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Research Article

Rings such that, for each unit u, $u - u^n$ belongs to the Jacobson radical

M. Tamer Koşan*[1], Troung Cong Quynh[2], Tülay Yıldırım[3], Jan Žemlička[4]

[1] Department of Mathematics, Faculty of Sciences, Gazi University, Ankara, Turkey
[2] Department of Mathematics, The University of Danang - University of Science and Education, 459 Ton Duc Thang, Danang city, Vietnam
[3] Department of Mathematics, Gebze Technical University, Kocaeli, Turkey
[4] Department of Algebra, Charles University, Faculty of Mathematics and Physics Sokolovska 83, 186 75 Praha 8, Czech Republic

Abstract

A ring R is said to be n-UJ if $u - u^n \in J(R)$ for each unit u of R, where $n > 1$ is a fixed integer. In this paper, the structure of n-UJ rings is studied under various conditions. Moreover, the n-UJ property is studied under some algebraic constructions.

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1. Introduction

Throughout the paper, all considered rings are associative and unital. For a ring R, the Jacobson radical, the set of nilpotent elements and the set of invertible elements of R are denoted by $J(R)$, $Nil(R)$ and $U(R)$, respectively. The symbols $M_n(R)$ and $T_n(R)$ stand for the $n \times n$ matrix ring and the $n \times n$ upper triangular matrix ring over R, respectively. $R[x]$ ($R[[x]]$, respectively) stands for the polynomial ring (the power series ring, respectively) over R. Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}_n$ be the ring of $\mathbb{Z}$ modulo n. We also use $\mathbb{N}$ to denote the set of natural numbers.

Recall that a ring R is called a UJ-ring ([12]) if $1 + J(R) = U(R)$ (see also, [6] and [19]). Let $n \in \mathbb{N}$. For a fixed integer $n > 1$, consider the following forms of the units of a ring R which belong to $J(R)$:

(1) $u - u^n \in J(R)$ for each $u \in U(R)$;

(2) For each $u \in U(R)$ there exists n such that $u - u^n \in J(R)$.

If a ring R satisfies the condition (1) (respectively, (2)), then we call R an n-UJ ring (respectively, an ∞-UJ ring). Notice that all UJ rings are n-UJ and every n-UJ ring is ∞-UJ. Let R be a UJ-ring. In [12, Proposition 1.3], it is shown that if R is a division ring, then $R \cong \mathbb{F}_2$. More generally, $R/J(R)$ is reduced and hence abelian.

*Corresponding Author.
Email addresses: mtamerkosan@gazi.edu.tr (M.T. Koşan), tcquynh@ued.udn.vn (T.C. Quynh), tyildirim@gtu.edu.tr (T. Yıldırım), zemlicka@karlin.mff.cuni.cz (J. Žemlička)
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The notions of n-UJ and ∞-UJ generalize 2-UJ rings introduced in the paper [5]. In this article, it will be shown that a division ring that is ∞-UJ is a field. Further, R is a UJ-ring iff there exists k such that R is a $(2^k + 1)$-UJ ring, $R/J(R)$ is reduced and $2 \in J(R)$ respectively.

When R is a UJ-ring with nil Jacobson radical, then R is a UU-ring (i.e., rings with unipotent units, equivalently $1 + Nil(R) = U(R)$) ([4]), we get that if R is an n-UJ ring and $n-1$ is a unit of R, then $J(R)$ contains $Nil(R)$. We also study the correspondence of the clean and n-UJ property which is similar to UJ property which were handled by Koşan, Leroy and Matczuk in [12, Section 3]. We obtain that, for a $(2n)$-UJ ring R, R is a semiregular ring iff R is an exchange ring iff R is a clean ring. Finally, the behavior of n-UJ property under some classical ring constructions, the trivial extension and the (trivial) Morita context are studied.

2. General properties of n-UJ rings

Definition 2.1. Let $n \in \mathbb{N}$. A ring R is said to be an n-UJ ring if $u - u^n \in J(R)$ for each $u \in U(R)$ where $n > 1$ is a fixed integer.

Definition 2.2. Let $n \in \mathbb{N}$. A ring R is said to be an ∞-UJ ring if for each $u \in U(R)$ there exists $n > 1$ such that $u - u^n \in J(R)$.

For $n \in \mathbb{N}$, consider the following sets:

$$\mathbb{U}_n(R) = \{u^{n-1} : u \in U(R)\} \subseteq U(R),$$
$$\mathbb{V}_n(R) = \{u \in U(R) : u^{n-1} \in 1 + J(R)\}.$$

We remark that $\mathbb{U}_n(R)$ and $\mathbb{V}_n(R)$ are subgroups of $U(R)$ if R is a commutative ring, but they need not be subgroups of $U(R)$ in the noncommutative case.

Lemma 2.3. *The following statements are equivalent for a ring R and $n \in \mathbb{N}$:*

(1) *R is an n-UJ ring;*
(2) *$\mathbb{V}_n(R) = U(R)$;*
(3) *$\mathbb{U}_n(R) \subseteq 1 + J(R)$;*
(4) *$U(R/J(R)) = \{\overline{u} = u + J(R/J(R) : \overline{u}^{n-1} = \overline{1}\} = \mathbb{V}_n(R/J(R))$.*

Proof. (1)⇒(2)⇒(3) They are obvious.

(3)⇒(4) If $\overline{u} \in U(R/J(R))$, there exists $u \in U$ such that $\overline{u} = u + J(R)$ and $u^{n-1} \in 1 + J(R)$. Hence $\overline{u}^{n-1} = \overline{1}$. The reverse inclusion is clear.

(4)⇒(1) Let $u \in U(R)$. Then $u^{n-1} \in 1 + J(R)$. Hence $1 - u^{n-1} \in J(R)$ which implies $u - u^n \in J(R)$, as desired. □

Note that every n-UJ ring is ∞-UJ. Furthermore, as an easy consequence of Lemma 2.3, we obtain:

Corollary 2.4. *A ring R is ∞-UJ if and only if $\bigcup_{n\in\mathbb{N}} \mathbb{V}_n(R) = U(R)$.*

In the following observation, we collect some general properties of n-UJ rings.

Proposition 2.5. *Let R be a ring and $n, m \in \mathbb{N}$, $n, m > 1$.*

(1) *If R is an n-UJ ring, then $2 \in J(R)$ if n is an even number.*
(2) *If R is an n-UJ ring and $n-1$ divides $m-1$, then R is an m-UJ ring.*
(3) *All UJ rings (in particular, any ring with trivial units, Boolean rings, free commutative and free noncommutative algebras over the field $\mathbb{F}_2$) are n-UJ.*

Proof. (1) Assume that R is an n-UJ ring with n an even number. Then $-1 = (-1)^{n-1} \in 1 + J(R)$, and so $2 \in J(R)$.

(2) This follows from Lemma 2.3(2) using the obvious fact that $\mathbb{V}_n \subseteq \mathbb{V}_m$ whenever $n - 1 | m - 1$.

(3) This is obvious by Lemma 2.3(3) since $\mathbb{U}_n \subseteq U(R) = 1 + J(R)$. □

Note that the claim of Proposition 2.5(1) for odd numbers generally fails. For instance, the ring $\mathbb{Z}_6$ is a 3-UJ ring with $2 \notin J(\mathbb{Z}_6)$.

Let us point out that, for any division ring R, we have $U(R) = R \setminus \{0\}$ and $J(R) = 0$. Hence a division ring R is n-UJ if and only if $u^{n-1} = 1$ for every $u \neq 0$.

Proposition 2.6. *Let $n \in \mathbb{N}$ such that $n > 1$.*

(1) *If R is a division ring which is ∞-UJ then R is a field.*
(2) *A field $\mathbb{F}$ is n-UJ iff there exist a prime p and $k \in \mathbb{N}$ such that $p^k - 1$ divides $n - 1$ and $\mathbb{F} \cong \mathbb{F}_{p^k}$, a field of p^k elements.*
(3) *A product of rings is n-UJ if and only if each component is n-UJ.*

Proof. (1) For each $u \in R$ there is $n(u) > 1$ such that $u^{n(u)} = u$. By [13, 12.10], Jacobson's Theorem, R is commutative.

(2) Let $\mathbb{F}$ be an n-UJ field. Then all nonzero elements of $\mathbb{F}$ are roots of the polynomial $x^{n-1} - 1$. Hence $\mathbb{F}$ is a finite field and there exist $k \in \mathbb{N}$ and a prime number p such that $\mathbb{F} \cong \mathbb{F}_{p^k}$, i.e. $\mathbb{F}$ is a field of p^k-elements. Finally $(p^k - 1)|(n - 1)$, since $U(F)$ is a cyclic group of order $p^k - 1$ all of whose elements have the exponent $n - 1$.

The reverse implication is clear.

Observe that R satisfies the polynomial identity $x^n - x = 0$. As R is a finite-dimensional algebra over $Z(R)$ by [10, Theorem 1], it is finite division ring, which is a field by Wedderburn Theorem. Thus $R = Z(R)$.

The reverse implication follows from (1).

(3) This follows from Lemma 2.3(3) and the facts

$$J(\prod_{i\in I} R_i) = \prod_{i\in I} J(R_i),$$

$$U(\prod_{i\in I} R_i) = \prod_{i\in I} U(R_i)$$

and

$$\mathbb{U}_n(\prod_{i\in I} R_i) = \prod_{i\in I} \mathbb{U}_n(R_i).$$

□

Example 2.7. (1) Let $p_1, \dots, p_r$ be prime numbers and $\epsilon_1, \dots, \epsilon_r \in \mathbb{N}$. Denote by n the least common multiple of $p_1^{\epsilon_1} - 1, \dots, p_r^{\epsilon_1} - 1$. Applying Proposition 2.6 we obtain that $\prod_i \mathbb{F}_{p_i^{\epsilon_i}}$ is an $(n + 1)$-UJ ring which is not m-UJ for every m such that n does not divide $m - 1$, in particular for any $m \leq n$.

(2) Let $R = \overline{\mathbb{F}_p}$ be an algebraic closure of the finite field $\mathbb{F}_p$ for a prime p. Then R is not an n-UJ ring for any $n \in \mathbb{N}$, but it is ∞-UJ.

The following example shows that the class of n-UJ rings is not closed under taking quotients.

Example 2.8. Recall $U(\mathbb{Z}) = \{1, -1\}$ and $J(\mathbb{Z}) = 0$. Hence $\mathbb{U}_n(\mathbb{Z}) = \{1\}$ for every odd number n, and so $\mathbb{Z}$ is an n-UJ ring. Nevertheless, for a prime p, the ring $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_p$ is not n-UJ for every n unless $p - 1$ divides $n - 1$ by Proposition 2.6(1).

Proposition 2.9. *For a ring R, the following observations hold:*

(1) *Let $I \subseteq J(R)$ be an ideal of R. Then R is an n-UJ ring if and only if R/I is an n-UJ ring.*
(2) *Let R be an n-UJ ring and T a subring of R. Then T is an n-UJ ring if $T \cap J(R) \subseteq J(T)$.*

Proof. (1) If $v \in U(R/I)$, then there exists an $u \in U(R)$ such that $u + I = v$ and by the hypothesis $u - u^n \in J(R)$. So one has $v - v^n \in J(R/I) = J(R)/I$.

On the other hand, recall that $(R/I)/J(R/I) \cong R/J(R)$. So R is an n-UJ ring if and only if $R/J(R)$ is an n-UJ ring by Lemma 2.3.

(2) Let $v \in U(T)$ $(\subseteq U(R))$. Since R is an n-UJ ring, we have $v^{n-1} - 1 \in J(R) \cap T \subseteq J(T)$. Therefore, T is an n-UJ ring. □

The following observation shows that the n-UJ property passes to corners.

Proposition 2.10. *If $n \in \mathbb{N}$ or $n = \infty$ and R is an n-UJ ring, then eRe is n-UJ for any $e^2 = e \in R$.*

Proof. Let $n \in \mathbb{N}$. For any $u \in U(eRe)$, we have $u + (1 - e) \in U(R)$ (with the inverse $v+(1-e)$ for $v \in eRe$ where $uv = e = vu$). By the hypothesis, $[u+(1-e)]-[u+(1-e)]^n \in J(R)$, so $u - u^n \in J(R)$. Thus $u - u^n \in eRe \cap J(R) = eJ(R)e = J(eRe)$, which implies that eRe is an n-UJ ring.

If $n = \infty$ and $u \in U(eRe)$ then we again have $u + (1 - e) \in U(R)$, hence there exists $m \in \mathbb{N}$ such that $[u + (1 - e)] - [u + (1 - e)]^m \in J(R)$. Thus $u - u^m \in J(eRe)$ and so eRe is an ∞-UJ ring. □

A ring R is reduced if R has no nonzero nilpotent elements, and the ring R is called abelian if every idempotent is central.

Proposition 2.11. *If R is an n-UJ ring and $n - 1 \in U(R)$, then $R/J(R)$ is reduced and so is abelian.*

Proof. Let $a + J(R)$ be a nilpotent element in $R/J(R)$. There exists a $k \in \mathbb{N}$ such that $a^k + J(R) = J(R)$, and so $a^k \in J(R)$.

We may assume $k \geq 2$. One can check that $a^{k-1} + J(R)$ is a nilpotent element of $R/J(R)$. Then $1 + a^{k-1}$ is a unit of R. Since R is an n-UJ ring, $(1 + a^{k-1})^{n-1} \in 1 + J(R)$. We can write $(1 + a^{k-1})^{n-1} = 1 + (n - 1)a^{k-1} + a^k.x$ for some $x \in R$. We have that $(1 + a^{k-1})^{n-1} \in 1 + J(R)$ and $n - 1 \in U(R)$ and obtain that $a^{k-1} \in J(R)$. Note that $a^{k-1} + J(R)$ is a nilpotent element of $R/J(R)$.

Repeating this process, we also have $a^{k-2} \in J(R)$. By the induction on k, we deduce that $a \in J(R)$. Thus, $R/J(R)$ is reduced and so is abelian. □

Corollary 2.12. *If R is an n-UJ ring with $n - 1 \in U(R)$, then $Nil(R) \subseteq J(R)$.*

The following example shows that the assumption "$n - 1 \in U(R)$" in Proposition 2.11 is not superfluous.

Example 2.13. Consider Bergman's example of UU-ring $R = \mathbb{F}_2\langle x, y\rangle/(x^2)$presented in [7, Example 2.5], where $\mathbb{F}_2\langle x, y\rangle$ is the free algebra generated by x and y. Recall that $0 = J(R) \subsetneq Nil(R)$ and $U(R) = 1 + \mathbb{Z}_2x + xRx$ by [7, Example 2.5], hence R is not reduced. Since $(U(R))^2 = (1 + \mathbb{Z}_2x + xRx)^2 = \{1\}$, we obtain that R is an example of a 3-UJ ring which is not reduced.

Theorem 2.14. *The following conditions are equivalent for a ring R:*

(1) *R is a UJ-ring.*
(2) *There exists k such that R is (2^k+1)-UJ, $R/J(R)$ is reduced and $2 \in J(R)$.*

Proof. (1) $\Rightarrow$ (2) This follows from the facts that UJ-rings are n-UJ, $R/J(R)$ is reduced and $2 \in J(R)$ by [12, Proposition 2.3].

(2) $\Rightarrow$ (1) Let u be a unit of R. Then $u^{2^k} \in 1 + J(R)$, and hence

$$(1+u)^{2^k} = 1 + u^{2^k} + 2v$$

for some $v \in R$. The assumption, $2 \in J(R)$, gives $(1+u)^{2^k} \in J(R)$. Since $R/J(R)$ is reduced, we have $1 + u \in J(R)$, which implies that R is a UJ-ring. □

$u \in U(R)$ is called n-torsion if $u^n = 1$ (see [8]).

Proposition 2.15. *If R is an n-UJ ring such that $U(R) = \{u \mid u \text{ is } n\text{-torsion}\}$, then R is a UJ ring.*

Proof. This is clear. □

Proposition 2.16. *Let R be a $(2k)$-UJ ring. If $J(R) = 0$ and every nonzero right ideal of R contains a nonzero idempotent, then R is reduced.*

Proof. Suppose that there exists non-zero $a \in R$ such that $a^2 = 0$. By [13], there is an idempotent $e \in RaR$ such that $eRe \cong M_2(T)$. Since R is a $(2k)$-UJ ring, eRe is as well by Proposition 2.10. Thus $M_2(T)$ is a $(2k)$-UJ ring, but this is a contradiction, since $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in U(M_2(T))$ and $A^{2k-1} = A$ or $A^{2k-1} = -A$. □

An element a in the ring R is said to be regular if there exists $b \in R$ such that $a = aba$. If all elements of R are regular, then R is called a regular ring.

Example 2.17. Consider the ring $R = \begin{pmatrix} \mathbb{F}_2 & \mathbb{F}_2 \\ \mathbb{F}_2 & \mathbb{F}_2 \end{pmatrix}$. It is easy to compute that $|U(R)| = 6$, hence $u^6 = 1$ for each $u \in U(R)$. Thus $u^7 - u \in J(R)$ for each $u \in U(R)$ which means that R is a 7-UJ ring. Moreover, $J(R) = 0$, since R is regular and every nonzero right ideal of R contains a nonzero idempotent. But, R is not reduced.

R is called a π-regular ring if for every $a \in R$ there exists a positive integer n such that $a^n \in a^n R a^n$.

An element x of the ring R is called n-potent if $x^n = x$, and R is n-potent if all its elements are n-potent.

Theorem 2.18. *The following statements are equivalent for a ring R.*

(1) *R is a regular $(2n)$-UJ ring.*
(2) *R is a π-regular, reduced and $(2n)$-UJ ring.*
(3) *R satisfies the polynomial identity $x^{2n} = x$ and it is commutative.*

Proof. (1) $\Rightarrow$ (2) Since R is regular, we get $J(R) = 0$ and every nonzero right ideal contains a nonzero idempotent. By [14], R is reduced and clearly all regular rings are π-regular.

(2) $\Rightarrow$ (3) Notice that reduced rings are abelian. By [2], R is strongly π-regular and $J(R) \subseteq Nil(R) = 0$. Let $x \in R$. By [18], there exist $e^2 = e \in R$ and $u \in U(R)$ such that $x = e + u$ and $xe = ex \in Nil(R) = 0$. Thus we have $x = x - xe = x(1-e) = u(1-e) = (1-e)u$. Since R is an $(2n)$-UJ ring, we get $x^{2n} = ((1-e)u)^{2n} = u^{2n}(1-e)^{2n} = u(1-e) = x$, as desired. Finally, recall that R is commutative by Jacobsons Theorem [13, 12.10].

(3) $\Rightarrow$ (1) Clearly, R is regular. Let $u \in U(R)$. Then $u^{2n} = u$ which implies that $u - u^{2n} \in J(R)$. Hence R is a $(2n)$-UJ ring. □

A ring R is semiregular ([16]) if $R/J(R)$ is regular and idempotents lift modulo $J(R)$, and R is exchange ([17]) if for each $a \in R$ there exists $e^2 = e \in aR$ such that $1-e \in (1-a)R$. Notice that semiregular rings are exchange.

R is called a clean ring if every element of R is a sum of an idempotent and a unit ([17]).

Theorem 2.19. *The following statements are equivalent for a $(2n)$-UJ ring R:*

(1) *R is a semiregular ring.*
(2) *R is an exchange ring.*
(3) *R is a clean ring.*

Proof. (1) $\Rightarrow$ (2) This is obvious, since every semiregular ring is an exchange ring.

(2) $\Rightarrow$ (3) By [9], R is clean if and only if $R/J(R)$ is clean and idempotents lift modulo $J(R)$. Proposition 2.16 implies that $R/J(R)$ is an exchange $(2n)$-UJ ring and $R/J(R)$ is abelian. By [17], $R/J(R)$ is clean and so R is clean.

(3) $\Rightarrow$ (1) Assume that R is a clean ring. Then idempotents lift modulo $J(R)$. By Theorem 2.18, we have that $R/J(R)$ is a regular ring. Thus, R is semiregular. □

Let us close this section with the following algebraic constructions.

Proposition 2.20. *Let R be a ring and $m \in \mathbb{N}$.*

(1) *R is an n-UJ ring if and only if $R[x]/x^m R[x]$ is an n-UJ ring.*
(2) *R is an n-UJ ring if and only if the power series ring $R[[x]]$ is an n-UJ ring.*

Proof. (1) This follows from Proposition 2.9(1) since $xR[x]/x^m R[x] \subseteq J(R[x]/x^m R[x])$ and $(R[x]/x^m R[x])/(xR[x]/x^m R[x]) \cong R$.

(2) Let us consider $(x) = xR[[x]]$ as an ideal of $R[[x]]$. Then $(x) \subseteq J(R[[x]])$. Since $R \cong R[[x]]/(x)$, the result follows from Proposition 2.9(1). □

Recall that a ring R is called 2-primal if its prime radical contains $Nil(R)$.

Proposition 2.21. *If the polynomial ring $R[x]$ is an n-UJ ring, then R is an n-UJ ring. The converse holds if R is 2-primal, $J(R)$ is nil and $n-1 \in U(R)$.*

Proof. Let $\pi : R[x] \longrightarrow R$ be a surjective ring homomorphism defined by $\pi(\sum_i a_i x_i) = a_0$. Then $\pi(J(R[x])) \subseteq J(R)$, hence $J(R[x]) \cap R \subseteq J(R)$. If $u \in U(R) \subseteq U(R[x])$, then $u - u^n \in J(R[x]) \cap R \subseteq J(R)$.

For the converse, assume R is a 2-primal n-UJ ring, $J(R)$ is nil and and $n-1 \in U(R)$. By [3, Proposition 2.6], $R[x]$ is 2-primal. We note also that $Nil(R) = J(R)$, $Nil(R[x]) = J(R[x])$ and $J(R[x]) = Nil(R)[x] = J(R)[x]$. Thus $R[x]/J(R[x]) \cong (R/J(R))[x]$ is reduced. As $R/J(R)$ is reduced by Proposition 2.11, $U(R/J(R)) = U(R[x]/J(R[x]))$. Finally, since $R/J(R)$ is an n-UJ ring, we get $R[x]/J(R[x])$ is an n-UJ ring and $R[x]$ is an n-UJ ring by Proposition 2.9. □

3. Extensions

Let R be a ring and M a bimodule over R. The trivial extension of R and M is

$$T(R, M) = \{(r, m) : r \in R \ \text{ and } \ m \in M\}$$

with an addition defined componentwise and a multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms).$$

The trivial extension $T(R, M)$ is isomorphic to the subring $\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R \ \text{and} \ m \in M\}$ of the formal 2×2 matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$ and also $T(R, R) \cong R[x]/(x^2)$.

We also note that the set of units of trivial extension $T(R,M)$ is

$$U(T(R,M)) = T(U(R),M)$$

by [1, Proposition 4.9 (2)] and

$$J(T(R,M)) = T(J(R),M)$$

by [1, Corollary 4.8 (2)].

A Morita context is a 4-tuple $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$, where A and B are rings, ${}_AM_B$ and ${}_BN_A$ are bimodules, and there exist context products $M \times N \to A$ and $N \times M \to B$ written multiplicatively as $(w,z) = wz$ and $(z,w) = zw$, such that $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is an associative ring with the obvious matrix operations .

A Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is called trivial if the context products are trivial, i.e., $MN = 0$ and $NM = 0$ (see [15, p. 1993]). We have

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong T(A \times B, M \oplus N),$$

where $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a trivial Morita context by [11].

Theorem 3.1. *Let R be a ring and let M be an (R,R) bimodule. Then R is an n-UJ ring if and only if the trivial extension $T(R,M)$ is an n-UJ ring.*

Proof. (:⇒) Let $\overline{u}$=$\begin{pmatrix} u & m \\ 0 & u \end{pmatrix} \in U(T(R,M)) = T(U(R),M)$ with $u \in U(R)$ and $m \in M$. We will show that $\overline{u} - \overline{u}^n \in J(T(R,M))$. In fact, we have $\overline{u}^n = \begin{pmatrix} u^n & m_1 \\ 0 & u^n \end{pmatrix}$ for some $m_1 \in M$. By the hypothesis, we have $\overline{u} - \overline{u}^n = \begin{pmatrix} u & m \\ 0 & u \end{pmatrix} - \begin{pmatrix} u^n & m_1 \\ 0 & u^n \end{pmatrix} = \begin{pmatrix} u-u^n & m-m_1 \\ 0 & u-u^n \end{pmatrix} \in J(T(R,M))$.

(⇐:) The converse is clear. □

Corollary 3.2. *Let S and R be rings and let M be an (R,S) bimodule. Then the formal triangular matrix ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is an n-UJ ring if and only if R and S are n-UJ rings.*

By [12, Page 5], the ring $M_n(R)$ is not UJ for any $n \geq 2$. But, the ring $\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ is a 7-UJ ring.

Corollary 3.3. *R is an n-UJ ring if and only if the upper triangular matrix ring $\mathbb{T}_n(R)$ is an n-UJ ring, $n \geq 1$.*

For a subring C of a ring D, the set

$$\mathcal{R}[D,C] := \{(d_1, \cdots, d_n, c, c, \cdots) : d_i \in D, c \in C, n \geq 1\},$$

with the addition and the multiplication defined componentwise is called the tail ring extension and denoted by $\mathcal{R}[D,C]$.

Example 3.4. $\mathcal{R}[D,C]$ is an n-UJ ring if and only if D and C are n-UJ rings.

Proof. (:⇒) Firstly, we prove that D is an n-UJ ring. Let $u \in U(D)$. Then $\overline{u} = (u,1,1,1,\cdots) \in U(\mathcal{R}[D,C])$. By the hypothesis, we have $\overline{u} - \overline{u}^n \in J(\mathcal{R}[D,C])$ for any $n \in \mathbb{N}$. Thus, $\overline{u} - \overline{u}^n = (u-u^n, 0, 0, \cdots) \in J(\mathcal{R}[D,C]) = R[J(D), J(C)]$. Hence $u - u^n \in J(D)$ which implies that D is an n-UJ ring.

To see that C is an n-UJ ring, we can take $v \in U(C)$ such that $\overline{v} = (1, \cdots, 1, v, v, \cdots) \in U(\mathcal{R}[D, C])$.

($\Leftarrow$:) Assume D and C are n-UJ rings. Let $\overline{u} = (u_1, u_2, \cdots, u_n, v, v, \cdots) \in U(\mathcal{R}[D, C])$, where $u_i, v \in U(R)$ for $1 \leq i \leq n$. Write

$$\begin{aligned} \overline{u} - \overline{u}^n &= (u_1, u_2, \cdots, u_n, v, v, \cdots) - (u_1, u_2, \cdots, u_n, v, v, \cdots)^n \\ &= (u_1 - u_1^n, u_2 - u_2^n, \cdots, u_n - u_n^n, v - v^n, v - v^n, \cdots). \end{aligned}$$

Then $u_i - u_i^n \in J(D)$ and $v - v^n \in J(C)$ imply $\overline{u} - \overline{u}^n \in \mathcal{R}[J(D), J(C)] = J(\mathcal{R}[D, C])$, as desired. ☐

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