ALKU Journal of Science 2019, 1(3): 156-164
e-ISSN: 2667-7814

Numerical Methods in Calculating Eigenvalues: Case Studies in Stability of Euler Columns

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Abstract

A comparative analysis of well renowned "Shooting Method" with another numerical method "Complementary Functions Method" (CFM) is presented for calculating eigenvalue (λ). Contrary to the shooting method hit and trial approach, CFM exploits the properties of linear ordinary differential equation (LODE). In the case of linear eigenvalue Boundary value Problem (BVP), CFM generates an algebraic equation system with one unknown "λ" and, alone root finding method is sufficient to give required eigenvalue. However, the Shooting Method create a system of algebraic equations containing two unknowns "λ" and "missing initial conditions", that demands an additional numerical technique along with root finding method. These radical differences between two approaches, sets the basis for this comparative investigation. As a case study in Linear Elastic Stability, different cases of Euler columns are investigated by finding eigenvalues for each case numerically, under both methods. Comparison is performed on the basis of results accuracy and cost effectiveness for both numerical techniques while solving linear stability problems.

Keywords: Shooting method, Complementary Functions Method (CFM), eigenvalue, linear ordinary differential equation (LODE), Linear Elastic Stability, Linear eigenvalue boundary value problem, Euler Columns.

1. INTRODUCTION

Solving eigenvalue boundary value problems is one of the basic requirements while dealing with mechanical stability cases. In such problems, non-availability of analytical or closed-form solutions leaves us only with numerical methods. Further, the function of used numerical methods is not limited to conversion of a BVP into IVP but also determining correct eigenvalues. This extension in task demands greater efficiency from the used numerical method.

Shooting method has been extensively used for solving the regular and Eigen value BVP due to its simple and direct approach. In general shooting method's main role is determining the missing initial conditions by suitable root finding numerical methods. In the case of eigenvalue BVP, Presence of an eigenvalue as an additional variable restricts the direct use of shooting method. D.J.Jone's [1] used an additional step to find eigenvalues by shooting method. Firstly use of the Least Square Method by incrementing the guessed λ from 0 to 100,000 in steps of 20 is employed to get the traces of first few eigenvalues. Later it is followed by Newton's iteration as a root finding method. Even with an additional step, Shooting Method with its direct guessing approach required massive number of iterations to reach the correct eigenvalue. Xi Chen [2] presented a generalized approach with a mathematical proof of using the Shooting Method for an

eigenvalue BVP. An equation derived from the Schrodinger equation is solved with the same guessing approach. In recent years use of Shooting Method for eigenvalue BVP arise from stability and free vibration problems can be seen commonly. Zhou & Zheng [3] use the Shooting Method while doing thermal buckling analysis of an elastic rod with fixed ends. A set of algebraic equations is generated containing two unknown's "λ" and "missing initial conditions" with a condition to satisfy boundary conditions. Here Newton-Raphson iteration is employed along with analytical continuation for "λ" to find the root of above equations. Li [4] extended thermal buckling problem by including vibration. Here Shooting Method is used to find two eigenvalues by the same approached mentioned earlier. Increase in the number of unknowns means increase in the number of guesses that will further reduce the efficiency of the Shooting Method. Zhu [5] introduce a technique to improve the initial guessing during root finding in shooting method for calculating bifurcation points. A perturbation form of original problem is considered and then bisection method is used to find the perturbation points. In [6] and [7] Shooting Method is used for solving FGM discs buckling and free vibration cases. In light of the above discussion, effectiveness of Shooting Method with its hit and trail approach especially for solving stability problems comes under question.
In past, CFM has been used by many researchers to solve mechanical and applied mathematics problems. [8], [9] & [10] CFM is used to solve both linear and non-linear BVP. [11] & [12] applied CFM to solve different instability problems. [13] - [16] CFM is used for solving different free and forced vibration cases of composite beams and arches. In [17] CFM is employed to perform transient analysis of composite parabolic arches. [18] Uses CFM to study the effect of the Poisson ratio on stresses of heterogeneous pressure vessels. [19] Applied CFM while studying the effect of uniform magnetic field on pressurized FG cylindrical and spherical vessels. In the present paper application area of CFM method is extended by using it first time for solving Buckling stability problem and also comparing with Shooting Method. Ability of CFM that it removes the need of guessing initial conditions while solving BVP makes it an extremely robust method. Further, it gives solution by separately finding complementary and particular solutions. In the case of eigenvalue BVP, Complementary and particular solutions generates a single algebraic eq. containing λ and solving it for root give us required eigenvalue.

2. GOVERNING EQUATIONS

Consider an axially loaded slender rod as an Euler Column of length "L" with flexural rigidity of "D=EI" under the compressive force "P". Differential boundary equations and analytical expression of eigenvalue for the buckling of Euler Columns under various boundary conditions are given by [1]. X and Y represent axial coordinate and deflection of rod respectively. Where $\lambda = P/D$ and bending moment is given by M_0.

$$\frac{dY}{dX} = Q\ (X, \lambda, Y) \quad , \qquad 0 < X < 1 \tag{1}$$

$$Y = \{\ y_1, y_2\ \}^T$$

Both ends hinged (H-H)

$$Q = \{y_2, -\lambda^2\, y_1\}^T \qquad y_1(0) = y_1(1) = 0 \tag{2}$$

$$\lambda = \frac{n\pi}{L} \tag{3}$$

Both ends fixed (F-F)

$$Q = \{y_2, -\lambda^2\, y_1 + \frac{M_0}{D}\}^T \qquad y_1(0) = y_2(0) = y_1(1) = y_2(1) = 0 \tag{4}$$

$$\lambda = \frac{2n\pi}{L} \tag{5}$$

One end fixed and one end hinged (F-H)

$$Q = \{y_{2,} - \lambda^2\, y_1 + \frac{M_0 X}{DL}\}^T \qquad y_1(0) = y_1(1) = y_2(1) = 0 \qquad (6)$$

$$\lambda = \frac{4.49n}{L} \qquad (7)$$

One end fixed and one end free (F-F')

$$Q = \{y_{2,} - \lambda^2\,(y_1 + \delta)\}^T \qquad y_1(0) = y_1(1) = 0\ ;\ y_1(1) = \delta \qquad (8)$$

$$\lambda = \frac{n\pi}{2L} \qquad (9)$$

3. NUMERICAL METHODS FORMULATION

The shooting method and complementary function method are used to determine the eigenvalue for each case numerically.

3.1. Shooting Method

By considering missing initial conditions as a multivariable function of given boundary conditions; the boundary value problem for all cases are converted into initial value problems.

From equation (1) and equation (2) the initial value problem for (H-H) is given as

$$(H-H) \qquad \begin{cases} \dfrac{dY}{dX} = Q\,(X,\lambda,Y) \\ \mathrm{Y}(0) = \{\,0,\beta\}^{\mathrm{T}} \end{cases} ,\quad 0 < X < 1 \qquad (10)$$

Here β is the guess value for the missing initial condition. For a certain value of β and λ, denote the solution of equation (10) as $y(X,\lambda,\beta)$ such that the value of β and λ satisfy the following algebraic equation

$$(H-H) \qquad (1;\lambda,\beta) = 0 \qquad (11)$$

Initial value problem Eq.10 is needed to be solve simultaneously with an Eq.11.An iterative scheme is set up by using RK4 method to integrate the Eq.(10) and secant method to find the root β of Eq.(11). Analytical continuation is employed on λ by increasing it values on each step from an infinitesimal value. Similarly for other Euler Columns an iterative scheme is formulated.

From equation (1) and equation (4) the initial value problem for (H-H) is given as

$$(F-F) \qquad \begin{cases} \dfrac{dY}{dX} = Q\,(X,\lambda,Y) \\ \mathrm{Y}(0) = \{\beta,0\}^{\mathrm{T}} \end{cases} ,\quad 0 < X < 1 \qquad (12)$$

$$(F-F) \qquad (1;\lambda,\beta) = 0 \qquad (13)$$

From equation (1) and equation (6) the initial value problem for (H-H) is given as

$$(F-H) \qquad \begin{cases} \dfrac{dY}{dX} = Q\,(X,\lambda,Y) \\ \mathrm{Y}(0) = \{0,\beta\}^{\mathrm{T}} \end{cases} , \quad 0 < X < 1 \tag{14}$$

$$(F-H) \qquad (1;\lambda,\beta) = 0 \tag{15}$$

From equation (1) and equation (8) the initial value problem for (H-H) is given as

$$(F-F') \qquad \begin{cases} \dfrac{dY}{dX} = Q\,(X,\lambda,Y) \\ \mathrm{Y}(0) = \{0,\beta\}^{\mathrm{T}} \end{cases} , \quad 0 < X < 1 \tag{16}$$

$$(F-F') \qquad (1;\lambda,\beta) - \delta = 0 \tag{17}$$

3.2. Complementary Function Method

Solution of n^{th} order Eigenvalue linear ordinary differential boundary value problem is composed of a_m Complementary solutions with c_m Constants and an expression for inhomogeneous solution b.

$$f(x,\lambda) = \sum_{i=1}^{m} \{\mathrm{a_m}(\mathrm{x},\lambda).\mathrm{c_m}\} + \mathrm{b}(\mathrm{x}) \qquad \mathrm{g} < x < h \tag{18}$$

Considering a family of linearly independent initial conditions

$$\{\mathrm{a_1}(\mathrm{g}), \dots, \mathrm{a_m}(\mathrm{g}), \mathrm{b}(\mathrm{g})\}^T = \{1 \dots 0\ \ 0, \dots, 0 \dots 1\ \ 0, 0 \dots 0\ \ 0\}^T \tag{19}$$

$$\mathrm{a_1}(\mathrm{g}) = \{\mathrm{a_1}(\mathrm{g})^0, \dots, \mathrm{a_1}(\mathrm{g})^{\mathrm{n}}\}\ ,\ \ \mathrm{a_m}(\mathrm{g}) = \{\mathrm{a_m}(\mathrm{g})^0, \dots, \mathrm{a_m}(\mathrm{g})^{\mathrm{n}}\}\ \ ,\ \ \mathrm{b}(\mathrm{g}) = \{\mathrm{b}(\mathrm{g})^0, \dots, \mathrm{b}(\mathrm{g})^{\mathrm{n}}\}$$

RK4 is employed as a numerical integrator to find values of $\mathrm{a_n}(\mathrm{h})$ and b(h). On forcing boundary conditions B_m generate a set of algebraic equations and solving them simultaneously gives $\mathrm{c_n}$.

$$\mathrm{A}\,(\lambda).\,\mathrm{C} = 0 \tag{20}$$

$$\mathrm{A}\,(\lambda) = \{\mathrm{a_1}(\mathrm{B_1}) \dots \mathrm{a_m}(\mathrm{B_1}),\ \mathrm{a_1}(\mathrm{B_m}) \dots \mathrm{a_m}(\mathrm{B_m})\}^{T}, \quad C = \{\mathrm{c_1}, \dots, \mathrm{c_m}\}^T$$

For a non-trivial solution $\mathrm{C} \neq 0$, hence

$$\mathrm{Det}|\mathrm{A}(\lambda)| = 0 \tag{21}$$

CFM provides an algebraic equation containing only a single variable and direct application of secant method as root finding algorithm on equation (21) will generate eigenvalue. On following the above mentioned numerical scheme single variable algebraic equations are generated for all Columns cases.

From equation (1) , equation (2) & equation (18-20)

$$(\boldsymbol{H}-\boldsymbol{H}) \qquad \mathrm{A}\,(\lambda).\,\mathrm{C}=0 \qquad (22)$$

$$\mathrm{A}\,(\lambda)=\{1 \quad 0,\ \mathrm{a}_1(1) \quad \mathrm{a}_2(1)\ \}^T, \quad C=\{\ \mathrm{c}_1, \mathrm{c}_2\}^T$$

From equation (1), equation (4) & equation (18-20)

$$(\boldsymbol{F}-\boldsymbol{F}) \qquad \mathrm{A}\,(\lambda).\,\mathrm{C}=0 \qquad (23)$$

$$\mathrm{A}\,(\lambda)=\{1 \quad 0 \quad 1,\ 0 \quad 1 \quad 0,\ \mathrm{a}_1(1) \quad \mathrm{a}_2(1) \quad 1\ \}^T, \quad C=\{\ \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}^T$$

From equation (1), equation (6) & equation (18-20)

$$(\boldsymbol{F}-\boldsymbol{H}) \qquad \mathrm{A}\,(\lambda).\,\mathrm{C}=0 \qquad (24)$$

$$\mathrm{A}\,(\lambda)=\{1 \quad 0 \quad 0, \mathrm{a}_1^1(1) \quad \mathrm{a}_2^1(1) \quad 1\ ,\ \mathrm{a}_1(1) \quad \mathrm{a}_1(1) \quad 1\ \}^T, \quad C=\{\ \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}^T$$

From equation (1), equation (8) & equation (18-20)

$$(\boldsymbol{F}-\boldsymbol{F}') \qquad \mathrm{A}\,(\lambda).\,\mathrm{C}=\mathrm{S} \qquad (25)$$

$$\mathrm{A}\,(\lambda)=\{1 \quad 0 \quad 1,\ 0 \quad 1 \quad 0,\ \mathrm{a}_1(1) \quad \mathrm{a}_2(1) \quad 1\ \}^T, \quad C=\{\ \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}^T, \quad S=\{0{,}0, \delta\}^T$$

4. NUMERICAL RESULTS & DISCUSSION

In order to establish a neutral medium for comparison; Secant method is applied as a root finding algorithm while Runge-kutta of 4th order is used as numerical integrator for both methods. “Intel(r) Core(TM) i7-6500u CPU @ 2.50 GHz & 2.60 GHz with 8GB Ram” operating system is used to perform iterations. Same initial guesses and tolerance values, iterations are carried out for finding first order (n=1) eigenvalue. For simplicity, Consider L = D = M = δ = 1. Accuracy of numerical results is given with respect to analytical results. Processing time for CFM is calculated for 10^{-6} accuracy and for Shooting Method it’s restricted to 10^{-3} due to high number of iterations.

Table 1. Numerical results comparison for (H-H)

	Shooting Method	**CFM**
Step Size	100	100
No of Iterations	$3x10^4$	6
Accuracy	10^{-3}	10^{-6}
Processing Time	28 min	0.1 Sec

Table 2. Numerical results comparison for (F-F)

	Shooting Method	**CFM**
Step Size	100	100
No of Iterations	$7x10^4$	8
Accuracy	10^{-3}	10^{-6}
Processing Time	35 min	0.2 Sec

Solving first two cases H-H, F-F by shooting method gives another challenge in selection of known initial conditions. In cases of giving common initial conditions for both cases from equation (10)

$$\mathrm{Y}(0) = \{\, 0, \beta\}^{\mathrm{T}} \tag{26}$$

Solution shape for F-F doesn't satisfy boundary conditions $y_2(0) = y_2(1) = 0$. Unless, a set of uncommon initial conditions considering for F-F from equation (11)

$$\mathrm{Y}(0) = \{\beta, 0\}^{\mathrm{T}} \tag{27}$$

It signifies another drawback of using shooting method for stability problems.

Table 3. Numerical results comparison for (F-F)

	Shooting Method	**CFM**
Step Size	100	100
No of Iterations	$5x10^4$	6
Accuracy	10^{-3}	10^{-6}
Processing Time	29 min	0.2 Sec

Table 1-3, depicts that shooting method needed a massive number of iterations along with huge processing time to find the missing initial conditions and eigenvalue with low accuracy of 10^{-3}. Conversely, CFM requires fewer numbers of iterations and insignificant amount of processing time for calculating eigenvalue with double accuracy of 10^{-6}.

Table 4. Numerical results comparison for (F-F′)

	Shooting Method	CFM
Step Size	100	100
No of Iterations	$2x10^4$	$2x10^3$
Accuracy	10^{-3}	10^{-3}
Processing Time	29 min	11 min

In fixed-free case since given boundary conditions are not homogeneous, from equation(8)

$$y_1(0) = y_1(1) = 0 \; ; \; y_1(1) = \delta$$

As a result, here CFM gives a non-homogenous system of algebraic equations, from equation (10)

$$A(\lambda).\, C = S$$

In order to solve the above non-homogenous system of algebraic equations, directly root finding method cannot be employed. Hence, equation (10) is solved iteratively for the successive values of λ until both sides become equal and this process in literature is called as analytical expansion. It leads to increase in the number of iterations and processing time for CFM but still its performance is better than shooting method.

5. CONCOLUSION

A comparative investigation is carried out based upon the results accuracy and cost effectiveness, between the two numerical methods namely, Shooting Method and Complementary Function Method (CFM) respectively, for finding the eigenvalues in different cases of Euler Columns. Although Shooting Method is an effective direct approach for solving Ordinary boundary value problems, but in case of eigenvalue problems due to the introduction of another variable “Eigenvalue” (λ), causes the significant increases in its number of iterations and processing time. On the other hand, CFM solves the same problem with double accuracy as of shooting method under fewer numbers of iterations and insignificant processing time.

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Geliş/Received: 20 Ara 2019/20 Dec 2019
Kabul Ediliş/Accepted: 07 Oca 2020/07 Jan 2020