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Ergodic Theorem in Grand Variable Exponent Lebesgue Spaces

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Abstract
We consider several fundamental properties of grand variable exponent Lebesgue spaces. Moreover, we discuss Ergodic theorems in these spaces whenever the exponent is invariant under the transformation.

Keywords: *Variable exponent grand Lebesgue space; Ergodic theorem; probability measure.*

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1. Introduction

In 1992, Iwaniec and Sbordone [14] introduced grand Lebesgue spaces $L^{p)}(\Omega)$, $(1 < p < \infty)$, on bounded sets $\Omega \subset \mathbb{R}^d$ with applications to differential equations. A generalized version $L^{p),\theta}(\Omega)$ appeared in Greco et al. [13]. These spaces has been intensively investigated recently due to several applications, see [2, 5, 9, 11, 15, 18]. Also the solutions of some nonlinear differential equations were studied in these spaces, see [10, 13]. The variable exponent Lebesgue spaces (or generalized Lebesgue spaces) $L^{p(\cdot)}$ appeared in literature for the first time in 1931 with an article written by Orlicz [17]. Kováčik and Rákosník [16] introduced the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^d)$ and Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^d)$ in higher dimensions Euclidean spaces. The spaces $L^{p(\cdot)}(\mathbb{R}^d)$ and $L^{p}(\mathbb{R}^d)$ have many common properties such as Banach space, reflexivity, separability, uniform convexity, Hölder inequalities and embeddings. A crucial difference between $L^{p(\cdot)}(\mathbb{R}^d)$ and $L^{p}(\mathbb{R}^d)$ is that the variable exponent Lebesgue space is not invariant under translation in general, see [6, Lemma 2.3] and [16, Example 2.9]. For more information, we refer [3, 7, 8]. Moreover, the space $L^{p(\cdot)}(\Omega)$ was studied by [1], where Ω is a probability space. The grand variable exponent Lebesgue space $L^{p(\cdot),\theta}(\Omega)$ was introduced and studied by Kokilashvili and Meskhi [15]. In this work, they established the boundedness of maximal and Calderon operators in these spaces. Moreover, the space $L^{p(\cdot),\theta}(\Omega)$ is not reflexive, separable, rearrangement invariant and translation invariant.

In this study, we give some basic properties of $L^{p(\cdot),\theta}(\Omega)$, and consider Birkhoff's Ergodic Theorem in the context of a certain subspace of the grand variable exponent Lebesgue space $L^{p(\cdot),\theta}(\Omega)$. So, we have more general results in sense to Gorka [12] in these spaces.

2. Notations and Preliminaries

Definition 2.1. Assume that (Ω, Σ, μ) is a probability space, that is, Σ is a σ-algebra and μ is a measure on Σ satisfying $\mu(\Omega) = 1$. Let $p(.) : \Omega \longrightarrow [1, \infty)$ be a measurable function (variable exponent) such that

$$1 \leq p^- = \operatorname*{essinf}_{x\in\Omega} p(x) \leq \operatorname*{esssup}_{x\in\Omega} p(x) = p^+ < \infty.$$

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The variable exponent Lebesgue space $L^{p(.)}(\Omega)$ is defined as the set of all measurable functions f on Ω such that $\varrho_{p(.)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \varrho_{p(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\},$$

where $\varrho_{p(.)}(f) = \int\limits_{\Omega} |f(x)|^{p(x)}\, d\mu\,(x)$. The space $L^{p(.)}(\Omega)$ is a Banach space with respect to $\|.\|_{p(.)}$. Moreover, the norm $\|.\|_{p(.)}$ coincides with the usual Lebesgue norm $\|.\|_{p}$ whenever $p(.) = p$ is a constant function. Let $p^{+} < \infty$. Then $f \in L^{p(.)}(\Omega)$ if and only if $\varrho_{p(.)}(f) < \infty$, see [16].

Definition 2.2. Let $\theta > 0$. The grand variable exponent Lebesgue spaces $L^{p(.),\theta}\,(\Omega)$ is the class of all measurable functions for which

$$\|f\|_{p(.),\theta} = \sup_{0<\varepsilon<p^{-}-1} \varepsilon^{\frac{\theta}{p^{-}-\varepsilon}}\, \|f\|_{p(.)-\varepsilon} < \infty.$$

When $p(.) = p$ is a constant function, these spaces coincide with the grand Lebesgue spaces $L^{p),\theta}\,(\Omega)$.

It is easy to see that we have

$$L^{p(.)} \hookrightarrow L^{p(.),\theta} \hookrightarrow L^{p(.)-\varepsilon} \hookrightarrow L^{1},\, 0 < \varepsilon < p^{-} - 1 \tag{2.1}$$

due to $|\Omega| < \infty$, see [4, 15, 18].

Remark 2.1. Let $C_0^{\infty}(\Omega)$ be the space of smooth functions with compact support in Ω. It is well known that $C_0^{\infty}(\Omega)$ is not dense in $L^{p(.),\theta}\,(\Omega)$, i.e., the closure of $C_0^{\infty}(\Omega)$ with respect to the $\|.\|_{p(.),\theta}$ norm does not coincide with the space $L^{p(.),\theta}\,(\Omega)$. Now, we denote $\left[L^{p(.)}\,(\Omega)\right]_{p(.),\theta}$ as the closure of $C_0^{\infty}(\Omega)$ in $L^{p(.),\theta}\,(\Omega)$. Hence this closure is obtained as

$$\left\{f \in L^{p(.),\theta}\,(\Omega) : \lim_{\varepsilon\to 0} \varepsilon^{\frac{\theta}{p^{-}-\varepsilon}}\, \|f\|_{p(.)-\varepsilon} = 0\right\}$$

, see [4, 13, 15]. Moreover, we have

$$C_0^{\infty}(\Omega) \subset L^{p(.)}\,(\Omega) \subset \left[L^{p(.)}\,(\Omega)\right]_{p(.),\theta} \text{ and } \left[L^{p(.)}\,(\Omega)\right]_{p(.),\theta} = \overline{C_0^{\infty}(\Omega)}.$$

Definition 2.3. Let (G, Σ, μ) be a measure space. A measurable function $T : G \longrightarrow G$ is called a measure-preserving transformation if

$$\mu\left(T^{-1}(A)\right) = \mu\,(A)$$

for all $A \in \Sigma$.

3. Main Results

In the following theorem, we obtain more general result than [12, Theorem 3.1] since $L^{p(.)}\,(\Omega) \subset \left[L^{p(.)}\,(\Omega)\right]_{p(.),\theta} \subset L^{p(.),\theta}\,(\Omega)$.

Theorem 3.1. *Let (Ω, Σ, μ) be a probability space and $T : \Omega \longrightarrow \Omega$ a measure preserving transformation. Moreover, if $p(.)$ is T-invariant, i.e., $p(T(.)) = p(.)$, then*

(i) The limit

$$f_{av}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f\left(T^{j}(x)\right)$$

exists for all $f \in L^{p(.),\theta}\,(\Omega)$ and almost each point $x \in \Omega$, and $f_{av} \in L^{p(.),\theta}\,(\Omega)$.

(ii) For every $f \in L^{p(.),\theta}\,(\Omega)$, we have

$$f_{av}(x) = f_{av}\,(T(x))\,, \tag{3.1}$$

$$\int\limits_{\Omega} f_{av} d\mu = \int\limits_{\Omega} f d\mu. \tag{3.2}$$

(iii) For all $f \in \left[L^{p(.)}(\Omega)\right]_{p(.),\theta}$*, we get*

$$\lim_{n\to\infty}\left\Vert f_{av}-\frac{1}{n}\sum_{j=0}^{n-1}f\circ T^{j}\right\Vert_{p(.),\theta}=0. \tag{3.3}$$

Proof. By (2.1), the existence of limit $f_{av}(x)$ for almost every point in Ω follows from the standard Birkhoof's Theorem, see [12]. By Fatou's Lemma and the definition of the norm $\Vert.\Vert_{p(.),\theta}$, we have

$$\begin{aligned}\int_{\Omega}|f_{av}(x)|^{p(x)-\varepsilon}\,d\mu &= \int_{\Omega}\left|\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}f\left(T^{j}(x)\right)\right|^{p(x)-\varepsilon}d\mu\\ &\le \int_{\Omega}\lim_{n\to\infty}\left(\frac{1}{n}\sum_{j=0}^{n-1}\left|f\left(T^{j}(x)\right)\right|\right)^{p(x)-\varepsilon}d\mu\\ &\le \liminf_{n\to\infty}\int_{\Omega}\left(\frac{1}{n}\sum_{j=0}^{n-1}\left|f\left(T^{j}(x)\right)\right|\right)^{p(x)-\varepsilon}d\mu\\ &\le \liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int_{\Omega}\left|f\left(T^{j}(x)\right)\right|^{p(x)-\varepsilon}d\mu\end{aligned}$$

for any $\varepsilon\in(0,p^{-}-1)$. Here, we used convexity and Jensen inequality in last step. Moreover, since T is a measure preserving map and $p(.)$ is T-invariant, we get

$$\int_{\Omega}|f(T(x))|^{p(x)-\varepsilon}\,d\mu=\int_{\Omega}|f(T(x))|^{p(T(x))-\varepsilon}\,d\mu=\int_{\Omega}|f(x)|^{p(x)-\varepsilon}\,d\mu.$$

It follows that

$$\int_{\Omega}|f_{av}(x)|^{p(x)-\varepsilon}\,d\mu\le\int_{\Omega}|f(x)|^{p(x)-\varepsilon}\,d\mu<\infty. \tag{3.4}$$

Thus, we obtain

$$\begin{aligned}\Vert f_{av}\Vert_{p(.),\theta} &= \sup_{0<\varepsilon<p^{-}-1}\varepsilon^{\frac{\theta}{p^{-}-\varepsilon}}\Vert f_{av}\Vert_{p(.)-\varepsilon}\\ &\le \sup_{0<\varepsilon<p^{-}-1}\varepsilon^{\frac{\theta}{p^{-}-\varepsilon}}\Vert f\Vert_{p(.)-\varepsilon}<\infty\end{aligned}$$

and $f_{av}\in L^{p(.),\theta}(\Omega)$. This completes *(i)*. By the Ergodic Theorem in the classical Lebesgue spaces (see [12]), we have (3.1) and (3.2) immediately. In order to prove (3.3), we assume that $f\in C_{0}^{\infty}(\Omega)$. Thus, $f\in L^{\infty}(\Omega)$ and

$$\begin{aligned}\lim_{n\to\infty}\left|f_{av}(x)-\frac{1}{n}\sum_{j=0}^{n-1}f\left(T^{j}(x)\right)\right|^{p(x)-\varepsilon} &= 0\text{, a.e.}\\ \Vert f_{av}\Vert_{L^{\infty}(\Omega)} &\le \Vert f\Vert_{L^{\infty}(\Omega)}\end{aligned}$$

for any $\varepsilon\in(0,p^{-}-1)$. Therefore, we have

$$\begin{aligned}\left|f_{av}(x)-\frac{1}{n}\sum_{j=0}^{n-1}f\left(T^{j}(x)\right)\right|^{p(x)-\varepsilon} &\le \left|\Vert f\Vert_{L^{\infty}(\Omega)}+\frac{1}{n}\sum_{j=0}^{n-1}\left\Vert f\left(T^{j}\right)\right\Vert_{L^{\infty}(\Omega)}\right|^{p(x)-\varepsilon}\\ &\le 2^{p^{+}}\left(\Vert f\Vert_{L^{\infty}(G)}+1\right)^{p^{+}-\varepsilon}\in L^{1}(\Omega).\end{aligned}$$

Hence, by Lebesgue dominated convergence theorem (see [7]), we have (3.3) and provided $f \in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $\left[L^{p(.)}(\Omega)\right]_{p(.),\theta}$ with respect to the norm $\|.\|_{p(.),\theta}$, for any $f \in \left[L^{p(.)}(\Omega)\right]_{p(.),\theta}$ and $\eta > 0$ there is a $g \in C_0^\infty(\Omega)$ such that

$$\|f - g\|_{p(.),\theta} < \eta. \tag{3.5}$$

By the previous step, there is an n_0 such that

$$\left\| g_{av} - \frac{1}{n} \sum_{j=0}^{n-1} g \circ T^j \right\|_{p(.)-\varepsilon} < \eta \tag{3.6}$$

for $n \geq n_0$ and $\varepsilon \in (0, p^- - 1)$. Hence, we have

$$\left\| g_{av} - \frac{1}{n} \sum_{j=0}^{n-1} g \circ T^j \right\|_{p(.),\theta} < \eta \tag{3.7}$$

by (3.6) and the definition of the norm $\|.\|_{p(.),\theta}$. This follows from (3.4), (3.5) and (3.7) that

$$\begin{aligned}
\left\| f_{av} - \frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j \right\|_{p(.),\theta} &\leq \|f_{av} - g_{av}\|_{p(.),\theta} + \left\| g_{av} - \frac{1}{n} \sum_{j=0}^{n-1} g \circ T^j \right\|_{p(.),\theta} \\
&\quad + \left\| \frac{1}{n} \sum_{j=0}^{n-1} (f - g) \circ T^j \right\|_{p(.),\theta} \\
&\leq 2\|f - g\|_{p(.),\theta} + \left\| g_{av} - \frac{1}{n} \sum_{j=0}^{n-1} g \circ T^j \right\|_{p(.),\theta} \\
&< \frac{\eta}{2} + \frac{\eta}{2} = \eta.
\end{aligned}$$

That is the desired result. □

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