Available online: January 6, 2020

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 511–527 (2020)
DOI: 10.31801/cfsuasmas.527231
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

# THE RELATIVELY OSCULATING DEVELOPABLE SURFACES OF A SURFACE ALONG A DIRECTION CURVE

RASHAD A. ABDEL-BAKY AND YASIN ÜNLÜTÜRK

Abstract. We construct a developable surface tangent to a surface along a curve on the surface. We call this surface as relatively osculating developable surface. We choose the curve as the tangent normal direction curve on which the new surface is formed in the Euclidean 3-space. We obtain some results about the existence and uniqueness, and the singularities of relatively osculating developable surfaces. We also give two invariants of curves on a surface which determine these singularities. We present two results for special curves such as asymptotic line and line of curvature which are rulings of the relatively osculating surface.

## 1. Introduction

One-parameter family of straight lines forms a surface called ruled surface in Euclidean space. It has been an interesting subject that is studied from the end of the 19th century until today. Applications of ruled surfaces have been extensively performed to computer-aided geometric design (CAGD), design of surfaces, technology of manufacture, simulation and rigid bodies [10],[11],[14].

Developable surfaces as a special kind of ruled surfaces are generally characterized by Gaussian curvature, that is, if Gaussian curvature of ruled surfaces becomes vanishing, ruled surfaces can be mapped onto the plane surfaces without distortion of curves: any curve from such a surface drawn onto the flat plane remains the same. Although all developable surfaces are ruled ones, but all ruled surfaces are not developable [11], [17]. Developable surfaces as a kind of ruled surfaces are classified into cylinders, cones or tangent surfaces of space curves [1], [3], [13], [14], [18].

As well known, the inner metric of a surface determines the Gaussian curvature, therefore all the lengths and angles on the surface remain invariant under bending. This feature is what makes developable surfaces important in manufacturing.

Received by the editors: February 14, 2019; Accepted: December 13, 2019.

2010 *Mathematics Subject Classification.* Primary 53A04, 53A05; Secondary 57R45.

*Key words and phrases.* Relatively osculating developable surfaces, curves on surfaces, ruled surfaces, direction curve, singularities.

Hence both ruled surfaces and developable surfaces have been paid attention in engineering, architecture, and design, etc. [15], [16].

Based upon a curve in a surface in Euclidean 3-space, a surface has been constructed to be a developable surface tangent to the surface along the curve. This geometric object has been said to be an osculating developable surface along the curve [8]. It has been known that an osculating developable surface is a ruled surface whose rulings are directed by the osculating Darboux vector field along the curve [8].

Singularities of ruled surfaces were studied in the Euclidean 3-space $\mathbb{R}^3$ by Izumiya and Takeuchi [4]. Izumiya and Takeuchi, in their survey of ruled surfaces [5], presented original results about curves in ruled surfaces in the Euclidean 3-space. They studied curves on ruled surfaces by choosing curves as cylindirical helices and Bertrand curves [6]. In their another paper [7], the notions of helices generalized to slant helices and conical geodesic curves were defined in $\mathbb{R}^3$. Also the tangential Darboux developable of a space curve was constructed and its singularities were examined. Interesting results about a geometric invariant of space curve which is closely regarded to singularities of the tangential Darboux developable of the original curve given by Izumiya et al. [7].

The motivation of this study is based on the works of Izumiya and Otani [8], and Hananoi and Izumiya [9]. In [8], the authors constructed osculating developable surface along the curve in the surface by taking a developable surface tangent to a surface forward a curve in the surface into consideration. Then they gave some results such as the uniqueness and the singularities of such a surface.

In [9], Hananoi and Izumiya studied a developable surface which remains normal to a surface along a curve on ruled surface. They had results such as the uniqueness and the singularities of relatively osculating developable surfaces. Recently, Markina and Raffaelli examined the same topic in $\mathbb{R}^{m+1}$. Taking a smooth curve $\gamma$ in an $m-$dimensional surface $M$ in $\mathbb{R}^{m+1}$, they gave some results about the existence and uniqueness of a flat surface $H$ having the same field of normal vectors as $M$ along $\gamma$ [12].

The paper is organized as follows: the next two sections present some preliminaries, and introductory relevant notation and terminology. In Sec. 3, new developable surfaces which remain tangent to the base surface are constructed along a tangent normal direction curve and some results such as invariants of $M_o$ characterizing contour generators of $M$ are given. The existence and uniqueness of the surface have been presented for these surfaces. We give two results for special curves such as asymptotic line and line of curvature which are rulings of the relatively osculating surface. Finally, illustrative examples have been given for the base surface and its osculating developable surface.

## 2. Preliminaries

Some notions, formulas and conclusions for space curves, and ruled surfaces in Euclidean 3-space $\mathbb{R}^3$ are presented in this section, so these basic information are available in the textbooks on differential geometry (See for instance Refs. [5], [7], [14]).

Let $M$ be a regular surface in $\mathbb{R}^3$ and let $\boldsymbol{\alpha}: I \subseteq \mathbb{R} \to M$ be a unit speed curve. At each point on $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, consider the following three vectors: the unit normal vector $\mathbf{e}_3(s)$ to the surface, the unit tangent vector $\mathbf{e}_1 = \mathbf{e}_1(s)$ to the curve and the tangent normal vector $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$. The vector $\mathbf{e}_2$ is tangent to the surface $M$ , but normal to the curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$. Then we have an orthonormal frame $\{\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)\}$ along $\boldsymbol{\alpha}$, which is called the Darboux frame along $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$. Darboux equations for this frame are given by:

$$\begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix},$$

or equivaently

$$\mathbf{e}_1' = \boldsymbol{\Omega}_n \times \mathbf{e}_1, \quad \mathbf{e}_2' = \boldsymbol{\Omega}_o \times \mathbf{e}_2, \quad \mathbf{e}_1' = \boldsymbol{\Omega}_r \times \mathbf{e}_3, \tag{1}$$

where

$$\boldsymbol{\Omega}_n = -\kappa_n \mathbf{e}_2 + \kappa_g \mathbf{e}_3, \quad \boldsymbol{\Omega}_r = \tau_g \mathbf{e}_1 + \kappa_g \mathbf{e}_3, \quad \boldsymbol{\Omega}_o = \tau_g \mathbf{e}_1 - \kappa_n \mathbf{e}_2, \tag{2}$$

are said to be the normal, the rectifying, and the osculating Darboux vector fields along $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, respectively [5]. The functions $\kappa_g(s), \kappa_n(s)$, $\tau_g(s)$ are entitled as geodesic curvature, normal curvature, and geodesic torsion of $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, respectively [13]. In terms of these quantities, the geodesics, asymptotic lines, and line of curvatures on a smooth surface can be determined, as loci along which $\kappa_g(s) = 0$, $\kappa_n = 0$, and $\tau_g(s) = 0$, respectively. The definitions of the spherical images of each Darboux vector fields are as follows:

$$\left.\begin{aligned} \mathbf{e}_n(\mathbf{s}) &= \frac{\boldsymbol{\Omega}_n}{\| \boldsymbol{\Omega}_n \|} = \frac{-\kappa_n \mathbf{e}_2 + \kappa_g \mathbf{e}_3}{\sqrt{\kappa_g^2 + \kappa_n^2}}, \text{ if } (\kappa_n, \kappa_g) \neq (0,0), \\ \mathbf{e}_r(\mathbf{s}) &= \frac{\boldsymbol{\Omega}_r}{\| \boldsymbol{\Omega}_r \|} = \frac{\tau_g \mathbf{e}_1 + \kappa_g \mathbf{e}_3}{\sqrt{\tau_g^2 + k_g^2}}, \text{ if } (\tau_g, \kappa_g) \neq (0,0), \\ \mathbf{e}_o(\mathbf{s}) &= \frac{\boldsymbol{\Omega}_o}{\| \boldsymbol{\Omega}_o \|} = \frac{\tau_g \mathbf{e}_1 - \kappa_n \mathbf{e}_2}{\sqrt{\tau_g^2 + k_n^2}}, \text{ if } (\tau_g, \kappa_n) \neq (0,0). \end{aligned}\right\} \tag{3}$$

On the other hand, it is known that

$$\kappa(s) = \sqrt{\kappa_g^2 + \kappa_n^2}, \quad \text{and} \quad \tau_g(s) = \frac{\kappa_n \kappa_g' - \kappa_g \kappa_n'}{\kappa_n^2 + \kappa_g^2} + \tau(s), \tag{4}$$

where $\kappa(s)$, and $\tau(s)$ are the curvature and the torsion of $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ as a space curve, respectively [13]. From now on, we shall often not write the parameter $s$ explicitly in our formulae.

2.1. **Ruled and developable surfaces.** A ruled surface in Euclidean 3-space $\mathbb{R}^3$ is a differentiable one-parameter set of straight lines. Such a surface has a parameterization of the form

$$\mathbf{P}(s,v) = \boldsymbol{\alpha}(s) + v\mathbf{e}(s), \;\; v \in \mathbb{R}, \tag{5}$$

where $\boldsymbol{\alpha}(s)$ is the base curve and $\mathbf{e}(s)$ is the unit vector giving the direction of the straight lines of the surface. The unit normal vector of the ruled surface $\mathbf{P}(s,v)$ at each point is defined by

$$\mathbf{n}(s,v) = \frac{\mathbf{P}_s \times \mathbf{P}_v}{\|\mathbf{P}_s \times \mathbf{P}_v\|} = \frac{\boldsymbol{\alpha}' \times \mathbf{e} + v\mathbf{e}' \times \mathbf{e}}{\|\boldsymbol{\alpha}' \times \mathbf{e} + v\mathbf{e}' \times \mathbf{e}\|}. \tag{6}$$

The base curve is not unique, since any curve of the form:

$$\mathbf{C}(s) = \boldsymbol{\alpha}(s) - \eta(s)\mathbf{e}(s) \tag{7}$$

may be used as its base curve, $\eta(s)$ is a smooth function. If there is a common perpendicular vector to two neighboring rulings on $\mathbf{P}(s,v)$, then the foot of the common perpendicular on the main ruling is said to be a central point. The locus of the central points is said to be the striction curve. In Eq. (7) if

$$\eta(s) = \frac{\langle \boldsymbol{\alpha}'(s), \mathbf{e}' \rangle}{\|\mathbf{e}'\|^2}, \tag{8}$$

then $\mathbf{C}(s)$ is named as the striction curve on the ruled surface and it is unique. In the case $\eta = 0$, the base curve is the striction curve. The distribution parameter of $\mathbf{P}(s,v)$ is defined by

$$\lambda(s) = \frac{\det(\boldsymbol{\alpha}', \mathbf{e}, \mathbf{e}')}{\|\mathbf{e}'\|^2}. \tag{9}$$

The parameter of distribution is a real integral invariant of a ruled surface and allows further classification of the ruled surface.

Developable surfaces are briefly introduced as special types of ruled surfaces. If the ruled surface $\mathbf{P}(s,v)$ is a developable one, then we have

$$\lambda(s) = 0 \Leftrightarrow \det(\boldsymbol{\alpha}', \mathbf{e}, \mathbf{e}') = 0. \tag{10}$$

Thus a volume formed by $\boldsymbol{\alpha}', \mathbf{e}$ and $\mathbf{e}'$ is vanishing, i.e, they are linearly dependent. This condition is satisfied provided that there are three non-identically vanishing functions $\eta(s)$, $\xi(s)$ and $\gamma(s)$ satisfying

$$\mu(s)\boldsymbol{\alpha}' + \beta(s)\mathbf{e} + \gamma(s)\mathbf{e}' = \mathbf{0}. \tag{11}$$

We has to analyze the following cases:
Case 1: $\mu = 0$

Since $\langle \mathbf{e}, \mathbf{e}' \rangle = 0$, it follows immediately that Eq. (11) is only satisfied when $\mathbf{e}$ is a constant vector, i.e., $\mathbf{P}(s,v)$ is a part of a cylinder.
Case 2: $\mu \neq 0$ from Eq. (11) it follows:

$$\boldsymbol{\alpha}' = \zeta(s)\mathbf{e} + \upsilon(s)\mathbf{e}', \tag{12}$$

where

$$\zeta(s) = -\frac{\beta}{\mu},\ \upsilon(s) = -\frac{\gamma}{\mu}.$$

Differentiating Eq. (7) and using Eq. (12), we get

$$\mathbf{C}'(s) = (\zeta(s) - \eta'(s))\,\mathbf{e}(s) + (\upsilon(s) - \eta(s))\,\mathbf{e}'. \tag{13}$$

The situation for $\mathbf{C}$ to be striction curve becomes equivalent to that the vectors $\mathbf{C}'$ and $\mathbf{e}'$ are perpendicular to each other. Therefore, we conclude that the ruling becomes parallel to the first differentiation of the striction curve which is also the tangent of the striction curve, i.e.

$$\mathbf{C}' = (\zeta(s) - \eta'(s))\mathbf{e}(s). \tag{14}$$

Thus we have to consider the following sub-case: $\zeta(s) = \eta'(s)$. In this case Eq. (14) yields to that $\mathbf{C} = \mathbf{C}_0$ is a constant vector. So, $\mathbf{P}(s,v)$ becomes a part of a cone as follows:

$$\mathbf{P}(s,v) = \mathbf{C}_0 + (\eta(s) + v)\mathbf{e}(s),\ v \in \mathbb{R}. \tag{15}$$

We now define the concept "contour generators". Let $M$ be an orientable surface and $\mathbf{n}$ a unit normal vector field on $M$. For a unit vector $\mathbf{x}$ in the unit sphere $\mathbb{S}^2 = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| = 1\right\}$, the normal contour generator of the orthogonal projection with the direction $\mathbf{x}$ is defined to be

$$\{\mathbf{p} \in M \mid < \mathbf{n}, \mathbf{x} >= 0\}. \tag{16}$$

Moreover, for a fixed point $\mathbf{c} \in \mathbb{R}^3$, the normal contour generator of the central projection with the center $\mathbf{c}$ is defined to be

$$\{\mathbf{p} \in M \mid < \mathbf{n}, \mathbf{p} - \mathbf{c} >= 0\}. \tag{17}$$

## 3. The relatively osculating developable surfaces

In this section, we present a relatively osculating developable surface along the $\mathbf{e}_2(s)$-direction curve

$$\boldsymbol{\beta}(s) = \int_0^s \mathbf{e}_2(s)ds$$

as follows:

$$M_o : \widetilde{\mathbf{P}}(s,v) = \boldsymbol{\beta}(s) + v\mathbf{e}_o(s), \tag{18}$$

where $v \in \mathbb{R}$, and

$$\mathbf{e}_o(s) = \frac{\tau_g \mathbf{e}_1 - \kappa_n \mathbf{e}_2}{\sqrt{\tau_g^2 + k_n^2}}$$

under the assumption $(\tau_g,\kappa_n) \neq (0,0)$. Firstly differentiating $\mathbf{e}_o$, we find

$$\mathbf{e}_o' = \left(\kappa_g + \frac{\kappa_n \tau_g' - \tau_g \kappa_n'}{\tau_g^2 + \kappa_n^2}\right)\left(\frac{\kappa_n \mathbf{e}_1 + \tau_g \mathbf{e}_2}{\sqrt{\tau_g^2 + \kappa_g^2}}\right), \tag{19}$$

and thus $\lambda(s) = 0$. This results that $M_o$ is a developable surface. Furthermore, we propose two invariants $\delta_o(s)$, and $\sigma_o(s)$ of $M_o$ as follows:

$$\delta_o = \kappa_g + \frac{\kappa_n \tau_g' - \tau_g \kappa_n'}{\tau_g^2 + \kappa_n^2}, \text{ and } \quad \sigma_o = -\left[\frac{\kappa_n}{\sqrt{\tau_g^2 + \kappa_n^2}} + \left(\frac{\tau_g}{\delta_o \sqrt{\tau_g^2 + \kappa_g^2}}\right)'\right], \tag{20}$$

where $\delta_o \neq 0$. We can also calculate that

$$\widetilde{\mathbf{P}}_s \times \widetilde{\mathbf{P}}_v = -\left(v\delta_o + \frac{\tau_g}{\sqrt{\tau_g^2 + \kappa_n^2}}\right)\mathbf{e}_3. \tag{21}$$

Hence, the normal vector of $M_o$ is in the same direction to the normal vector of $M$. This is the reason why we name $M_o$ the relatively osculating developable surface of $M$ along $\boldsymbol{\beta}(s)$.

On the other hand, the invariants $\delta_o(s)$, and $\sigma_o(s)$ of $M_o$ describe contour generators of $M$ as follows:

**Theorem 1**. Let $M_o$ be the relatively osculating developable surface of $M$ expressed by Eq. (18). Then we have the following:
(A) The following are equivalent:
(1) $M_o$ is a cylinder,
(2) $\delta_o(s) = 0$,
(3) $\boldsymbol{\beta} = \boldsymbol{\beta}(s)$ is a contour generator with respect to an orthogonal projection.
(B) If $\delta_o(s) \neq 0$, then the following are equivalent:
(1) $M_o$ is a cone,
(2) $\sigma_o(s) = 0$,
(3) $\boldsymbol{\beta} = \boldsymbol{\beta}(s)$ is a contour generator with respect to a central projection.

**Proof**. (A) From Eq. (18), it is obvious that $M_o$ is a cylinder if and only if $\mathbf{e}_o(s)$ is constant, i.e. $\delta_o(s) = 0$. Therefore, the condition (1) becomes equivalent to the situation (2). Suppose that the condition (3) holds. Then there exists a fixed vector $\mathbf{x} \in \mathbb{S}^2$ such that $\langle \mathbf{e}_3, \mathbf{x} \rangle = 0$. So there are $a$, $b \in \mathbb{R}$ such that $\mathbf{x} = a\mathbf{e}_1 + b\mathbf{e}_2$. Since $\langle \mathbf{e}_3', \mathbf{x} \rangle = 0$, we have $-a\kappa_n - b\tau_g = 0$, so that we have $\mathbf{x} = \pm\mathbf{e}_o(s)$. Namely, the situation (1) holds. Suppose that $\mathbf{e}_o(s)$ is constant. Then we choose $\mathbf{x} = \mathbf{e}_o(s) \in \mathbb{S}^2$. By the definition of $\mathbf{e}_o(s)$, we have $\langle \mathbf{x}, \mathbf{e}_3 \rangle = 0$. Hence the condition (1) entails the situation (3).

(B) The situation (1) determines that the singular value set of $M_n$ is a constant vector. Thus, in view of Eqs. (8), (9), and from Eq. (19), we have

$$\mathbf{C}'(s) = -\left[\frac{\kappa_n}{\sqrt{\tau_g^2+\kappa_n^2}} + \left(\frac{\tau_g}{\delta_o\sqrt{\tau_g^2+\kappa_g^2}}\right)'\right]\mathbf{e}_o(s) = -\sigma_o(s)\mathbf{e}_o(s).$$

Then $M_o$ is a cone if and only if $\sigma_o(s) = 0$. It follows that the situations (1) and (2) are equivalent. By the definition of the the central projection means that there is a fixed point $\mathbf{c} \in \mathbb{R}^3$ such that $\langle \mathbf{e}_3, \boldsymbol{\beta} - \mathbf{c}\rangle = 0$. If the condition (1) holds, then $\mathbf{C}(s)$ is constant. For the constant point $\mathbf{c} = \mathbf{C}(s)$, we have

$$\langle \mathbf{e}_3, \boldsymbol{\beta} - \mathbf{c}\rangle = \langle \mathbf{e}_3, \boldsymbol{\beta} - \mathbf{C}\rangle = \left\langle \mathbf{e}_3, \frac{\langle \boldsymbol{\beta}', \mathbf{e}_o'\rangle}{\|\mathbf{e}_o'\|^2}\ \mathbf{e}_o \right\rangle = \langle \mathbf{e}_3, \mathbf{e}_o\rangle = 0. \tag{22}$$

This implies that (3) holds. On the contrary, by (3), there is a fixed point $\mathbf{c} \in \mathbb{R}^3$ such that $\langle \mathbf{e}_3\ , \boldsymbol{\beta} - \mathbf{c}\rangle = 0$. Differentiating both side of Eq. (22), we have

$$0 = \langle \mathbf{e}_3, \boldsymbol{\beta} - \mathbf{c}\rangle' = \langle -\kappa_n\mathbf{e}_1 - \tau_g\mathbf{e}_2\ , \boldsymbol{\beta} - \mathbf{c}\rangle, \tag{23}$$

so we may write $\boldsymbol{\beta} - \mathbf{c} = f(s)\mathbf{e}_o(s)$, where $f(s)$ is a differentiable function. Differentiating Eq. (23) again, we have:

$$0 = \langle \mathbf{e}_3, \boldsymbol{\beta} - \mathbf{c}\rangle'' = \langle -\kappa_n\mathbf{e}_1 - \tau_g\mathbf{e}_2, \mathbf{e}_2\rangle + \left\langle -\left(\kappa_n\mathbf{e}_1 + \tau_g\mathbf{e}_2\right)', \boldsymbol{\beta} - \mathbf{c}\right\rangle,$$

or equivaently,

$$0 = \langle \mathbf{e}_3, \boldsymbol{\beta} - \mathbf{c}\rangle'' = -\tau_g + f\delta_o\sqrt{\tau_g^2+\kappa_n^2}.$$

It follows that

$$\mathbf{c} = \boldsymbol{\beta}(\mathbf{s}) - \frac{\tau_g}{\delta_o\sqrt{\tau_g^2+\kappa_n^2}}\mathbf{e}_o(s) = \boldsymbol{\beta} - \frac{\langle\ \boldsymbol{\beta}', \mathbf{e}_o'\rangle}{\|\ \mathbf{e}_o'\|^2}\mathbf{e}_o(s) = \mathbf{C}(s).$$

Therefore, $\mathbf{C}(s)$ is constant, so that (1) holds ⊡.

**Theorem 2** (Existence and uniqueness). Let $M \subset \mathbb{R}^3$ be a regular surface and $\boldsymbol{\beta} : I \to M \subset \mathbb{R}^3$ be a unit-speed curve given by $\boldsymbol{\beta} = \int \mathbf{e}_2(s)ds$ with $\kappa_n^2 + \tau_g^2 \neq 0$. Then there exists uniquely a relatively osculating developable surface represented by Eq. (18).

**Proof**. For the existence, we have the relatively osculating developable surface along $\boldsymbol{\beta} = \boldsymbol{\beta}(s)$ represented by Eq. (18). On the other hand, since $M_o$ is a ruled surface, we suppose that

$$M_o : \widetilde{\mathbf{P}}(s, v) = \boldsymbol{\beta}(s) + v\boldsymbol{\zeta}(s), \tag{24}$$

where $v \in \mathbb{R}$, with $(\tau_g,\kappa_n) \neq (0, 0)$, and

$$\boldsymbol{\zeta}(s) = \zeta_1(s)\mathbf{e}_1 + \zeta_2(s)\mathbf{e}_2 + \zeta_3(s)\mathbf{e}_3,\ \boldsymbol{\zeta}'(s) \neq \mathbf{0}.$$

It can be immediately seen that $M_o$ is developable if and only if

$$\det(\boldsymbol{\beta}^{'},\boldsymbol{\zeta},\boldsymbol{\zeta}^{'})=0\Leftrightarrow\zeta_3\zeta_1^{'}-\zeta_1\zeta_3^{'}-\zeta_2\left(\zeta_1\tau_g+\zeta_3\kappa_g\right)+\kappa_n\left(\zeta_3^2+\zeta_1^2\right)=0. \tag{25}$$

Conversely, since $M_o$ is a relatively osculating developable surface along $\boldsymbol{\beta}=\boldsymbol{\beta}(s)$, we have

$$\left(\widetilde{\mathbf{P}}_s\times\widetilde{\mathbf{P}}_v\right)(s,v)=\psi\left(s,v\right)\mathbf{e}_3. \tag{26}$$

Also, the normal vector $\widetilde{\mathbf{P}}_s\times\widetilde{\mathbf{P}}_v$ at the point $(s,0)$ is

$$\left(\widetilde{\mathbf{P}}_s\times\widetilde{\mathbf{P}}_v\right)(s,0)=\zeta_3\mathbf{e}_1-\zeta_1\mathbf{e}_3. \tag{27}$$

By means of Eqs. (26) and (27) we find:

$$\zeta_3=0,\text{ and }\zeta_1=-\psi\left(s,0\right), \tag{28}$$

which follows from Eq. (25) that

$$-\zeta_1\left(\zeta_1\kappa_n+\zeta_2\tau_g\right)=0. \tag{29}$$

If $(s,0)$ is a regular point (i.e., $\psi\left(s,0\right)\neq0$), then $\zeta_1(s)\neq0$. Thus, we have

$$\zeta_2=-\frac{\kappa_n}{\tau_g}\zeta_1,\text{ with }\tau_g\neq0. \tag{30}$$

Therefore, we obtain

$$\boldsymbol{\zeta}(s)=\zeta_1\mathbf{e}_1-\frac{\kappa_n}{\tau_g}\zeta_1\mathbf{e}_2=\frac{\zeta_1}{\cos\varphi}\mathbf{e}_o(s), \tag{31}$$

where $(\tau_g,\kappa_n)\neq(0,0)$, and $\varphi\neq\dfrac{\pi}{2}$. It follows that $\boldsymbol{\zeta}(s)$ becomes equal to the direction of $\mathbf{e}_o(s)$. If $\tau_g\neq0$ (i.e., $\varphi\neq\dfrac{\pi}{2}$), we have the same result as the above case.

On the other hand, suppose that $M_o$ has a singular point at $(s_0,0)$. Then $\psi\left(s_0,0\right)=\zeta_1(s_0)=\zeta_3(s_0)=0$, and we have $\boldsymbol{\zeta}(s_0)=\zeta_2(s_0)\mathbf{e}_2(s_0)$. If the singular point $\boldsymbol{\beta}(s_0)$ is in the closure of the set of points where the relatively osculating developable surface along $\boldsymbol{\beta}(s)$ is regular, then there is a point $\boldsymbol{\beta}(s)$ in any neighborhood of $\boldsymbol{\beta}(s_0)$ such that the uniqueness of the relatively osculating developable surface is satisfied at $\boldsymbol{\beta}(s)$. Passing to the limit $s\to s_0$, uniqueness of the relatively osculating surface holds at $s_0$. Assume that there is an open interval $J\subseteq I$ such that $M_o$ is singular at $\boldsymbol{\beta}(s)$ for any $s\in J$. Then $\mathbf{P}(s,v)=\boldsymbol{\beta}(s)+v\zeta_2(s)\ \mathbf{e}_2(s)$ for any $s\in J$. This means that $\zeta_1(s)=\zeta_3(s)=0$ for $s\in J$. It follows that

$$\left(\widetilde{\mathbf{P}}_s\times\widetilde{\mathbf{P}}_v\right)(s,v)=-v\zeta_2^2\left(\tau_g\mathbf{e}_1+\kappa_g\mathbf{e}_3\right)\ . \tag{32}$$

Thus the above vector is directed to $\mathbf{e}_3$, i.e. $\widetilde{\mathbf{P}}_s\times\widetilde{\mathbf{P}}_v\parallel\mathbf{e}_3(s)$ if and only if $\kappa_g\neq0$ and $\tau_g=0$ for any $s\in J$. In this case, $\mathbf{e}_0(s)=\pm\mathbf{e}_3$. This determines that uniqueness holds $\boxdot$.

**Proposition 1**. Let $M_o$ be the relatively osculating developable surface expressed by Eq. (18) with $(\tau_g,\kappa_n) \neq (0,0)$. If there are two osculating developable surfaces along $\boldsymbol{\beta}(s)$, then $\boldsymbol{\beta}(s)$ is a straight line.

**Proof**. Assume that $(\tau_g,\kappa_n) \neq (0,0)$, the relatively osculating developable surface along the direction curve $\boldsymbol{\beta}(s)$ is unique by Theorem 2. If $\kappa_n = \tau_g = 0$, then $\boldsymbol{\beta}(s)$ is a plane curve. In this case, a plane $\Pi$ at $\boldsymbol{\beta}(s_0)$ is a relatively osculating developable surface along $\boldsymbol{\beta}(s)$. If there is another relatively osculating developable surface $M_o$ along $\boldsymbol{\beta}(s)$, then $M_o$ is tangent to $\Pi$ along $\boldsymbol{\beta}(s)$. By definition, $\Pi$ is tangent to $M_o$ along a ruling of $M_o$, which is $\boldsymbol{\beta}(s)$. Thus $\boldsymbol{\beta}(s)$ is a line. If $\kappa_n = \tau_g = 0$ at an isolated point $s_0 \in I$ except at $s_0$, then there is a point $s \in I$ in any neighborhood of $s_0$ such that the uniqueness of the relatively osculating developable surface is satisfied at $s \in I$. Passing to the limit $s \to s_0$, uniqueness of the relatively osculating developable surface is satisfied at $s_0 \in I$ $\boxdot$.

**Proposition 2**. Let $M_o$ be the relatively osculating developable surface expressed by Eq. (18) with $(\tau_g,\kappa_n) \neq (0,0)$. Then $\kappa_n = \tau_g = 0$ if and only if $\boldsymbol{\beta}(s)$ is a ruling of $M_o$.

**Proof**. In general, the torsion of the curve $\boldsymbol{\beta}(s)$ as a space curve is given by

$$\tau_{\boldsymbol{\beta}}(s) := \frac{\det(\boldsymbol{\beta}',\ \boldsymbol{\beta}'', \boldsymbol{\beta}''')}{\left\|\boldsymbol{\beta}' \times \boldsymbol{\beta}''\right\|^2} = -\kappa_n + \frac{\kappa_g \tau_g' - \tau_g \kappa_g'}{\tau_g^2 + \kappa_g^2} \ . \tag{33}$$

Assuming that $\kappa_n = \tau_g = 0$, the torsion $\tau_{\boldsymbol{\beta}}$ becomes constantly equal to 0. Thus, $\boldsymbol{\beta}(s)$ becomes a plane curve. Moreover, we have $\mathbf{e}_3' = -\kappa_n\, \mathbf{e}_1 - \tau_g \mathbf{e}_2 = 0$. The assumption that $M_o$ is an osculating developable surface implies that $M_o$ is a plane generated by $\boldsymbol{\beta}(s)$. Thus $\boldsymbol{\beta}(s)$ is a line. For the converse, we assume that $\boldsymbol{\beta}(s) = \int_0^s \mathbf{e}_2(s)ds$ is a ruling of the osculating developable $M_o$. Since $\boldsymbol{\beta}(s)$ is a ruling in $\mathbb{R}^3$; $\mathbf{e}_2$ is a constant vector. The supposition that $M_o$ is a developable surface determines that $\mathbf{e}_3' = \mathbf{0}$. Thus, by the Darboux equations we have $\kappa_n = \tau_g = 0$ $\boxdot$.

Therefore, we can give the following corollaries:

**Corollary 1**. The relatively osculating developable surface $M_o$ represented by Eq. (18) is a non-cylindrical if and only if $\delta_o(s) \neq 0$.

**Proof**. It is a straighforward result from the definition of non-cylindirical ruled surface.

**Corollary 2**. The relatively osculating developable surface $M_o$ represented by Eq. (18) is a tangential developable if and only if $\delta_o(s) \neq 0$, and $\sigma_o(s) \neq 0$.

**Proof**. According to the proof of Theorem 1, when $\delta_o(s) \neq 0$, and $\sigma_o(s) \neq 0$, we have $\mathbf{e}_o' \neq \mathbf{0}$, and $\mathbf{C}' \neq \mathbf{0}$. Since $\det(\boldsymbol{\beta}', \mathbf{e}_o, \mathbf{e}_o') = 0$, $< \mathbf{C}', \mathbf{e}_o' >=0$ and $< \mathbf{e}_o, \mathbf{e}_o' >= 0$, we find $\mathbf{C}' \| \mathbf{e}_o$. This determines that the surface $M_o$ is a tangent surface $\boxdot$.

3.1. **Special curves on a surface.** Based on the Theorem 3.3 of Ref. [8], we divide the singularities of relatively osculating developable surfaces $M_o$ forward special curves by using the two invariants $\delta_o$, and $\sigma_o$ in the following:

(A). If $\kappa_n = 0$, then $\alpha$ is an asymptotic line on $M$, and

$$M_o : \widetilde{\mathbf{P}}(s,v) = \boldsymbol{\beta}(s) + v\mathbf{e}_1(s), v \in \mathbb{R}. \tag{34}$$

In this case, we obtain the invariants as follows:

$$\delta_o = \kappa_g, \quad \text{and} \quad \sigma_o = -\left(\frac{\tau_g}{\delta_o\sqrt{\tau_g^2 + \kappa_g^2}}\right)'.$$

**Corollary 3**. Let $M_o$ be the relatively osculating developable surface expressed by Eq. (34). Then we have the following:
(1) $M_o$ is non-singular at points $\widetilde{\mathbf{P}}(s_0, v_0)$ if and only if $v_0 \neq 0$.
(2) $M_o$ is locally diffeomorphic to Cuspidal edge $CE$ at points $\widetilde{\mathbf{P}}(s_0, v_0)$ if and only if $v_0 = -\kappa_g^{-1}(s_0) \neq 0$, and $\kappa_g'(s_0) \neq 0$.
(3) $M_o$ is locally diffeomorphic to Swallowtail $SW$ at points $\widetilde{\mathbf{P}}(s_0, v_0)$ if and only if $v = -\kappa_g^{-1}(s_0) \neq 0$, $\kappa_g'(s_0) = 0$, and $\left(\kappa_g^{-1}\right)''(s_0) \neq 0$.
Here,

$$\left.\begin{array}{l} CE = \left\{(x_1, x_2, x_3) | x_1 = u,\ x_2 = v^2\ ,\ x_3 = v^3\right\},\ \text{(see Fig. 1)}. \\ SW = \left\{(x_1, x_2, x_3) | x_1{=}u,\ x_2{=}3v^2{+}uv^2,\ x_3{=}4v^3{+}2uv\right\},\ \text{(see Fig. 2)} \end{array}\right\}$$

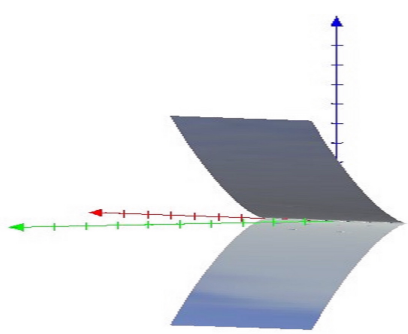

FIGURE 1. Cuspidal edge.

**Proof**. Singularities of the relatively osculating developable surface expressed by Eq. (34) are

$$\widetilde{\mathbf{P}}_s \times \widetilde{\mathbf{P}}_v = -\left(v\kappa_g + 1\right)\mathbf{e}_3. \tag{35}$$

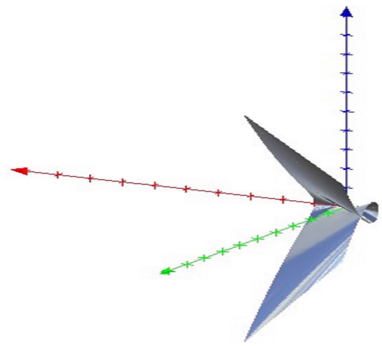

Figure 2. Swallowtail.

Therefore, $\widetilde{\mathbf{P}}(s_0, v_0)$ is non-singular if and only if $\widetilde{\mathbf{P}}_s \times \widetilde{\mathbf{P}}_v \neq \mathbf{0}$. This condition is equivalent to $v_0 = -\kappa_g^{-1}(s_0)$. This completes the proof of assertion (1). If there is a parameter $s_0$ such that

$$v_0 = -\kappa_g^{-1}(s_0), \quad \text{and} \quad v_0' = \frac{\kappa_g'(s_0)}{\kappa_g^2(s_0)} \neq 0 \ \left(\text{ i.e. } \kappa_g' \neq 0\right),$$

then $M_o$ is locally diffeomorphic to $CE$ at $\widetilde{\mathbf{P}}(s_0, v_0)$. This completes the proof of assertion (2). We also have, if there is a parameter $s_0$ such that

$$v_0 = -\kappa_g^{-1}(s_0), \quad v_0' = \frac{\kappa_g'(s_0)}{\kappa_g^2(s_0)} = 0, \quad \text{and} \quad \left(\kappa_g^{-1}\right)''(s_0) \neq 0,$$

then $M_o$ is locally diffeomorphic to $SW$ at points $\widetilde{\mathbf{P}}(s_0, v_0)$. This concludes the proof of affirmation (3).

(B). If $\tau_g = 0$, then $\boldsymbol{\alpha}$ becomes a line of curvature on $M$, and

$$M_o : \widetilde{\mathbf{P}}(s, v) = \boldsymbol{\beta}(s) - v\mathbf{e}_2(s), v \in \mathbb{R}, \tag{36}$$

which is recognized as the tangent surface of $\boldsymbol{\beta}(s)$. In this case we obtain the invariants as $\delta_o = \kappa_g$, and $\sigma_o = -1$.

**Corollary 4**. Let $M_o$ be the relatively osculating developable surface expressed by Eq. (36). Then we have the following:

(1) $M_o$ is non-singular at points $\mathbf{P}(s_0, v_0)$ if and only if $v_0 \neq 0$.

(2) $M_o$ is locally diffeomorphic to $CE$ at points $\widetilde{\mathbf{P}}(s_0, v_0)$ if and only if $v_0 = -\kappa_g^{-1}(s_0) \neq 0$, and $\kappa_g'(s_0) \neq 0$.

(3) $M_o$ is locally diffeomorphic to $SW$ at points $\widetilde{\mathbf{P}}(s_0, v_0)$ if and only if $v = -\kappa_g^{-1}(s_0) \neq 0$, $\kappa_g'(s_0) = 0$, and $\left(\kappa_g^{-1}\right)''(s_0) \neq 0$.

**Proof**. Singularities of the relatively osculating developable surface expressed by Eq. (36) are

$$\widetilde{\mathbf{P}}_s \times \widetilde{\mathbf{P}}_v = -v\kappa_g \mathbf{e}_3. \tag{37}$$

Therefore, $\widetilde{\mathbf{P}}(s_0, v_0)$ is non-singular if and only if $\widetilde{\mathbf{P}}_s \times \widetilde{\mathbf{P}}_v \neq \mathbf{0}$. This condition is equivalent to $v_0 = -c\kappa_g^{-1}(s_0)$, $c \neq 0$. This completes the proof of assertion (1). If there is a parameter $s_0$ such that

$$v_0 = -c\kappa_n^{-1}(s_0),\ c \neq 0, \quad \text{and} \quad v_0' = \frac{c\kappa_n'(s_0)}{\kappa_n^2(s_0)} \neq 0,\ (\text{i.e. } \kappa_n' \neq 0)\,,$$

then $M_o$ is locally diffeomorphic to $CE$ at $\widetilde{\mathbf{P}}(s_0, v_0)$. This finishes the proof of affirmation (2). Again, if there exists a parameter $s_0$ such that

$$v_0 = -c\kappa_g^{-1}(s_0),\ c \neq 0, \quad v_0' = \frac{c\kappa_g'(s_0)}{\kappa_g^2(s_0)} = 0, \quad \text{and} \quad \left(\kappa_g^{-1}\right)''(s_0) \neq 0,$$

then $M_o$ is locally diffeomorphic to $SW$ at pointgs $\widetilde{\mathbf{P}}(s_0, v_0)$. This finishes the proof of affirmation (3) ⊡.

3.1.1. *Curves on the unit sphere.* We now deal with the case when $M$ is the unit sphere $\mathbb{S}^2 = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\|^2 = 1\right\}$. Let $\boldsymbol{\alpha} : I \subseteq \mathbb{R} \to \mathbb{S}^2$ be a unit speed curve. In this case, we have $\mathbf{t}(s) = \boldsymbol{\alpha}'$, $\mathbf{g}(s) = \boldsymbol{\alpha} \times \mathbf{t}$, and since $s$ is a natural parameter of $\boldsymbol{\alpha}(s)$, it follows that $\|\mathbf{t}\| = 1$, and the frame $\{\boldsymbol{\alpha} = \boldsymbol{\alpha}(s), \mathbf{t}(s), \mathbf{g}(s)\}$ forms a moving orthonormal frame fitted to each point of the spherical curve $\boldsymbol{\alpha}(s)$. This frame is said to be the Darboux frame relative to $\mathbf{x}(s)$. By construction, the Darboux formula is

$$\begin{bmatrix} \boldsymbol{\alpha}' \\ \mathbf{t}' \\ \mathbf{g}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \gamma \\ 0 & -\gamma & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \mathbf{t} \\ \mathbf{g} \end{bmatrix}, \tag{38}$$

where $\gamma = \gamma(s)$ is the geodesic curvature of $\boldsymbol{\alpha}(s)$. It follows that $\delta_o = \gamma$, $\sigma_o = \pm 1$, $\mathbf{e}_0 = \pm\mathbf{g}(s)$, and $\boldsymbol{\beta}(s) = \int_0^s \mathbf{g}(s)ds$. Thus, we have:

$$M_o : \widetilde{\mathbf{P}}(s, v) = \boldsymbol{\beta}(s) + v\mathbf{g}(s),\ v \in \mathbb{R}, \tag{39}$$

which is recognized as the tangent developable surface of $\boldsymbol{\beta}(s)$. Then we have the following lemma as a result of Corollary 4.

**Lemma 1**. Let $M_o$ be the tangent developable expressed by Eq. (39). Then we have the following:
(1) $M_o$ is non-singular at points $\beta(s_0)$ if and only if $v_0 \neq 0$.
(2) $M_o$ is locally diffeomorphic to $CE$ at points $\mathbf{P}(s_0, v_0)$ if and only if $v_0 = -\gamma^{-1}(s_0) \neq 0$, and $\gamma'(s_0) \neq 0$.
(3) $M_o$ is locally diffeomorphic to $SW$ at points $\mathbf{P}(s_0, v_0)$ if and only if $v = -\gamma^{-1}(s_0) \neq 0$, $\gamma'(s_0) = 0$, and $\left(\gamma^{-1}\right)''(s_0) \neq 0$.

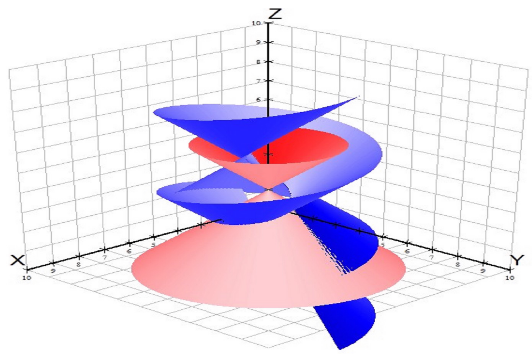


FIGURE 3.

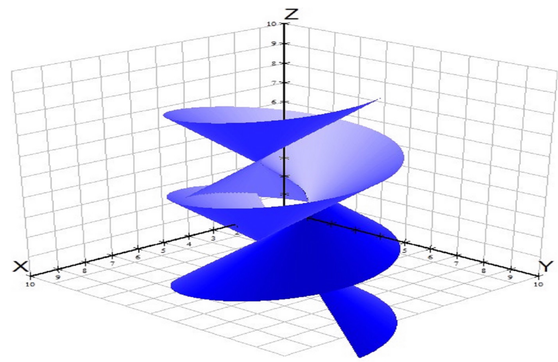


FIGURE 4.

4.

**Proposition 2**. THE RELATIVELY OSCULATING DEVELOPABLE SURFACE $M_o$ REPRESENTED BY EQ. (39) IS A CYLINDRICAL IF $\boldsymbol{\alpha}(s)$ IS A GREAT CIRCLE.

**Proof**. ASSUME THAT $\boldsymbol{\alpha}(s)$ BECOMES A GREAT CIRCLE. THEN $\gamma(s) = 0$, AND $\mathbf{g}(s)$ IS CONSTANT. THEREFORE, $M_o$ IS A CIRCULAR CYLINDER.

4.1. **Examples.** We close this section with some examples:

**Example 1.** Let the base surface $M$ be given as the following parameterization:

$$\mathbf{P}(s,v) = (\cos s - \frac{1}{\sqrt{2}}v\cos s, \sin s - \frac{1}{\sqrt{2}}v\sin s, \frac{v}{\sqrt{2}}). \tag{40}$$

The directrix curve $\boldsymbol{\beta}$ of the relatively osculating developable surface is $\boldsymbol{\beta} = (-\frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\cos s, \frac{s}{\sqrt{2}})$. The normal curvature and geodesic torsion of the base

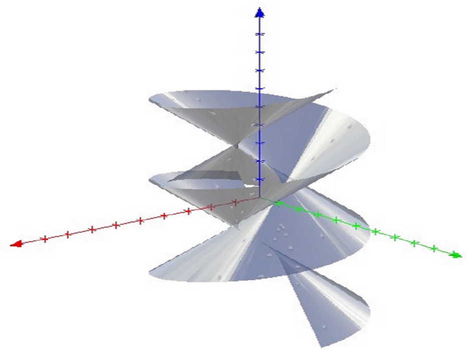

FIGURE 5.

curve are, respectively, computed as $\kappa_n = -\frac{1}{\sqrt{2}}$, and $\tau_g = 0$. Then the ruling line $\mathbf{e}_o$ of the relatively osculating developable surface is obtained as $\mathbf{e}_o = (\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\sin s, -\frac{1}{\sqrt{2}})$. As a result, the relatively osculating developable surface $M_o$ is given with the parameterization:

$$\widetilde{\mathbf{P}}(s,v)=(-\frac{1}{\sqrt{2}}\sin s+\frac{1}{\sqrt{2}}v\cos s, \frac{1}{\sqrt{2}}\cos s+\frac{1}{\sqrt{2}}v\sin s, \frac{s}{\sqrt{2}}-\frac{v}{\sqrt{2}}). \tag{41}$$

The base surface given by (40) and the relatively osculating developable surface given by (41) have been together plotted in Fig. 3. The relatively osculating developable surface given by (41) has been alone illustrated in Fig 4. The relatively osculating developable surface has been illustrated by reflecting surface in Fig. 5.

**Example 2.** Given the base surface $M$ as follows:

$$\mathbf{P}(s,v)=(\cos\frac{s}{\sqrt{2}}-\frac{1}{\sqrt{2}}v\sin\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}+\frac{1}{\sqrt{2}}v\cos\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}+\frac{v}{\sqrt{2}}). \tag{42}$$

The directrix curve $\boldsymbol{\beta}$ of the relatively osculating developable surface is $\boldsymbol{\beta} = (\sqrt{2}\sin\frac{s}{\sqrt{2}}, -\sqrt{2}\cos\frac{s}{\sqrt{2}}, 0)$. The normal curvature and geodesic torsion of the base curve are, respectively, computed as $\kappa_n = 0$, and $\tau_g = \frac{1}{2}$. Then the ruling line $\mathbf{e}_o$ of the relatively osculating developable surface is obtained as $\mathbf{e}_o = (-\frac{1}{2}\sin\frac{s}{\sqrt{2}}, \frac{1}{2}\cos\frac{s}{\sqrt{2}}, \frac{1}{2})$. As a result, the relatively osculating developable surface $M_o$ is given with the below parameterization:

$$\widetilde{\mathbf{P}}(s,v) = (\sqrt{2}\sin\frac{s}{\sqrt{2}} - \frac{v}{2}\sin\frac{s}{\sqrt{2}}, -\sqrt{2}\cos\frac{s}{\sqrt{2}} + \frac{v}{2}\cos\frac{s}{\sqrt{2}}, \frac{v}{2}). \tag{43}$$

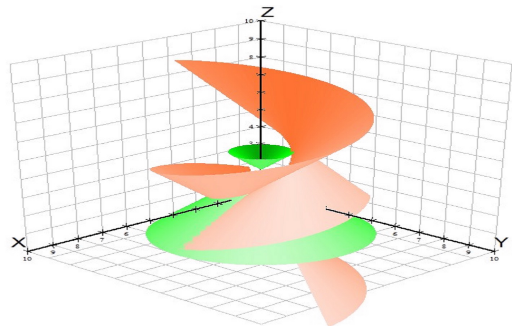


FIGURE 6.

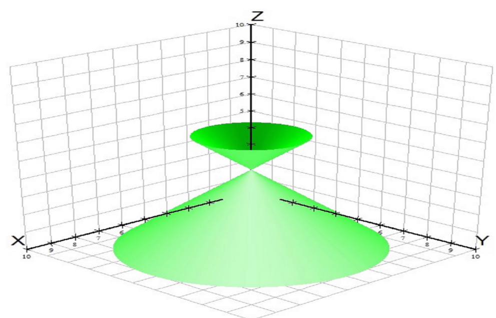


FIGURE 7.

The base surface given by (42) and the relatively osculating developable surface given by (43) have been together plotted in Fig. 6. The relatively osculating developable surface given by (42) has been alone illustrated in Fig. 7. The relatively osculating developable surface has been illustrated by reflecting surface in Fig. 8.

## 5. Conclusion

In this work, we have constructed a developable surface tangent to a surface forward a curve in the surface which we defined it as relatively osculating developable surface. We have chosen the curve as the tangent normal direction curve on which the new surface is formed in Euclidean space. We have obtained some results about the existence and uniqueness, and the singularities of such developable surfaces. We have also given two invariants of curves on a surface which describe these singularities. We have given two results for special curves such as asymptotic line and line of curvature which are rulings of the relatively osculating developable surface.

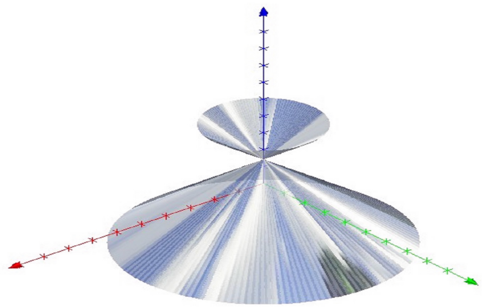

Figure 8.

**Acknowledgement.** Authors would like to thank referee(s) for their review and contribution.

## References

[1] Abbena, E., Salamon, S. and Gray, A. Modern differential geometry of curves and surfaces with Mathematica, Chapman and Hall/CRC; 3 edition, June 21, 2006.

[2] Cipolla, R. and Giblin, P. J., Visual Motion of Curves and Surfaces, Cambridge Univ. Press, 2000.

[3] Hathout, F., Bekar, M. and Yaylı, Y., Ruled surfaces and tangent bundle of unit 2-sphere, *International Journal of Geometric Methods in Modern Physics*, 14(10) (2017),1750145.

[4] Izumiya, S. and Takeuchi, N., Singularities of ruled surfaces in $\mathbb{R}^3$. Math. *Proceedings of Cambridge Philosophical Soc.*,vol 130 (2001),1–11

[5] Izumiya, S. and Takeuchi, N., Geometry of Ruled Surfaces, Applicable *Mathematics in the Goldon Age* (ed. J. C. Misra), 305–308, Narosa Pulishing House, New Delhi, 2003.

[6] Izumiya, S. and Takeuchi, N., Special curves and ruled surfaces, *Beitr. Algebra Geom.* 44(1) (2003), 203-212.

[7] Izumiya, S. and Takeuchi, N., New special curves and developable surfaces, *Turk J Math,* 28(2) (2004), 153-163.

[8] Izumiya S. and Otani, S., Flat approximations of surfaces along curves, *Demonstr. Math.* 48(2) (2015) 1–7.

[9] Hananoi, S. and Izumiya, S., Normal developable surfaces of surfaces along curves. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics,* 147(1) (2017), 177-203.

[10] Hoschek, J. and Pottman, H., Interpolation and approximation with developable B-spline surfaces , in *Mathematical Methods for curves and surfaces,* ed. by M. Dæhlen, T. Lyche and L.L. Schumacker, Vanderbilt Univ. Press (1995), 255–264

[11] Lawrence, S., Developable Surfaces: Their History and Application, *Nexus Netw. J.* (2011) 13: 701-714.

[12] Markina, I.and Raffaelli, M., Flat approximations of hypersurfaces along curves, manuscripta math. (2018). https://doi.org/10.1007/s00229-018-1072-6.

[13] Milman, R. S. and Parker, G. D., Elements of Differential Geometry, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1977.

[14] Porteous, I. R., Geometric Differentiation for the Intelligence of Curves and Surfaces (second edition). Cambridge University Press, Cambridge, 2001.

[15] Sasaki, T., Projective Differential Geometry and Linear Homogeneous Differential Equations, Rokko Lectures in Mathematics, Kobe University, vol 5, 1999

[16] Schneider, M., Interpolation with Developable Strip-Surfaces Consisting of Cylinders and Cones, in *Mathematical Methods for curves and surfaces II*, ed. by M. Dæhlen, T. Lyche and L.L. Schumacker, Vanderbilt Univ. Press, (1998), 437–444.

[17] Ushakov, V., Developable surfaces in Euclidean space, *J. Austral. Math. Soc.* Ser. A 66 (1999), 388-402.

[18] Vaisman, I., A First Course in Differential Geometry, Pure and Applied Mathematics, A Series of Monograph and Textbooks, Marcel Dekker, 1984.

*Current address*: Department of Mathematics, Sciences Faculty for Girls, University of Jeddah, P.O. Box 126300, Jeddah 21352, SAUDI ARABIA.

*E-mail address*: rbaky@live.com

ORCID Address: https://orcid.org/0000-0001-7016-9280

*Current address*: Department of Mathematics, Kırklareli University, 39100, Kırklareli,TURKEY.

*E-mail address*: yasinunluturk@klu.edu.tr

ORCID Address: https://orcid.org/0000-0002-6809-1266