad h_{22} = \overline{\rho} e^{-2\int \frac{d\overline{\rho}}{\rho-\overline{\rho}}}. \tag{3.8}$$

Thus g_{11}, g_{22} , and h_{11}, h_{22} are expressible as functions of ρ or $\overline{\rho}$, and consequently they are functions of one another. It is clear after simple manipulation that these magnitudes satisfy the Gauss's equation.

3.1. **Weingarten line congruence (W-line congruence).** A line congruence in Euclidean 3-space E^3 is a two-parameter set of straight lines. Such a congruence has a parameterization in the form [14]:

$$L : \mathbf{y}(u,v,\lambda) = \mathbf{p}(u,v) + \lambda\boldsymbol{\xi}(u,v), \|\boldsymbol{\xi}\| = 1, \tag{3.9}$$

where $\mathbf{p}(u,v)$ is its base surface (the surface of reference) and $\boldsymbol{\xi}(u,v)$ is the unit vector giving the direction of the straight lines of the congruence, λ being a parameter on each line. The equations

$$u = u(t), \ v = v(t), \quad u'^2 + v'^2 \neq 0, \tag{3.10}$$

define a ruled surface belonging to the line congruence. The ruled surface is called a developable if and only if

$$det[\boldsymbol{\xi}(t), \boldsymbol{\xi}'(t), \mathbf{p}(t)] = 0. \tag{3.11}$$

This is a quadratic equation for u', v'. If it has two real and distinct roots, then the solutions of this equation define two distinct families of developable ruled surfaces. In the generic case, each family consists of the tangent lines to a surface, and these two surfaces M and M^* are called the focal surfaces of the line congruence. The line congruence gives a mapping $f : M \to M^*$ with the property that the line congruence consists of lines which are tangent to both M and M^* and joining $\mathbf{p} \in M$ to $f(\mathbf{p} \in M^*)$. This simple construction plays a fundamental role in the theory of the transformation of surfaces. The classical *Bäckland* theorem studies the transformations of surfaces of constant negative Gaussian curvature in 3-dimensional Euclidean space E^3 by realizing them as the focal surfaces of a pseudo-spherical (p.s.) line congruence. The integrability theorem says that we can construct a new surface in E^3 with constant negative Gaussian curvature from a given one.

We can rephrase this in more current terminology as follows:

Definition 1. *Let L be a line congruence in 3-dimensional Euclidean space E^3 with focal surfaces M, M^* and let $f : M \to M^*$ be the function defined above. The line congruence is called a p.s. line congruence if*
(i) the distance $\|\mathbf{pp}^*\| = r$ is a constant independent of $\mathbf{p}$,
(ii) the angle between the two normals at $\mathbf{p}$ and $\mathbf{p}^*$ is a constant independent of $\mathbf{p}$.

Theorem 1. *(Bäckland 1875): Suppose that L is a p.s. line congruence in E^3 with the focal surfaces M and M^*. Then both focal surfaces have constant negative Gaussian curvature equal to $-sin^2\theta/r^2$ (such surfaces are called p.s. surfaces).*

There is also an integrability theorem:

Theorem 2. *Suppose M is a surface in E^3 of constant negative Gaussian curvature $K = -sin^2\theta/r^2$, where $r > 0$ and $0 < \theta < \pi$ are constants. Given any unit vector $\mathbf{e} \in M_p$, which is not a principal direction, there exists a unique surface M^* and p.s. congruence $f : M \to M^*$ such that if $\mathbf{p}^* = f(\mathbf{p})$, we have $\mathbf{pp}^* = r\mathbf{e}$ and θ is the angle between the normals at $\mathbf{p}$ and $\mathbf{p}^*$.*

Thus one can construct one-parameter family of new surface of constant negative Gaussian curvature from a given one, the results by varying r.

One of the problems of the theory of line congruences is to classify the categories of them which have the property such that this representation preserves the asymptotic curves between the two focal surfaces. This leads to the following definitions for a Weingarten line congruence (W-line congruence):

Definition 2. *A W-line congruence is a line congruence which preserves the asymptotic curves between its focal surfaces.*

Equivalently, for a W-line congruence the second fundamental forms of the two surfaces are proportional.

Corollary 1. *A p.s. line congruence is a W-line congruence.*

Theorem 3. *Consider a line congruence generated by the normals along a regular non-spherical and non-developable surface M in Euclidean 3-Space E^3. If the generators of this congruence are preserving the asymptotic curves on their focal surfaces, then the Gaussian curvatures of the focal surfaces satisfying the relation:*

$$K_x \ \overline{K}_{\overline{\mathbf{x}}} = \frac{1}{(\ \overline{\rho} - \rho)^4}. \tag{3.12}$$

Hence at the corresponding points the curvature is of the same kind.

Proof. Let II_x and $\overline{II}_{\overline{\mathbf{x}}}$ be the second fundamental forms of the focal surfaces $\mathbf{x} = \mathbf{x}(u_1, u_2)$, and $\overline{\mathbf{x}} = \overline{\mathbf{x}}(u_1, u_2)$, respectively. By equations (3.7), we have:

$$\left.\begin{array}{r} II_x = -\frac{\rho_1}{\rho} ds^2 + \frac{\rho\ \overline{\rho}_1}{\overline{\rho}^2} d\overline{s}^2, \\ \overline{II}_{\overline{\mathbf{x}}} = \frac{\overline{\rho}\rho_2}{\rho^2}\ ds^2 - \frac{\overline{\rho}_2}{\overline{\rho}} d\overline{s}^2. \end{array}\right\} \tag{3.13}$$

Then the proportionality of the second fundamental forms means $II_x = \lambda\ \overline{II}_{\overline{\mathbf{x}}}$; $\lambda \in \mathbb{R}$, which is equivalent to the following condition on the invariants:

$$\rho_1 \overline{\rho}_2 - \rho_2 \overline{\rho}_1 = 0 \Rightarrow \frac{\overline{\rho}_1}{\rho_1} = \frac{\overline{\rho}_2}{\rho_2}. \tag{3.14}$$

From this relation, it follows that

$$K_x \ \overline{K}_{\overline{\mathbf{x}}} = \frac{\overline{\rho}_1 \rho_2}{\rho_1\ \overline{\rho}_2 (\ \overline{\rho} - \rho)^4} = \frac{1}{(\ \overline{\rho} - \rho)^4}, \tag{3.15}$$

as claimed. $\square$

Example 1. *As an example of a p.s. surface, pseudo-sphere can be given as a focal surface of a p.s. line congruence (so a W-line congruence):*

$$\mathbf{x}(u, v) = (sech(u) \cos(v), sech(u) sin(v), u - tanh(u))$$

Let $\left\{ \mathbf{e}_1 = \frac{\mathbf{x}_u}{\| \mathbf{x}_u \|}, \mathbf{e}_2 = \frac{\mathbf{x}_v}{\| \mathbf{x}_v \|}, \mathbf{e}_3 = \mathbf{x}_u \times \mathbf{x}_v \right\}$ *be an orthonormal frame of the surface* $\mathbf{x}$*, where superscript shows the partial derivatives. Then the W-line congruence is represented by (for simplicity for the equations we can chose)*

$$\mathbf{L}(u, v, \mu) = \mathbf{x}(u, v) + \mu \mathbf{e}_1(u, v),$$

where $\mu \in \mathbb{R}$.

The other focal surface can be given as $\overline{\mathbf{x}} = \mathbf{x} + r\mathbf{e}_1$ *where* $r = \| \overline{\mathbf{x}} - \mathbf{x} \|$. *Therefore, assuming* $r = \frac{\sqrt{2}}{2}$ *the second focal surface is (see Fig.1)*

$$\overline{\mathbf{x}} = \left(\left(\frac{2-\sqrt{2}}{2} \right) sech(u)\cos(v), \left(\frac{2-\sqrt{2}}{2} \right) sech(u) sin(v), u - \left(\frac{2-\sqrt{2}}{2} \right) tanh(u) \right).$$

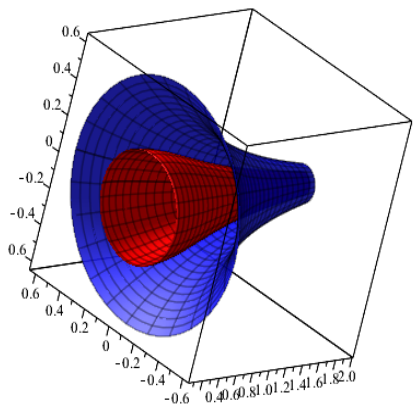

FIGURE 1. Focal surfaces of the line congruence.

3.1.1. *W-surfaces.* From equation (3.14), we see that ρ and $\overline{\rho}$ are connected by functional relation as:

$$f(\rho, \overline{\rho}) = \overline{\rho} - \rho = c, \quad c \in \mathbb{R}. \tag{3.16}$$

Surfaces with this property are called W-surfaces. From this relation it follows that

$$\frac{\overline{\rho}_1}{\rho_1} = \frac{\overline{\rho}_2}{\rho_2} = 1.$$

Therefore, by using the expressions of the Gaussian curvatures in (3.4), we have

$$K_x \ = \overline{K}_{\overline{\mathbf{x}}} = -\frac{1}{c^2}. \tag{3.17}$$

We know that the Gaussian curvature of the focal surfaces of the W-line congruence equal to:

$$\frac{-sin^2\tau}{r^2} = -\frac{1}{c^2} \Rightarrow \overline{\rho} - \rho = |\frac{r}{sin\tau}| \tag{3.18}$$

Surfaces with constant negative Gaussian curvatures are called pseudo-spherical surfaces and they are a result of the sine-Gordon partial differential equation, [8, 12]. Hence, when this functional relation is substituted into (3.7), and (3.8), we obtain:

$$\left.\begin{array}{l} g_{11}=\rho^2 e^{\frac{2\rho}{c}},\ g_{12}=0, \quad g_{22}=\overline{\rho}^2 e^{\frac{-2\overline{\rho}}{c}}, \\ h_{11}=\rho e^{-\frac{2\rho}{c}}, \quad h_{12}=0, \qquad h_{22}=\overline{\rho} e^{\frac{-2\overline{\rho}}{c}}. \end{array}\right\} \tag{3.19}$$

Combining the above analysis with the fact that the Gauss and Mainardi-Codazzi equations are the only independent algebraic equations among the fundamental invariants k, $\overline{k}$, and q, $\overline{q}$ and following Bonnet's theorem. Then we may state the following theorem:

Theorem 4. *Among the line congruence in the Euclidean space E^3, the only line congruence whose focal surfaces are pseudo-spherical surfaces, and these surfaces can be geodesically mapped upon the plane, is W-line congruence.*

Now, the second fundamental form of the director surface M is given from the equation:

$$II=\frac{1}{\rho}ds^2+\frac{1}{\overline{\rho}}d\overline{s}^2, \tag{3.20}$$

If we consider the possibility of the following corresponding $II=\lambda\ II_x;\ \lambda\in\mathbb{R}$, which is equivalent to the following condition on the invariants:

$$\rho_1\overline{\rho}_1+\overline{\rho}\rho_1=0. \tag{3.21}$$

That is

$$\frac{\partial}{\partial s}(\rho\overline{\rho})=0. \tag{3.22}$$

This means that the Gaussian curvature of the director surface M is constant along the lines of curvature $u_2-const,$. Hence, the following theorem can be given:

Theorem 5. *A necessary and sufficient condition for the Gaussian curvature of the director surface of the congruence C_N is constant along one set of lines of curvature is that the second fundamental forms of the focal surface, conjugate to this set, and the director surface are proportional.*

3.2. **Degenerate focal surfaces.** Now, we proceed to show the case for which the line congruence C_N degenerate into ruled surface. Since one of the families of lines of curvature on a surface are plane curves, they are circular: In this case, either sheet of the centro-surface may degenerate into a curve , i.e. $\mathbf{x}(u_1,c)$, or $\overline{\mathbf{x}}(\overline{c},u_2)$. In such a case the edge of regression of the developable ruled surface generated by the normals along a line of curvature becomes a single point of that curve. Then, from equations (3.1), the focal surface F is a curve if and only if

$$C:\ \mathbf{x}(u_1,c)=\mathbf{r}(u_1,u_2)+\frac{1}{k}\mathbf{e}_3(u_1,u_2),\ k(u_1,u_2)\neq 0,\ c\in\mathbb{R}, \tag{3.23}$$

or

$$C:\ \mathbf{r}(u_1,u_2)=\mathbf{x}(u_1,c)-\frac{1}{k}\mathbf{e}_3(u_1,u_2),\ k(u_1,u_2)\neq 0. \tag{3.24}$$

Because of $<\mathbf{e}_3, d\mathbf{r}>=0$, then we have

$$<\mathbf{e}_3,(\frac{\partial \mathbf{x}}{\partial s}+\frac{k_1}{k^2}\mathbf{e}_3)>=0, \tag{3.25}$$

and

$$<\mathbf{e}_3,-\frac{\overline{k}}{k}\mathbf{e}_2+\frac{k_2}{k^2}\mathbf{e}_3>=0, \tag{3.26}$$

which is equivalent to

$$-\frac{k_1}{k^2}=<\mathbf{e}_3,\frac{\partial \mathbf{x}}{\partial s}>=\left\|\frac{\partial \mathbf{x}}{\partial s}\right\|\cos\varphi,\ k_2=0, \tag{3.27}$$

where φ is the inclination of the normal to M with the tangent to the curve C. Since then

$$\begin{aligned}\frac{\partial}{\partial \overline{s}}(\left\|\frac{\partial \mathbf{x}}{\partial s}\right\|\cos\varphi) &= \frac{\partial}{\partial \overline{s}}(-\frac{k_1}{k^2}),\\ &= \frac{\partial}{\partial \overline{s}}(\frac{\partial}{\partial s}\frac{1}{k}),\\ &= \frac{\partial}{\partial s}(\frac{\partial}{\partial \overline{s}}\frac{1}{k}),\\ &= \frac{\partial}{\partial s}(-\frac{k_2}{k^2})=0,\end{aligned} \tag{3.28}$$

it follows that $\varphi=\varphi(u_1)$, i.e. is a function of u_1 only. Thus the normals to M, which meet at a point of the curve C, form a right circular cone whose semi-vertical angle φ changes as the point moves along the curve. These intersecting normals emanate from line of curvature ($u_2 = const.$) on M , which must then be circular. Thus the surface M has a system of circular lines of curvature. The sphere described with center at the point of concurrence of the normals, and passing through the feet of these normals, will touch M along one of the circular lines of curvature. Thus M is the envelope of a one-parameter family of spheres with centers on the curve C, i.e. M is a canal surface.

By similar argument, we can also have $\overline{k}_1=0$ for the focal surface $\overline{F}$ degenerate into a curve. Hence, both systems of lines of curvature of M are circular lines of curvature ($k_2=0,\ \overline{k}_1=0$), and each sheet of the focal surfaces degenerate to a curve. From the preceding arguments, it follows that each of these curves lies on a one-parameter family of circular cones whose axes are tangents to the other curve. Surfaces of this nature are called Dupin's cyclids. Then as a result:

Theorem 6. *For the line congruence C_N, a necessary and sufficient condition for the focal surfaces degenerate into curves is that the director surface is Dupin cyclide. More explicitly, we have the following:*

$$k_2=0,\ \overline{k}_1=0.$$

3.3. **Generalized middle focal surface.** The Gaussian curvature has the important property of remaining invariant if the surface is subject to an arbitrary bending. A bending is defined as any deformation for which the arc length and angles of all curves on the surface are left invariant. In equation (2.5), as $\mu=\mu(u_1,u_2)$

is a differentiable function with continuos partial derivatives of a certain order the regular surface

$$G:\mathbf{y}(u_1,u_2)=\mathbf{r}(u_1,u_2)+\mu(u_1,u_2)\mathbf{e}_3(u_1,u_2), \tag{3.29}$$

define the graph of the function $\mu=\mu(u_1,u_2)$ on the surface $M:\mathbf{r}=\mathbf{r}(u_1,u_2)$. For each fixed $t\in(\varepsilon,-\varepsilon)$, we define the generalized of the middle focal surface as:

$$\mathbf{y}(u_1,u_2)=\mathbf{r}(u_1,u_2)+t\mu\mathbf{e}_3(u_1,u_2);\ \mu=\frac{(\rho+\overline{\rho})}{2}. \tag{3.30}$$

Thus μ is signed mean distance between the two focal surfaces $\mathbf{x}=\ \mathbf{x}(u_1,u_2)$, and $\overline{\mathbf{x}}=\ \overline{\mathbf{x}}(u_1,u_2)$, and the lines of C_N generate the corresponding between the surfaces M and G.

Two surfaces that can be transformed into each other by bending are called applicable to each other. Equivalently, we will determine whether the generating lines of the congruence C_N establish an area preserving representation between M and G, i.e. it is necessary and sufficient condition for the following condition to be satisfied:

$$|A(G)-A(M)|\to\min, \tag{3.31}$$

where $A(G)$ and $A(M)$ are the element areas on the surfaces M and G. So, we have to calculate

$$A(G)=\iint\limits_U\sqrt{g_{11}^G\ g_{22}^G-(g_{12}^G)^2}\int du_1du_2, \tag{3.32}$$

where g_{11}^G, g_{22}^G, and g_{12}^G are the coefficients of the first fundamental form of the surfaces G. By making use of the equations (2.2), (2.3), and (3.26), we obtain

$$\left.\begin{array}{l} g_{11}^G=<\mathbf{y}_1,\mathbf{y}_1>=g_{11}-2t\mu h_{11}+t^2(\mu^2k^2+\mu_1^2),\\ g_{12}^G=<\mathbf{y}_1,\mathbf{y}_2>=t^2\mu_1\mu_2,\\ g_{22}^G=<\mathbf{y}_2,\mathbf{y}_2>=g_{22}-2t\mu h_{22}+t^2(\mu^2\overline{k}^2+\mu_2^2). \end{array}\right\} \tag{3.33}$$

It follows that if ε is sufficiently small subject to the relations $\varepsilon^2=\varepsilon^3=...=0$, then we obtain

$$g_{11}^Gg_{22}^G=g_{11}g_{22}(1-4t\mu H), \tag{3.34}$$

or as

$$\sqrt{g_{11}^G\ g_{22}^G}=\sqrt{g_{11}g_{22}}(1-2t\mu H), \tag{3.35}$$

where we have used

$$\sqrt{1-4t\mu H}=1-2t\mu H. \tag{3.36}$$

Substituting equation (3.31) into the element area formula (3.28), then

$$A(G)=A(M)-2t\mu\iint\limits_U Hdsd\overline{s}. \tag{3.37}$$

where H denotes to the mean curvatures of the director surface M. With $\rho+\overline{\rho}\neq 0$ on the surface M, it means there is no change of sign of the mean curvature: It exists a real number $m > 0$, with $|\rho + \overline{\rho}| > 0$ for all $\mu \in (u_1, u_2) \in U$. Therefore, the function $(\rho + \overline{\rho})$ is bounded, and the relation (3.27) is hold.

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