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Some Results on Harmonic Type Sums

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ABSTRACT

In this study, we consider the summatory function of convolutions of the Möbius function with harmonic numbers, and we show that these summatory functions are linked to the distribution of prime numbers. In particular, we give infinitely many asymptotics which are consequences of the Riemann hypothesis. We also give quantitative estimate for the moment function which counts non-integer hyperharmonic numbers. Then, we obtain the asymptotic behaviour of hyperharmonics.

Keywords: *Möbius function, hyperharmonic numbers, Dirichlet series*

Harmonik Tipi Toplamlar Üzerine Bazı Sonuçlar

ÖZET

Bu çalışmada, Möbius fonksiyonunun harmonik sayılarla konvolüsyonunun toplamsal fonksiyonunu ele alacağız ve bu toplamsal fonksiyonun asalların dağılımı ile ilişkili olduğunu göstereceğiz. Özel olarak, Riemann hipotezinin sonucu olan sonsuz çoklukta asimptotik vereceğiz. Ayrıca tamsayı olmayan hiperharmoniklerin sayaç fonksiyonunun momentleri için niceliksel bir kestirim vereceğiz. Sonra da hiperharmoniklerin asimptotik davranışını elde edeceğiz.

Anahtar Kelimeler: *Möbius fonksiyonu, hiperharmonik sayılar, Dirichlet serileri*

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I. INTRODUCTION

In this note, first we study the interaction between the Möbius function and hyperharmonic numbers. In particular, we work on the summatory function of convolutions of the Möbius function with hyperharmonic numbers, and we show that these summatory functions are related to the distribution of prime numbers. We write

$$f(x) = O_t(g(x))$$

or

$$f(x) \ll_t (g(x))$$

to emphasize that the big-O constant may depend on the finite tuple t. The summatory function

$$\sum_{n \leq x} \mu(n)$$

of the Möbius function has been studied extensively. For instance, for any fixed but arbitrary $\varepsilon > 0$, the collection of estimates

$$M(x) = O_\varepsilon\left(x^{\frac{1}{2}+\varepsilon}\right) \tag{1}$$

is equivalent of the Riemann hypothesis. Even the estimate $M(x) = o(x)$ is known to be equivalent to the Prime Number Theorem (see Chapter 4 of [2]), which states that

$$\pi(x) \sim \frac{x}{\log x},$$

where $\pi(x) = |\{p \leq x : p \text{ is prime}\}|$ is the prime counting function.

Now, we define harmonic numbers. Harmonic numbers are defined by the sequence of partial sums of the harmonic series, namely

$$h_n = \sum_{k=1}^{n} \frac{1}{k}$$

for $n \geq 1$. These numbers have been studied recurrently and attracted considerable attention. For instance, it was shown in [7] that there is no harmonic number which is an integer except 1. It is well-known that

$$h_n = \log n + \gamma + O(1/n) \tag{2}$$

and a finer one is

$$h_n \sim \log n + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \cdots \tag{3}$$

as n tends to infinity, where γ is Euler's constant and B_m is the mth Bernoulli number. Next, we define hyperharmonic numbers. Hyperharmonic numbers were first defined in the book of Conway and Guy

[3] and they generalize harmonic numbers. The nth hyperharmonic number of order $r \geq 2$ is defined recursively by

$$h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)},$$

where $h_n^{(1)} = h_n$. By [3], one has that $h_n^{(r)}$ can be expressed in terms of binomial coefficients and harmonic numbers with the formula

$$h_n^{(r)} = \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1}). \tag{4}$$

Equation (4) gives the order of growth of $h_n^{(r)}$

$$h_n^{(r)} = O_r(n^{r-1} \log n). \tag{5}$$

The summatory function of the von Mangoldt function $\Lambda(n)$ plays a central role in number theory and it is known that the estimate

$$\sum_{n \leq x} \Lambda(n) \sim x \tag{6}$$

is equivalent of the Prime Number Theorem. Moreover, the Riemann hypothesis is equivalent to the estimates

$$\sum_{n \leq x} \Lambda(n) = x + O_\varepsilon\left(x^{\frac{1}{2}+\varepsilon}\right) \tag{7}$$

where $\varepsilon > 0$. Next, we recall Wiener-Ikehara theorem (see [6]). This theorem states that if we have non-negative real numbers $a(n)$ and its Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

is analytic in $\Re(s) \geq b$, with a simple pole of residue c at b, then we have

$$\sum_{n \leq x} a(n) \sim \frac{cx^b}{b}.$$

It is a Tauberian theorem, and it also yields the Prime Number Theorem, as the Riemann Zeta function $\zeta(s)$ does not vanish on the line $\sigma = 1$, see Chapter 13 of [4].

In this paper, motivated by convolutions of the Möbius function, we begin by focusing on the arithmetic function

$$\alpha_r(n) = n^{r-1}\mu(n) * h_n^{(r)}$$

and its summatory function

$$S_r(x) = \sum_{n \le x} \alpha_r(n).$$

At a first sight, the behaviour of $S_r(x)$ is not clear, seems chaotic, even it is not obvious the sum is positive after a while. We will see that this sum is actually connected to the distribution of prime numbers. Note that the trivial estimate for $S_r(x)$ is $O_r(x^r \log^2 x)$ which can be seen as follows: as $\mu(n)$ is bounded by 1 and by Theorem 3.10 of [2]

$$|S_r(x)| \le \sum_{n \le x} \left(n^{r-1} * h_n^{(r)}\right) = \sum_{n \le x} n^{r-1} H_r\left(\frac{x}{n}\right) \tag{8}$$

where

$$H_r(\mathrm{x}) = \sum_{\mathrm{n} \le \mathrm{x}} h_n^{(r)}.$$

By (5) and partial summation (Abel's identity, Theorem 4.2 of [2]), we see that

$$H_r(x) = \sum_{n \le x} h_n^{(r)} \ll_r \sum_{n \le x} n^{r-1} \log n \ll_r x^r \log x. \tag{9}$$

By (8), (9) and partial summation again, one infers that

$$|S_r(x)| \ll_r \sum_{n \le x} n^{r-1} \frac{x^r}{n^r} \log \frac{x}{n} \ll_r x^r \log^2 x. \tag{10}$$

Now we state our first theorem which is better than the trivial estimate (10). Furthermore, assuming the Riemann hypothesis, one can control the remainder term of $S_r(x)$ for every $r \ge 1$ and we obtain infinitely many asymptotics which are implied by the Riemann hypothesis.

Theorem 1.1. Let $r \ge 1$ be given. Then we have

$$S_r(x) \sim \frac{x^r}{r!}.$$

Conditionally, if the Riemann hypothesis holds, then

$$S_r(x) = \frac{x^r}{r!} + O_{r,\varepsilon}\left(x^{r-\frac{1}{2}+\varepsilon}\right).$$

Note that when $r = 1$, the results of the above theorem are reminiscent of equations (6) and (7) respectively.

Now let

$$S(x) = |\{(n, r) \in [0, x] \times [0, x] : h_n^{(r)} \notin \mathbb{Z}\}|.$$

In other words, the function $S(x)$ counts the number of pairs (n, r) in the finite rectangle $[0, x] \times [0, x]$ where the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. In [5], it was obtained that

$$S(x) = x^2 + O\left(x^{\frac{2.475}{1.475}}\right),$$

which means that non-integer hyperharmonics have the full asymptotic in the first quadruple. Recently in [1], the previous result was improved and obtained that

$$S(x) = x^2 + O_A\left(\frac{x^{\frac{2000}{1475}}}{(\log x)^A}\right). \tag{11}$$

Our second result is about the asymptotic of kth moments of the counting function $S(x)$:

Theorem 1.2. Let

$$T_k(x) = \sum_{n\le x}\left(1-\frac{n}{x}\right)^k S(n).$$

Then we have

$$T_k(x) = c_k x^3 + O_{k,A}\left(\frac{x^{\frac{3475}{1475}}}{(\log x)^A}\right)$$

where

$$c_k = \frac{2}{(k+1)(k+2)(k+3)}.$$

Our next result is the asymptotic of $h_n^{(r)}$:

Proposition 1.3. Let $r, \ell \ge 2$ be given natural numbers. For sufficiently large n, there are explicitly computable constants $a_{r,k}$, $b_{r,k}$, $c_{r,j}$ where $0 \le k \le r-2$ and $1 \le j \le \ell - 1$ such that

$$h_n^{(r)} = \frac{n^{r-1}\log n}{(r-1)!} + \frac{(\gamma - h_{r-1})n^{r-1}}{(r-1)!} + \sum_{k=0}^{r-2}\left(a_{r,k}n^k \log n + b_{r,k}n^k\right) + \sum_{j=1}^{\ell-1}\frac{c_{r,j}}{n^j} + O_{r,\ell}\left(\frac{1}{n^\ell}\right),$$

where γ denotes Euler's constant.

II. PRELIMINARIES

To obtain our second result, we need the following lemma.

Lemma 2.2. Let A be any positive real number. For any $\theta > -1$ and sufficiently large x, we have

$$H(x,\theta,A) = \int_2^x \frac{t^\theta dt}{\log^{\mathrm{A}} t} \ll_{\theta,A} \frac{x^{\theta+1}}{\log^A x}.$$

Proof. Integration by parts yields that

$$H(x,\theta,A) = \int_2^x \frac{t^\theta dt}{\log^A t} \ll_{\theta,A} \frac{x^{\theta+1}}{\log^A x} + \int_2^x \frac{t^\theta dt}{\log^{A+1} t}.$$

If we divide the latter integral into two parts via x^{ε} for $\varepsilon = \frac{\theta+1}{\theta+2}$, then we get that

$$\begin{aligned} H(x,\theta,A) &\ll_{\theta,A} \frac{x^{\theta+1}}{\log^A x} + \int_2^{x^\varepsilon} \frac{t^\theta dt}{\log^{A+1} t} + \int_{x^\varepsilon}^{x} \frac{t^\theta dt}{\log^{A+1} t} \\ &\ll_{\theta,A} \frac{x^{\theta+1}}{\log^A x} + \int_2^{x^\varepsilon} t^\theta dt + \frac{1}{\log(x^\varepsilon)} \int_{x^\varepsilon}^{x} \frac{t^\theta dt}{\log^{A} t} \\ &\ll_{\theta,A} \frac{x^{\theta+1}}{\log^A x} + \frac{x^{\varepsilon(\theta+1)}}{\theta+1} + \frac{H(x,\theta,A)}{\log x}. \end{aligned}$$

This gives the desired result. ■

III. PROOF OF THEOREM 1.1

Let $r \geq 1$ be fixed. For the first part of the theorem, we apply the Wiener-Ikehara theorem, as it yields the result directly in a clear way. By (2) and (4), we see that

$$\mathrm{h}_{\mathrm{n}}^{(\mathrm{r})} = \binom{n+r-1}{r-1} (\log(n+r-1) + \gamma + \mathrm{s}(n) - \mathrm{h}_{r-1})$$

where $s(n) = O(1/n)$. As we have

$$\log(n+r-1) = \log\left(n\left(1+\frac{r-1}{n}\right)\right) = \log n + \log\left(1+\frac{r-1}{n}\right) = \log n + O_r(1/n),$$

we obtain that

$$h_n^{(r)} = \binom{n+r-1}{r-1} \left(\log n + a_r + O_r(1/n)\right) \qquad (12)$$

where $a_r = \gamma - h_{r-1}$. Observe that

$$\binom{n+r-1}{r-1} = \frac{(n+1)(n+2)\cdots(n+r-1)}{(r-1)!} = \frac{n^{r-1}}{(r-1)!} + \frac{r(r-1)n^{r-2}}{2(r-1)!} + \cdots + 1 \qquad (13)$$

is a polynomial in n of order $r-1$. Thus by (12) and (13), we have

$$h_n^{(r)} = \frac{n^{r-1}\log n}{(r-1)!} + \frac{a_r n^{r-1}}{(r-1)!} + \frac{b_r n^{r-2}\log n}{(r-1)!} + O_r(n^{r-2}) \qquad (14)$$

where $b_r = r(r-1)/2$. Note also that, $b_1 = 0$ and $b_r > 0$ when $r \geq 2$. As $\alpha_r(n) = n^{r-1}\mu(n) * h_n^{(r)}$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\log n}{n^s} = -\zeta'(s) \qquad (15)$$

for $\Re(s) > 1$ and by (14), one obtains for $\Re(s) > r$ that

$$\Omega_r(s) = \sum_{n=1}^{\infty} \frac{\alpha_r(n)}{n^s} = \sum_{n=1}^{\infty} \frac{n^{r-1}\mu(n)}{n^s} \cdot \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^s} = \frac{1}{\zeta(s-r+1)} \cdot \sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n^s}$$

$$= \frac{1}{(r-1)!}\left(-\frac{\zeta'(s-r+1)}{\zeta(s-r+1)} + a_r - \frac{b_r\zeta'(s-r+2)}{\zeta(s-r+1)} + \frac{G_r(s)}{\zeta(s-r+1)}\right) \tag{16}$$

where

$$G_r(s) = \sum_{n=1}^{\infty} \frac{g_r(n)}{n^s}$$

and

$$g_r(n) = O_r(n^{r-2}). \tag{17}$$

As the Riemann Zeta function $\zeta(s)$ does not vanish on the line $\Re(s) = 1$, we get that $\Omega_r(s)$ can be extended to an analytic function in $\Re(s) \geq r$, with a simple pole at r. Note that

$$\zeta(s) = \frac{1}{s-1} + A(s)$$

where $A(s)$ is analytic in $\Re(s) \geq 1$ (actually $A(s)$ is an entire function). Therefore, $\Omega_r(s)$ has residue

$$\frac{1}{(r-1)!}$$

at r as we have

$$-\frac{\zeta'(s-r+1)}{\zeta(s-r+1)} = \frac{1}{s-r} + K(s),$$

where $K(s)$ is analytic in $\Re(s) \geq r$. Since for $\Re(s) \geq 1$ we have

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)},$$

the coefficients of

$$-\frac{\zeta'(s-r+1)}{\zeta(s-r+1)}$$

are non-negative, as they are $n^{r-1}\Lambda(n)$. By (15) and (17), the coefficients $\beta_r(n) = b_r n^{r-2}\log n + g_r(n)$ of the series $-b_r\zeta'(s-r+2) + G_r(s)$ satisfy

$$\beta_r(n) \leq 2b_r n^{r-2}\log n \tag{18}$$

for sufficiently large n. As $\mu(n)$ is bounded by 1, for n large enough we see that

$$|n^{r-1}\mu(n) * \beta_r(n)| \leq n^{r-1} * 2b_r n^{r-2}\log n$$

by (18). Therefore, the coefficients of

$$A_r(s) = -\frac{\zeta'(s-r+2)}{\zeta(s-r+1)} + \frac{G_r(s)}{\zeta(s-r+1)} - 2b_r\zeta'(s-r+2)\zeta(s-r+1)$$

are non-negative after a while. Hence, the coefficients of

$$B_r(s) = \Omega_r(s) - 2b_r\zeta'(s-r+2)\zeta(s-r+1)$$

are non-negative after a while, it is analytic in $\Re(s) \geq r$ with a simple pole of residue

$$\frac{1}{(r-1)!} - 2b_r\zeta'(2)$$

at r. Similarly, the Dirichlet series $A_r(s)$ and $-2b_r\zeta'(s-r+2)\zeta(s-r+1)$ are analytic in $\Re(s) \geq r$ with a simple pole of residue $-2b_r\zeta'(2)$ at r. So, we may apply the Wiener-Ikehara theorem to the functions $A_r(s)$, $B_r(s)$ and $-2b_r\zeta'(s-r+2)\zeta(s-r+1)$ to conclude the desired asymptotic

$$S_r(x) \sim \frac{x^r}{r!}.$$

Now we prove the second part of the theorem. Suppose that the Riemann hypothesis holds. Then as given in the introduction before, we have

$$\psi(x) = \sum_{n\leq x} \Lambda(n) = x + O_\varepsilon\left(x^{\frac{1}{2}+\varepsilon}\right) \tag{19}$$

and

$$M(x) = \sum_{n\leq x} \mu(n) = O_\varepsilon\left(x^{\frac{1}{2}+\varepsilon}\right). \tag{20}$$

By (16), we see that

$$S_r(x) = \frac{a_r}{(r-1)!} + \frac{1}{(r-1)!}\left(\sum_{n\leq x} n^{r-1}\,\Lambda(n) + \sum_{n\leq x}\big(\mu(n) * \beta_r(n)\big)\right) \tag{21}$$

where $\beta_r(n) = b_r n^{r-2}\log n + g_r(n)$ and $g_r(n) = O_r(n^{r-2})$. By partial summation and (19) we have that

$$\begin{aligned}\sum_{n\leq x} n^{r-1}\Lambda(n) &= x^{r-1}\psi(x) - (r-1)\int_1^x \psi(t)t^{r-2}\,dt \\ &= x^r - (r-1)\frac{x^r}{r} + O_{r,\varepsilon}\left(x^{r-\frac{1}{2}+\varepsilon}\right) = \frac{x^r}{r} + O_{r,\varepsilon}\left(x^{r-\frac{1}{2}+\varepsilon}\right).\end{aligned} \tag{22}$$

Now we estimate the sum $B(x) = \sum_{n\leq x}\big(\mu(n) * \beta_r(n)\big)$. Note that

$$B(x) = \sum_{n\leq x} \beta_r(n) M\left(\frac{x}{n}\right). \tag{23}$$

By (18), (20) and (23), we obtain that

$$B(x) \ll_{r,\varepsilon} \sum_{n\leq x} n^{r-2}\log n \cdot \left(\frac{x}{n}\right)^{\frac{1}{2}+\varepsilon} \ll_{r,\varepsilon} x^{r-\frac{1}{2}+\varepsilon}. \tag{24}$$

Combining (21), (22) and (24), we get that

$$S_r(x) = \frac{x^r}{r!} + O_{r,\varepsilon}\left(x^{r-\frac{1}{2}+\varepsilon}\right). \tag{25}$$

This completes the proof. ■

IV. PROOF OF THEOREM 1.2

By (11), we know that $S(n) = n^2 + R(n)$ where $R(n) = O_A\left(\frac{n^{\frac{2000}{1475}}}{(\log n)^A}\right)$. Thus,

$$\begin{aligned} T_k(x) &= \sum_{n\le x}\left(1-\frac{n}{x}\right)^k S(n) \\ &= \sum_{n\le x}\left[\left(\sum_{i=0}^{k}\binom{k}{i}(-1)^i\frac{n^i}{x^i}\right)\cdot\left(n^2+R(n)\right)\right] \\ &= \sum_{n\le x} n^2 - \frac{k}{x}\sum_{n\le x} n^3 + \cdots + \frac{(-1)^k}{x^k}\sum_{n\le x} n^{k+2} + O_{k,A}\left(\sum_{i=0}^{k}\frac{1}{x^i}\sum_{2\le n\le x} n^i\cdot\frac{n^{\frac{2000}{1475}}}{(\log n)^A}\right). \end{aligned} \tag{26}$$

Note that for any positive integer j, the sum

$$\sum_{n\le x} n^j = p_j([x])$$

where p_j is a polynomial of degree $j+1$ and its leading coefficient is $\frac{1}{j+1}$. Thus by (26), we see that

$$T_k(x) = c_k x^3 + O_k(x^2) + O_{k,A}\left(\sum_{i=0}^{k}\frac{1}{x^i}\sum_{2\le n\le x} n^i\frac{n^{\frac{2000}{1475}}}{(\log n)^A}\right) \tag{27}$$

where

$$c_k = \sum_{i=0}^{k}\frac{\binom{k}{i}(-1)^i}{i+3}. \tag{28}$$

If we put

$$G(x,\theta,A) = \sum_{2\le n\le x}\frac{n^\theta}{(\log n)^A} \ll_{\theta,A} \int_2^x \frac{t^\theta dt}{(\log t)^A},$$

we get by Lemma 2.2 that

$$G(x,\theta,A) \ll_{\theta,A} \frac{x^{\theta+1}}{(\log x)^A}. \tag{29}$$

Combining (27) and (29), we arrive at the asymptotic

$$T_k(x) = c_k x^3 + O_{k,A}\left(\frac{x^{\frac{3475}{1475}}}{(\log x)^A}\right). \tag{30}$$

Finally, to finish the theorem we show that $c_k$in (28) is $\frac{2}{(k+1)(k+2)(k+3)}$. As we have

$$x^2(1-x)^k = \sum_{i=0}^{k}\binom{k}{i}(-1)^i x^{i+2},$$

we see that $c_k = \int_0^1 x^2(1-x)^k\,dx$. By change of variables $u = 1-x$, we obtain that

$$c_k = \int_0^1 u^k(1-u)^2\,du$$

and the previous definite integral is $\frac{2}{(k+1)(k+2)(k+3)}$. ■

V. PROOF OF PROPOSITION 1.3

By (3) and the fundamental equation (4) of hyperharmonic numbers, we know that

$$\begin{aligned} h_n^{(r)} &= \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1}) \\ &\sim \frac{(n+1)\cdots(n+r-1)}{(r-1)!} \\ &\quad \cdot\left(\log(n+r-1) + \gamma + \frac{1}{2(n+r-1)} - \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(n+r-1)^{2k}} - h_{r-1}\right). \end{aligned} \tag{31}$$

If we see the binomial term as a polynomial in n, then we observe that

$$\frac{(n+1)\cdots(n+r-1)}{(r-1)!} = P_r(n) = \sum_{i=0}^{r-1} d_{r,i}n^i, \tag{32}$$

with $d_{r,r-1} = \frac{1}{(r-1)!}$. For n is sufficiently large and any positive integer t, we have

$$\frac{1}{(n+r-1)^t} = \frac{1}{n^t}\cdot\left(\frac{1}{1+\frac{r-1}{n}}\right)^t = \frac{1}{n^t}\cdot\left(\sum_{j=0}^{\infty}(-1)^j\frac{(r-1)^j}{n^j}\right)^t = \sum_{j=0}^{\infty}\frac{\beta_{j,r,t}}{n^{j+t}}, \tag{33}$$

where $\beta_{j,r,t}$'s are explicitly computable constants. Therefore we have

$$\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(n+r-1)^{2k}} = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{B_{2k}\beta_{j,r,2k}}{2kn^{j+2k}} = \sum_{k=2}^{\ell+r-1}\frac{\theta_{k,r}}{n^k} + O_{r,\ell}\left(\frac{1}{n^{\ell+r-1}}\right),$$

for some $\theta_{k,r}$'s which are explicitly computable constants. As n is sufficiently large, we also obtain that

$$\log(n+r-1) = \log n + \log\left(1+\frac{r-1}{n}\right) = \log n + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{r-1}{n}\right)^k$$
$$= \log n + \sum_{k=1}^{\ell+r-1}\frac{\alpha_{k,r}}{n^{j}} + O_{r,\ell}\left(\frac{1}{n^{\ell+r-1}}\right) \quad (34)$$

with explicitly computable constants $\alpha_{j,r}$'s. Plugging in equations (32), (33) and (34) into equation (31), we derive that

$$h_n^{(r)} = P_r(n)\cdot\left(\log n + (\gamma - h_{r-1}) + \sum_{k=1}^{\ell+r-1}\frac{\alpha_{k,r}}{n^k} + \frac{1}{2n}\sum_{k=0}^{\ell+r-1}\frac{\beta_{k,r,1}}{n^k} - \sum_{k=2}^{\ell+r-1}\frac{\theta_{k,r}}{n^k} + O_{r,\ell}\left(\frac{1}{n^{\ell+r-1}}\right)\right)$$
$$= \left(\sum_{i=0}^{r-1} d_{r,i}n^i\right)\cdot\left(\log n + (\gamma - h_{r-1}) + \sum_{k=1}^{\ell+r-1}\frac{\kappa_{k,r}}{n^k} + O_{r,\ell}\left(\frac{1}{n^{\ell+r-1}}\right)\right)$$

where $\kappa_{k,r} = \alpha_{k,r} + \frac{\beta_{k-1,r,1}}{2} - \theta_{k,r}$ with $\theta_{1,r} = 0$. Hence we get that

$$h_n^{(r)} = \frac{n^{r-1}\log n}{(r-1)!} + \frac{(\gamma - h_{r-1})n^{r-1}}{(r-1)!} + \sum_{i=0}^{r-2} d_{r,i}n^i\big(\log n + (\gamma - h_{r-1})\big) + \sum_{i=0}^{r-1}\sum_{k=1}^{\ell+r-1}\frac{d_{r,i}\kappa_{k,r}}{n^{k-i}} + O_{r,\ell}\left(\frac{1}{n^{\ell}}\right).$$

If we rearrange the terms after doing the corresponding calculations, we deduce the desired result of the proposition. ■

VI. CONCLUSION

In this note, we have seen that convolutions of the Möbius function with harmonic type sums and their summatory functions were closely related to the prime number theory and the Riemann Zeta function. Besides, asymptotic of the moment function related to the non-integer hyperharmonic numbers and the asymptotic of hyperharmonics were obtained.

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