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ON THE K_a-CONTINUITY OF REAL FUNCTIONS

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Abstract. The aim of the present paper is to define K_a-continuity which is associated to the number sequence $a = (a_n)$ and to give some new results.

1. Introduction and Preliminaries

Robbins proposed a problem and he asked readers to show that a function $f : \mathbb{R} \to \mathbb{R}$ with the following property has to be linear:

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f\left(x_k\right) = f\left(x_0\right) \text{ whenever } \lim_n \frac{1}{n}\sum_{k=1}^{n} x_k = x_0,\ x_0 \in \mathbb{R},$$

in 1946 ([9]). Solution by R. C. Buck [10] was published in 1948 (the problem was also solved by five others). Since then, different type continuities defined and studied by authors. Antoni and Salat [3] defined the concept of $A-$continuity for real functions based on $A-$summability. After that the notion of $F-$continuity based on almost convergence ($F-$convergence) was introduced in the paper [11] by Öztürk. This method studied by Borsik and Salat [4] and they remark that almost convergence and $A-$summability are not equivalent. Also some authors studied different concepts of continuity [2, 10, 12, 13].

Let $A = (a_{nk})$ be an infinite matrix of real numbers and $x = (x_n)$ be a number sequence. The sequence $\left(A\left(x\right)_n\right)$ where $A\left(x\right)_n = \sum_{k=1}^{\infty} a_{nk} x_k$ is called the $A-$transform of x whenever the series converges for $n = 1, 2, 3, ...$ The sequence x is said to be $A-$summable to l if the sequence $\left(A\left(x\right)_n\right)$ converges to l and we write $A - \lim_n x_n = l$. A is called regular if $\lim_n x_n = l$ implies $A - \lim_n x_n = l$ ([5, 6]).

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A sequence (x_n) of real numbers is said to be almost convergent ($F-$convergent) to number l if

$$\lim_p \frac{1}{p}\sum_{k=1}^{p} x_{n+k} = l$$

holds uniformly in $n = 1, 2, 3, ...$ and we write $F - \lim_n x_n = l$[8].

Definition 1. *Let $A = (a_{nk})$ be a regular matrix of real numbers and (x_n) be a number sequence. A function $f : \mathbb{R} \to \mathbb{R}$ is $A-$continuous at a point $x_0 \in \mathbb{R}$ if $A - \lim_n f(x_n) = f(x_0)$ whenever $A - \lim_n x_n = x_0$([2, 3]).*

Definition 2. *A function $f : \mathbb{R} \to \mathbb{R}$ is $F-$continuous at a point $x_0 \in \mathbb{R}$ if $F - \lim_n f(x_n) = f(x_0)$ whenever $F - \lim_n x_n = x_0$.*

In the present paper, we study the concept of K_a-continuity based on K_a-convergence, was defined by Lazic and Jovovic [7]. It is now natural to ask: Is the K_a-continuity a special case of $A-$continuity or do K_a-continuity and $F-$continuity contain each other? In general the answer is no. Simple examples show that these continuity methods do not contain each other. Namely, these methods are overlap.

We now recall some definitions and properties:

The notion of K_a-convergence was defined by Lazic and Jovovic [7] in 1993, which is obviously associated to the matrix $A = (a_{nk})$,

$$A = \begin{pmatrix} a_1 & 0 & 0 & 0 & ... \\ a_2 & a_1 & 0 & 0 & \\ a_3 & a_2 & a_1 & 0 & \\ . & & & & \\ . & & & & \end{pmatrix}.$$

Let $a = (a_n)$ and (x_n) be number sequences, set $y_n = \sum_{i=1}^{n} a_{n-i+1} x_i \ (n = 1, 2, 3, ...)$, then we say that (y_n) is the K_a-*transformation* of the (x_n).

Definition 3. [7] *The sequence (x_n) of real numbers is said to be K_a-convergent to the number l if, its K_a-transformation (y_n) converges to the number l, i.e. $\lim_n y_n = l$. This limit is denoted by $K_a - \lim_n x_n = l$.*

Proposition 4. [7] *Let $a = (a_n)$ be a number sequence and the series $\sum a_n$ be absolutely convergent, i.e.*

$$\sum_{n=1}^{\infty} |a_n| < \infty. \tag{1}$$

(i) *If* (x_n) *is convergent,* $\lim_n x_n = l$ *and the condition (1) is satisfied then,*

$$K_a - \lim_n x_n = l \sum_{n=1}^{\infty} a_n .$$

(ii) *The convergence method* K_a *is regular if and only if the condition (1) and*

$$\sum_{n=1}^{\infty} a_n = 1 \tag{2}$$

are valid (for more properties and details, see also [7]*).*

Now, we will give examples which show that K_a-convergence and almost convergence do not imply each other.

Example 5. *Let* $a = (a_n) = (2, 2, -2, 0, 0, ...)$ *and let*

$$x = (x_i) = (1, 0, 1, -1, 2, -3, 5, -8, ...) \left[x_i = x_{i-2} - x_{i-1} \text{ for } i \geq 3 \right].$$

Then,

$$(y_k) = \left(\sum_{i=1}^{k} a_{k-i+1} x_i \right) = (2, 2, 0, 0, ...) .$$

Therefore $K_a - \lim_n x_n = 0$*. However,* $F - \lim_n x_n$ *does not exist. Also, observe that* $\sum_{n=1}^{\infty} a_n = 2$ *and* K_a *is not regular.*

Example 6. *Let* $a = (a_n) = (1, 0, 1, 0, 0, ...)$ *and let*

$$(x_i) = \left(1, 1, \frac{1}{2^3}, \frac{1}{2^4}, 1, 1, \frac{1}{2^7}, \frac{1}{2^8}, 1, 1, ... \right).$$

Then,

$$(y_n) = \left(\sum_{i=1}^{n} a_{n-i+1} x_i \right) = \left(1, 1, 1 + \frac{1}{2^3}, 1 + \frac{1}{2^4}, \frac{1}{2^3} + 1, \frac{1}{2^4} + 1, ... \right).$$

Hence $K_a - \lim_n x_n = 1$*. However,* $F - \lim_n x_n \neq 1$*. Also, observe that* $\sum_{n=1}^{\infty} a_n = 2$ *and* K_a *is not regular.*

Now, we introduce the notion of K_a-continuity.

Definition 7. *Let* $a = (a_n)$ *and* (x_n) *be number sequences. The function* $f : \mathbb{R} \to \mathbb{R}$ *is* K_a-*continuous at a point* $x_0 \in \mathbb{R}$ *if* $K_a - \lim_n f(x_n) = f(x_0)$ *whenever* $K_a - \lim_n x_n = x_0$.

Lemma 8. *If (f_n) is a sequence of K_a-continuous functions defined on a subset D of $\mathbb{R}$, $\sum_{n=1}^{\infty}|a_n| = M \neq 0$ and (f_n) is uniformly convergent to a function f, then f is K_a-continuous on D.*

Proof. Let (x_n) be a K_a-convergent sequence and $\varepsilon > 0$. Since (f_n) is uniformly convergent, then there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2(M+1)}$ for all $x \in D$, whenever $n \geq N$. As f_N is K_a-continuous, there exists a positive integer N_1, greater than N, such that $\left|\sum_{i=1}^{n} a_{n-i+1} f_N(x_i) - f_N(x_0)\right| < \frac{\varepsilon}{2}$ for $n \geq N_1(\varepsilon)$. Then, for all $n \geq N_1$, we get

$$\begin{aligned}\left|\sum_{i=1}^{n} a_{n-i+1} f(x_i) - f(x_0)\right| &\leq \left|\sum_{i=1}^{n} a_{n-i+1}\left(f(x_i) - f_N(x_i)\right)\right| \\ &\quad + \left|\sum_{i=1}^{n} a_{n-i+1} f_N(x_i) - f_N(x_0)\right| \\ &\quad + |f_N(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{2(M+1)} M + \frac{\varepsilon}{2} + \frac{\varepsilon}{2(M+1)} = \varepsilon.\end{aligned}$$

This completes the proof. □

2. Main Results

In this section we prove our main theorems.

Theorem 9. *Let $a = (a_n)$ be a number sequence and $\sum_{n=1}^{\infty}|a_n| < \infty$. If $f : \mathbb{R} \to \mathbb{R}$ is K_a-continuous at a point $x_0 \in \mathbb{R}$, then f is a linear function.*

Proof. Let $\sum_{n=1}^{\infty} a_n = N$ and $N \neq 0$. First, we can assume that $g : \mathbb{R} \to \mathbb{R}$ is K_a-continuous at a point 0 and $g(0) = 0$ as a special case.

Let $x = (b, c, d, b, c, d, ...)$ such that $b, c, d \in \mathbb{R}$ and $b + c + d = 0$ and let $a = (a_n) = (1, 1, 1, 0, 0, ...)$. Then the sequence K_a-convergent to 0. Indeed,

$$\left(\sum_{i=1}^{n} a_{n-i+1} x_i\right) = (b, b + c, 0, 0, ...).$$

This means $K_a - \lim_n x_n = 0$. According to assumption, we have $K_a - \lim_n g(x_n) = g(0) = 0$, i.e., the sequence $(g(x_n)) = (g(b), g(c), g(d), ...)$ is K_a-convergent to

0. Also, by a direct calculation, we can see that

$$\begin{aligned}&\left(\sum_{i=1}^{n} a_{n-i+1} g\left(x_i\right)\right)\\ =\quad &\left(g\left(b\right), g\left(b\right)+g\left(c\right), g\left(b\right)+g\left(c\right)+g\left(d\right), g\left(c\right)+g\left(d\right)+g\left(b\right), \ldots\right).,\end{aligned}$$

$K_a-\lim_n g\left(x_n\right)=g\left(b\right)+g\left(c\right)+g\left(d\right)$. Hence

$$g\left(b\right)+g\left(c\right)+g\left(d\right)=0 \tag{3}$$

Since $d=-b-c$, we get $g\left(-b-c\right)=-g\left(b\right)-g\left(c\right)$. Putting $c=0$ we have

$$g\left(-b\right)=-g\left(b\right) \quad \left(b\in\mathbb{R}\right) \tag{4}$$

Let $x, y\in\mathbb{R}$ arbitrary. Put $d=x+y$, $b=-x$, $c=-y$ then $b+c+d=0$ and according to (3) and (4), we get

$$g\left(x+y\right)=-g\left(-x\right)-g\left(-y\right)=g\left(x\right)+g\left(y\right),\ g\left(nx\right)=ng\left(x\right).$$

If a sequence $\left(x_n\right)$ is K_a-convergent to zero, so that $\lim_n \sum_{i=1}^{n} a_{n-i+1}x_i=0$, then it can be seen that

$$\lim_n \sum_{i=1}^{n} a_{n-i+1} g\left(x_i\right)=\lim_n g\left(\sum_{i=1}^{n} a_{n-i+1}x_i\right)=0.$$

Hence g is continuous in the usual sense at zero. On the basis of well known knowledge on Cauchy equation we get $g\left(x\right)=Cx$ for $x\in\mathbb{R}$, C being a constant (p. 44-45, [1]).

Now, we shall discuss the general case. Let $f:\mathbb{R}\to\mathbb{R}$ be K_a-continuous at a point $x_0\in\mathbb{R}$. We write new coordinates $x'=x-x_0$, $y'=Ny-f\left(x_0\right)$. Put $g\left(x'\right)=Nf\left(x\right)-f\left(x_0\right)$. It is easy to see that from the K_a-continuity of f at x_0 the K_a-continuity of g at 0 follows. Hence, g has the form $g\left(x'\right)=C'x'$, i.e., $Nf\left(x\right)-f\left(x_0\right)=C'x'=C'\left(x-x_0\right)=C'x-C'x_0$, $f\left(x\right)=\frac{C'}{N}x+\frac{-C'x_0+f\left(x_0\right)}{N}=Cx+B$ where $C=\frac{C'}{N}$ and $B=\frac{-C'x_0+f\left(x_0\right)}{N}$. The proof is finished. □

Theorem 10. *Let $a=\left(a_n\right)$ be a number sequence, $\sum_{n=1}^{\infty}\left|a_n\right|<\infty$ and $f:\mathbb{R}\to\mathbb{R}$ have the following property:*

there exists such a point $x_0\in\mathbb{R}$ that the following implication

$$K_a-\lim_n x_n=x_0 \Rightarrow \lim_n f\left(x_n\right)=\frac{f\left(x_0\right)}{N}, \tag{5}$$

where $N=\sum_{n=1}^{\infty} a_n\left(N\neq 0\right)$, is valid. Then f is a constant function.

Proof. From (5) and Proposition 4, we have

$$K_a - \lim_n x_n = x_0 \Rightarrow K_a - \lim_n f(x_n) = f(x_0).$$

Hence f is K_a-continuous at a point $x_0 \in \mathbb{R}$. The Theorem 9 says that f is linear. Put $b = x_0 - 1$, $c = x_0 + 1$ and $a = (a_n) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, ...\right)$. Then the sequence $(x_n) = (b, c, b, c, ...)$ is K_a-convergent to x_0, i.e.,

$$\left(\sum_{i=1}^{n} a_{n-i+1} x_i\right) = \left(\frac{x_0 - 1}{2}, x_0, x_0, ...\right),$$

$K_a - \lim_n x_n = x_0$. It follows from (5) that

$$(f(x_n)) = (f(b), f(c), f(b), f(c), ...)$$

converges. The last statement yields

$$f(b) = f(c). \tag{6}$$

Since f is a linear function it follows from (6) that f is a constant function. □

We note that if $\sum_{n=1}^{\infty} a_n = 1$ then the matrix $A = (a_{nk})$ given via the sequence $a = (a_n)$ is regular. In that case, the K_a-continuity is a special case of $A-$continuity. But, here $\sum_{n=1}^{\infty} |a_n| < \infty$ and therefore our main theorems Theorem 9 and Theorem 10 are not a consequence of the results concerning the $A-$continuity.

References

[1] Aczel, J., Vorlesungen über Funktionalgleichungen und ihre Anwendungen, VEB Deutsch. Verlag der Wissenschaften, Berlin, 1961.

[2] Antoni, J., On the $A-$continuity of real functions II, *Math. Slovaca,* 36 (1986), 283-288.

[3] Antoni, J., Salat, T., On the $A-$continuity of real functions, *Acta Math. Univ. Comenian.,* 39 (1980), 159-164.

[4] Borsik, J., Salat, T., On $F-$continuity of real functions, *Tatra Mountains Math. Publ.*, 2 (1993), 37-42.

[5] Boos, J., Classical and modern methods in summability, Oxford Science Publications, 2000.

[6] Hardy, G. H., Divergent series, Oxford Univ. Press, London, 1949.

[7] Lazic, M., Jovovic, V., Cauchy's operators and convergence methods, *Univ. Beograd. Publ. Elektrothen Fak.,* 4 (1993), 81-87.

[8] Lorentz, G. G., A contribution to the theory of divergent sequences, *Acta Math.,* 80 (1948), 167-190.

[9] Robbins, H., Problem 4216, *Amer. Math. Monthly*, 53 (1946), 470-471.

[10] Problem 4216 (1946, 470) Amer. Math. Monthly, Propesed H. Robins. Solution by R. C. Buck, *Amer.Math.Monthly,* 55 (1948) 36.

[11] Öztürk, E., On almost continuity and almost $A-$continuity of real functions, *Commun. Fac. Sci. Univ. Ankara, Ser. A1,* 32 (1983), 25-30.

[12] Posner, E. C., Summability preserving functions, *Proc. Amer. Math. Soc.*, 12 (1961), 73-76.

[13] Savaş, E., Das, G., On the $A-$continuity of real functions, *İstanbul Üniv. Fen. Fak. Mat. Der.*, 53 (1994), 61-66.

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