OSCILLATION RESULTS FOR SECOND ORDER HALF-LINEAR FUNCTIONAL DYNAMIC EQUATIONS WITH UNBOUNDED NEUTRAL COEFFICIENTS ON TIME SCALES

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ABSTRACT. This study aims to present some new sufficient conditions for the oscillatory behavior of solutions to a class of second order half-linear functional dynamic equations with mixed neutral term i.e., the neutral term contains both retarded and advanced arguments. The results obtained are applicable in the case where the studied equation has unbounded neutral coefficients and they are new even for the linear case. Illustrative examples are also provided.

1. INTRODUCTION

In this study, we are concerned with the oscillation of second order half-linear mixed neutral dynamic equations of the form

$$\left(r(t)\left(y^{\Delta}(t)\right)^{\beta}\right)^{\Delta} + \sum_{i=1}^{n} q_i(t)x^{\beta}\left(h_i(t)\right) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}} \tag{1.1}$$

where $n \geq 1$ is an integer, β is a ratio of positive odd integers, $\mathbb{T}$ is a time scale unbounded above with $t_0 \in \mathbb{T}$, and

$$y(t) := x(t) + p_1(t)x\left(\tau_1(t)\right) + p_2(t)x\left(\tau_2(t)\right). \tag{1.2}$$

For some basic facts on time scale calculus and dynamic equations on time scales, one may consult the excellent texts by Bohner and Peterson [8, 9]. Throughout this study it is assumed that the reader is familiar with time scale calculus, and the following conditions are always satisfied:

(i) $q_i : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ are nonnegative rd-continuous functions such that not all of the $q_i(t)$ vanish in a neighborhood of infinity for $i = 1, 2, ..., n$ and $r : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is a positive rd-continuous function with

$$\int_{t_0}^{\infty} r^{-1/\beta}(s)\Delta s = \infty;$$

(ii) $\tau_1, \tau_2, h_i : [t_0, \infty)_{\mathbb{T}} \to \mathbb{T}$ are rd-continuous functions such that τ_1 and τ_2 are strictly increasing, $\tau_1(t) < t$, $\tau_2(t) > t$ and $\lim_{t \to \infty} \tau_1(t) = \lim_{t \to \infty} h_i(t) = \infty$, for $i = 1, 2, ..., n$;

and, either

(iii$_{\mathfrak{a}}$) $p_1, p_2 : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ are rd-continuous functions with $p_1(t) \geq 0$, $p_2(t) \geq 1$ and $p_2(t) \not\equiv 1$ eventually;

or

(iii$_{\mathfrak{b}}$) $p_1, p_2 : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ are rd-continuous functions with $p_2(t) \geq 0$, $p_1(t) \geq 1$ and $p_1(t) \not\equiv 1$ eventually.

Wherever we write "$t \geq t_n$" we mean "$t \in [t_n, \infty)_{\mathbb{T}}$".

By a solution of equation (1.1) we mean a function $x \in C_{rd}([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$ which has the properties $y \in C_{rd}^1([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$ and $r\left(y^\Delta\right)^\beta \in C_{rd}^1([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$, and satisfies equation (1.1) on $[T_x, \infty)_{\mathbb{T}}$. Without further mention, we will assume throughout that every solution $x(t)$ of (1.1) under consideration here is continuable to the right and nontrivial, i.e., $x(t)$ is defined on some ray $[T_x, \infty)_{\mathbb{T}}$, for some $T_x \geq t_0$, and

$$\sup\{|x(t)| : t \geq T_1\} > 0 \quad \text{for all } T_1 \geq T_x.$$

We make the standing hypothesis that (1.1) admits such solutions. Such a solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros on $[T_x, \infty)_{\mathbb{T}}$ and otherwise it is called *nonoscillatory*. Equation (1.1) is said to be *oscillatory* if all its solutions are *oscillatory*.

The study of oscillation of the solutions of neutral differential and difference equations presents a strong theoretical interest. One reason for this is that they arise in several areas of applied mathematics including circuit theory, bifurcation analysis, population dynamics, stability theory, the dynamics of delayed network systems and others. Besides, these equations are used in the analysis of computer networks containing lossless transmission lines, as in high speed networks where lossless transmission lines serve to connect switching circuits in the network. Also, second-order neutral delay differential equations are of great interest in biology in explaining self-balancing of the human body and in robotics in constructing biped robots [11, 13, 15, 25]. Interested readers can refer to the books by Hale [15] and, Kolmanovskii and Myshkis [22] for more applications in science and technology.

Note that equation (1.1) with $\mathbb{T} = \mathbb{R}$, $n = 2$, $h_1(t) < t$ and $h_2(t) > t$ were encountered in the study of vibrating masses attached to an elastic bar [15, 23]. Since it has some direct applications in science, the oscillatory behavior of equation (1.1) and it's special and more general forms have been studied by numerous authors utilizing different methods. In reviewing the related literature, most of such results are concerned with the cases where the functions $p_j(t)$ are constant or bounded functions, for $j = 1, 2$; see for example [6, 10, 12, 16, 19, 23, 28, 30, 36, 37] and the

references cited therein. However, to the best of our knowledge, there does not appear to be any oscillation results for second order mixed neutral dynamic equations in the case where the neutral term includes unbounded neutral coefficients.

Motivated by the papers mentioned above, the purpose of this study is to establish some new oscillation criteria that can be applied in the case where equation (1.1) has unbounded neutral coefficients, that is, $p_1(t) \to \infty$ and/or $p_2(t) \to \infty$ as $t \to \infty$. Moreover, it should be also pointed out that the results obtained here are new even for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, for the linear case when $\beta = 1$, for $r(t) = 1$, for $n = 1$ and for discrete deviating arguments such as $\tau_1(t) = t - a$, $\tau_2(t) = t + b$ and $h_i(t) = t \mp c$ with $a, b, c > 0$ and $i = 1, 2, ..., n$.

For convenience, we will use the following notations:

$$\gamma = \frac{\beta + 1}{\beta}, \qquad \theta_+(t) := \max\{0, \theta(t)\} \quad \text{and} \quad A(t, t_1) := \int_{t_1}^{t} r^{-1/\beta}(s)\Delta s,$$

for any rd-continuous function θ.

The following lemma is required in our main results. Since the proof is standard we omit the details here.

Lemma 1.1. *Assume that conditions* (i)-(iii$_a$) *(or* (i), (ii), (iii$_b$)*) hold and $x(t)$ is an eventually positive solution of* (1.1). *Then $y(t)$ satisfies $y(t) > 0$, $y^\Delta(t) > 0$ and $\left(r(t)\left(y^\Delta(t)\right)^\beta\right)^\Delta < 0$, for all t large enough.*

Lemma 1.2. [17] *If X and Y are nonnegative and $\lambda > 1$, then*

$$\lambda XY^{\lambda-1} - X^\lambda \leq (\lambda - 1)Y^\lambda,$$

where equality holds if and only if $X = Y$.

2. Oscillation results when (iii$_a$) holds

In this section, we establish some new criteria for the oscillation of (1.1) in the cases where $h_i(t) \leq \tau_2(t)$ and $h_i(t) \geq \tau_2(t)$ for $i = 1, 2, ..., n$, respectively. For notational purposes, we let

$$\psi(t) := \frac{1}{p_2(\tau_2^{-1}(t))}\left[1 - \frac{1}{p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} - \frac{p_1(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}\right]$$

where τ_2^{-1} denotes the inverse function of τ_2, and throughout this section we assume that $\psi(t) > 0$ for all sufficiently large t.

Theorem 2.1. *Assume that conditions* (i)-(iii$_a$) *hold and $h_i(t) \leq \tau_2(t)$ for $i = 1, 2, ..., n$. If there exists a positive function $\eta \in C_{rd}^1\left([t_0, \infty)_\mathbb{T}, \mathbb{R}\right)$ such that*

$$\limsup_{t \to \infty} \int_{T}^{t}\left(\Psi_1(s) - \frac{r(s)\left(\eta^\Delta(s)\right)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^\beta(s)}\right)\Delta s = \infty \qquad (2.1)$$

for $T > t_1$ where $t_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large and

$$\Psi_1(t) = \eta(t) \left(\sum_{i=1}^{n} q_i(t) \psi^\beta(h_i(t)) \frac{A^\beta\left(\tau_2^{-1}(h_i(t)), t_1\right)}{A^\beta(t, t_1)} \right),$$

then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$ and $x(h_i(t)) > 0$ for $t \geq t_1$ and $i = 1, 2, ..., n$. The proof if $x(t)$ is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper. In view of Lemma 1.1, it is obvious that

$$y(t) > 0, \ y^\Delta(t) > 0 \text{ and } \left(r(t)\left(y^\Delta(t)\right)^\beta\right)^\Delta < 0 \quad \text{for } t \geq t_1.$$

Since $r(t)\left(y^\Delta(t)\right)^\beta$ is decreasing, we have

$$\begin{aligned}
y(t) &= y(t_1) + \int_{t_1}^{t} \frac{\left(r(s)\left(y^\Delta(s)\right)^\beta\right)^{1/\beta}}{r^{1/\beta}(s)} \Delta s \\
&\geq \left(r(t)\left(y^\Delta(t)\right)^\beta\right)^{1/\beta} A(t, t_1) \quad \text{for } t \geq t_1.
\end{aligned} \tag{2.2}$$

Thus

$$\left(\frac{y(t)}{A(t, t_1)}\right)^\Delta \leq 0, \tag{2.3}$$

i.e., $y(t)/A(t, t_1)$ is nonincreasing on $[t_1, \infty)_{\mathbb{T}}$. From the definition of $y(t)$, we get

$$\begin{aligned}
x(t) &= \frac{1}{p_2(\tau_2^{-1}(t))} \left(y(\tau_2^{-1}(t)) - x(\tau_2^{-1}(t)) - p_1(\tau_2^{-1}(t)) x(\tau_1(\tau_2^{-1}(t))) \right) \\
&= \frac{y(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))} - \frac{y(\tau_2^{-1}(\tau_2^{-1}(t))) - x(\tau_2^{-1}(\tau_2^{-1}(t)))}{p_2(\tau_2^{-1}(t)) p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} \\
&\quad + \frac{p_1(\tau_2^{-1}(\tau_2^{-1}(t))) x(\tau_1(\tau_2^{-1}(\tau_2^{-1}(t))))}{p_2(\tau_2^{-1}(t)) p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} \\
&\quad - \frac{p_1(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))} \left[\frac{y(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t)))) - x(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}{p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))} \right. \\
&\quad \left. - \frac{p_1(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t)))) x(\tau_1(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t)))))}{p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))} \right] \\
&\geq \frac{y(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(t))} - \frac{y(\tau_2^{-1}(\tau_2^{-1}(t)))}{p_2(\tau_2^{-1}(t)) p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} - \frac{p_1(\tau_2^{-1}(t)) y(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}{p_2(\tau_2^{-1}(t)) p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}.
\end{aligned} \tag{2.4}$$

Using the fact that the functions y, τ_1 and τ_2 are strictly increasing, and noting that $\tau_1(t) < t < \tau_2(t)$, we get

$$y(\tau_2^{-1}(\tau_2^{-1}(t))) < y(\tau_2^{-1}(t)) \tag{2.5}$$

and

$$y(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t)))) < y(\tau_2^{-1}(t)). \tag{2.6}$$

Using (2.5) and (2.6) in (2.4) gives

$$x(t) \geq \frac{1}{p_2(\tau_2^{-1}(t))}\left[1 - \frac{1}{p_2(\tau_2^{-1}(\tau_2^{-1}(t)))} - \frac{p_1(\tau_2^{-1}(t))}{p_2(\tau_2^{-1}(\tau_1(\tau_2^{-1}(t))))}\right] y(\tau_2^{-1}(t)) \tag{2.7}$$

for $t \geq t_1$. Since $\lim_{t\to\infty} h_i(t) = \infty$, we can choose $t_2 \geq t_1$ such that all $h_i(t) \geq t_1$ for $t \geq t_2$, where $i = 1, 2, ..., n$. Thus, from (2.7) we obtain

$$x(h_i(t)) \geq \psi(h_i(t))y\left(\tau_2^{-1}(h_i(t))\right), \qquad i = 1, 2, ..., n \tag{2.8}$$

for $t \geq t_2$. Using (2.8) in (1.1) gives

$$\left(r(t)\left(y^\Delta(t)\right)^\beta\right)^\Delta + \sum_{i=1}^{n} q_i(t)\psi^\beta\left(h_i(t)\right)y^\beta\left(\tau_2^{-1}(h_i(t))\right) \leq 0. \tag{2.9}$$

Define the function $\omega(t)$ by the Riccati substitution

$$\omega(t) := \eta(t)\frac{r(t)\left(y^\Delta(t)\right)^\beta}{y^\beta(t)} \quad \text{for } t \geq t_2. \tag{2.10}$$

Clearly $\omega(t) > 0$ and from (2.9) we see that

$$
\begin{aligned}
\omega^\Delta(t) &= \frac{\eta(t)}{y^\beta(t)}\left(r(t)\left(y^\Delta(t)\right)^\beta\right)^\Delta + \left(\frac{\eta(t)}{y^\beta(t)}\right)^\Delta r\left(\sigma(t)\right)\left(y^\Delta\left(\sigma(t)\right)\right)^\beta \\
&\leq \eta^\Delta(t)\frac{r\left(\sigma(t)\right)\left(y^\Delta\left(\sigma(t)\right)\right)^\beta}{y^\beta\left(\sigma(t)\right)} - \eta(t)\frac{r\left(\sigma(t)\right)\left(y^\Delta\left(\sigma(t)\right)\right)^\beta\left(y^\beta(t)\right)^\Delta}{y^\beta(t)y^\beta\left(\sigma(t)\right)} \\
&\quad -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^\beta\left(h_i(t)\right)\frac{y^\beta\left(\tau_2^{-1}(h_i(t))\right)}{y^\beta(t)}\right) \quad \text{for } t \geq t_2, \tag{2.11}
\end{aligned}
$$

where $\sigma(t)$ is the forward jump operator on time scale $\mathbb{T}$. Using the fact $y(t)/A(t, t_1)$ is nonincreasing, and noting that $h_i(t) \leq \tau_2(t)$ implies $\tau_2^{-1}(h_i(t)) \leq t$, we obtain

$$\frac{y\left(\tau_2^{-1}(h_i(t))\right)}{y(t)} \geq \frac{A\left(\tau_2^{-1}(h_i(t)), t_1\right)}{A\left(t, t_1\right)}. \tag{2.12}$$

for $i = 1, 2, ..., n$. Substituting (2.12) into (2.11) gives

$$
\begin{aligned}
\omega^\Delta(t) &\leq \eta^\Delta(t)\frac{r\left(\sigma(t)\right)\left(y^\Delta\left(\sigma(t)\right)\right)^\beta}{y^\beta\left(\sigma(t)\right)} - \eta(t)\frac{r\left(\sigma(t)\right)\left(y^\Delta\left(\sigma(t)\right)\right)^\beta\left(y^\beta(t)\right)^\Delta}{y^\beta(t)y^\beta\left(\sigma(t)\right)} \\
&\quad -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^\beta\left(h_i(t)\right)\frac{A^\beta\left(\tau_2^{-1}(h_i(t)), t_1\right)}{A^\beta\left(t, t_1\right)}\right) \quad \text{for } t \geq t_2. \tag{2.13}
\end{aligned}
$$

From [8, Theorem 1.90], we obtain

$$\left(y^\beta(t)\right)^\Delta \geq \begin{cases} \beta y^{\beta-1}\big(\sigma(t)\big)y^\Delta(t), & \text{if } 0 < \beta \leq 1, \\ \beta y^{\beta-1}(t)y^\Delta(t), & \text{if } \beta > 1. \end{cases} \qquad (2.14)$$

If $0 < \beta \leq 1$, then we have from (2.13) and (2.14) that

$$\begin{aligned} \omega^\Delta(t) \quad \leq \quad & \eta^\Delta(t)\frac{r\big(\sigma(t)\big)\big(y^\Delta\big(\sigma(t)\big)\big)^\beta}{y^\beta\big(\sigma(t)\big)} - \beta\eta(t)\frac{r\big(\sigma(t)\big)\big(y^\Delta\big(\sigma(t)\big)\big)^\beta y^\Delta(t)}{y^\beta(t)y\big(\sigma(t)\big)} \\ & -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^\beta\big(h_i(t)\big)\frac{A^\beta\big(\tau_2^{-1}(h_i(t)),t_1\big)}{A^\beta\big(t,t_1\big)}\right). \end{aligned} \qquad (2.15)$$

If $\beta > 1$, then we have from (2.13) and (2.14) that

$$\begin{aligned} \omega^\Delta(t) \quad \leq \quad & \eta^\Delta(t)\frac{r\big(\sigma(t)\big)\big(y^\Delta\big(\sigma(t)\big)\big)^\beta}{y^\beta\big(\sigma(t)\big)} - \beta\eta(t)\frac{r\big(\sigma(t)\big)\big(y^\Delta\big(\sigma(t)\big)\big)^\beta y^\Delta(t)}{y(t)y^\beta\big(\sigma(t)\big)} \\ & -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^\beta\big(h_i(t)\big)\frac{A^\beta\big(\tau_2^{-1}(h_i(t)),t_1\big)}{A^\beta\big(t,t_1\big)}\right). \end{aligned} \qquad (2.16)$$

Using the fact that $y(t)$ is increasing and $r(t)\big(y^\Delta(t)\big)^\beta$ is decreasing, we get $y(t) \leq y\big(\sigma(t)\big)$ and $y^\Delta(t) \geq r^{1/\beta}(\sigma(t))y^\Delta\big(\sigma(t)\big)/r^{1/\beta}(t)$, respectively. Thus, combining (2.15) and (2.16) we obtain for $\beta > 0$ and $t \geq t_2$,

$$\begin{aligned} \omega^\Delta(t) \quad \leq \quad & \eta^\Delta(t)\frac{r\big(\sigma(t)\big)\big(y^\Delta\big(\sigma(t)\big)\big)^\beta}{y^\beta\big(\sigma(t)\big)} - \beta\eta(t)\frac{r^\gamma\big(\sigma(t)\big)\big(y^\Delta\big(\sigma(t)\big)\big)^{\beta+1}}{r^{1/\beta}(t)y^{\beta+1}\big(\sigma(t)\big)} \\ & -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^\beta\big(h_i(t)\big)\frac{A^\beta\big(\tau_2^{-1}(h_i(t)),t_1\big)}{A^\beta\big(t,t_1\big)}\right) \\ = \quad & \eta^\Delta(t)\frac{\omega\big(\sigma(t)\big)}{\eta\big(\sigma(t)\big)} - \beta\frac{\eta(t)}{r^{1/\beta}(t)}\frac{\omega^\gamma\big(\sigma(t)\big)}{\eta^\gamma\big(\sigma(t)\big)} \\ & -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^\beta\big(h_i(t)\big)\frac{A^\beta\big(\tau_2^{-1}(h_i(t)),t_1\big)}{A^\beta\big(t,t_1\big)}\right). \end{aligned} \qquad (2.17)$$

If we apply Lemma 1.2 with

$$X = \frac{[\beta\eta(t)]^{1/\gamma}}{\left[r^{1/\beta}(t)\eta^\gamma\big(\sigma(t)\big)\right]^{1/\gamma}}\omega\big(\sigma(t)\big) \ \text{ and } \ Y = \left[\frac{\beta}{\beta+1}\frac{\left[r^{1/\beta}(t)\eta^\gamma\big(\sigma(t)\big)\right]^{1/\gamma}}{[\beta\eta(t)]^{1/\gamma}}\frac{\eta^\Delta(t)}{\eta\big(\sigma(t)\big)}\right]^\beta$$

we see that

$$\eta^\Delta(t)\frac{\omega\big(\sigma(t)\big)}{\eta\big(\sigma(t)\big)} - \beta\frac{\eta(t)}{r^{1/\beta}(t)}\frac{\omega^\gamma\big(\sigma(t)\big)}{\eta^\gamma\big(\sigma(t)\big)} \leq \frac{1}{(\beta+1)^{\beta+1}}\frac{r(t)\big(\eta^\Delta(t)\big)^{\beta+1}}{\eta^\beta(t)}. \qquad (2.18)$$

Using (2.18) in (2.17) gives

$$\omega^\Delta(t) \le \frac{r(t)\big(\eta^\Delta(t)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^\beta(t)} - \eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^\beta\big(h_i(t)\big)\frac{A^\beta\big(\tau_2^{-1}(h_i(t)),t_1\big)}{A^\beta\big(t,t_1\big)}\right).$$

Integrating the latter inequality from t_2 to t yields

$$\int_{t_2}^{t}\left(\eta(s)\left(\sum_{i=1}^{n} q_i(s)\psi^\beta\big(h_i(s)\big)\frac{A^\beta\big(\tau_2^{-1}(h_i(s)),t_1\big)}{A^\beta\big(s,t_1\big)}\right) - \frac{r(s)\big(\eta^\Delta(s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^\beta(s)}\right)\Delta s \le \omega(t_2)$$

which contradicts (2.1) and completes the proof.   □

Following, we give an oscillation criterion for (1.1) by using the integral averaging technique due to Philos [27]. First we need to introduce, the function class $\mathcal{P}$. Let $D_0 \equiv \{(t,s) \in \mathbb{T}^2 : t > s \ge t_0\}$, $D \equiv \{(t,s) \in \mathbb{T}^2 : t \ge s \ge t_0\}$ and $H,h \in C_{rd}(D,\mathbb{R})$. The function $H \in C_{rd}(D,\mathbb{R})$ is said to belongs to the class $\mathcal{P}$ if

($\mathcal{P}_1$) $H(t,t) = 0$ for $t \ge t_0$ and $H(t,s) > 0$ on D_0,
($\mathcal{P}_2$) H has a nonpositive rd-continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ on D_0
          with respect to second variable and satisfies

$$H^{\Delta_s}(t,s) + H(t,s)\frac{\eta^\Delta(s)}{\eta(\sigma(s))} = \frac{h(t,s)}{\eta(\sigma(s))}H^{1/\gamma}(t,s),$$

where the function η is as in Theorem 2.1.

Theorem 2.2. *Assume that conditions* (i)-(iii$_a$) *hold and* $h_i(t) \le \tau_2(t)$ *for* $i = 1,2,...,n$. *Suppose also that there exist functions* $\eta \in C_{rd}^1([t_0,\infty)_\mathbb{T},\mathbb{R}^+)$ *and* $H,h \in C_{rd}(D,\mathbb{R})$ *with* H *belongs to the class* $\mathcal{P}$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t}\left[H(t,s)\,\Psi_1(s) - \frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^\beta(s)}\right]\Delta s = \infty, \qquad (2.19)$$

where $\Psi_1(t)$ *is as in Theorem 2.1 and* $t_* > t_1$ *for sufficiently large* $t_1 \in [t_0,\infty)_\mathbb{T}$. *Then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)_\mathbb{T}$ such that $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$ and $x(h_i(t)) > 0$ for $t \ge t_1$ and $i = 1,2,...,n$. Proceeding as in the proof of Theorem 2.1, we again arrive at (2.17) for $t \ge t_2$. In view of ($\mathcal{P}_1$) and ($\mathcal{P}_2$), it follows from (2.17) that

$$\int_{t_2}^{t} H(t,s)\eta(s)\left(\sum_{i=1}^{n} q_i(s)\psi^\beta\big(h_i(s)\big)\frac{A^\beta\big(\tau_2^{-1}(h_i(s)),t_1\big)}{A^\beta\big(s,t_1\big)}\right)\Delta s \le -\int_{t_2}^{t} H(t,s)\omega^\Delta(s)\Delta s$$

$$+ \int_{t_2}^{t} H(t,s)\eta^\Delta(s)\frac{\omega\big(\sigma(s)\big)}{\eta\big(\sigma(s)\big)}\Delta s - \int_{t_2}^{t} H(t,s)\beta\frac{\eta(s)}{r^{1/\beta}(s)}\frac{\omega^\gamma\big(\sigma(s)\big)}{\eta^\gamma\big(\sigma(s)\big)}\Delta s.$$

Using the integration by parts formula on time scales, we obtain

$$\int_{t_2}^t H(t,s)\,\Psi_1(s)\Delta s \le H(t,t_2)\omega(t_2) + \int_{t_2}^t H^{\Delta_s}(t,s)\omega\big(\sigma(s)\big)\Delta s$$

$$+ \int_{t_2}^t H(t,s)\eta^\Delta(s)\frac{\omega\big(\sigma(s)\big)}{\eta\big(\sigma(s)\big)}\Delta s - \int_{t_2}^t H(t,s)\beta\frac{\eta(s)}{r^{1/\beta}(s)}\frac{\omega^\gamma\big(\sigma(s)\big)}{\eta^\gamma\big(\sigma(s)\big)}\Delta s$$

$$\le H(t,t_2)\omega(t_2) + \int_{t_2}^t \frac{h_+(t,s)}{\eta\big(\sigma(s)\big)}H^{1/\gamma}(t,s)\omega\big(\sigma(s)\big)\Delta s$$

$$- \int_{t_2}^t H(t,s)\beta\frac{\eta(s)}{r^{1/\beta}(s)}\frac{\omega^\gamma\big(\sigma(s)\big)}{\eta^\gamma\big(\sigma(s)\big)}\Delta s. \qquad (2.20)$$

Applying Lemma 1.2 with

$$X = \frac{[H(t,s)\beta\eta(s)]^{1/\gamma}}{\big[r^{1/\beta}(s)\eta^\gamma\,(\sigma(s))\big]^{1/\gamma}}\omega\big(\sigma(s)\big) \ \ \text{and} \ \ Y = \left[\frac{\beta}{\beta+1}\frac{\big[r^{1/\beta}(s)\eta^\gamma\,(\sigma(s))\big]^{1/\gamma}}{[\beta\eta(s)]^{1/\gamma}}\frac{h_+(t,s)}{\eta\,(\sigma(s))}\right]^\beta$$

we obtain,

$$\frac{h_+(t,s)}{\eta\big(\sigma(s)\big)}H^{1/\gamma}(t,s)\omega\big(\sigma(s)\big) - H(t,s)\beta\frac{\eta(s)}{r^{1/\beta}(s)}\frac{\omega^\gamma\big(\sigma(s)\big)}{\eta^\gamma\big(\sigma(s)\big)} \le \frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^\beta(s)}.$$

Substituting the latter inequality into (2.20), we conclude that

$$\int_{t_2}^t \left[H\,(t,s)\,\Psi_1\,(s) - \frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^\beta(s)}\right]\Delta s \le H(t,t_2)\omega(t_2), \qquad (2.21)$$

which contradicts (2.19). This proves the theorem. $\qquad\square$

From Theorem 2.2, we immediately have the following oscillation criterion.

Corollary 2.1. *Suppose that all conditions of Theorem 2.2 are satisfied with (2.19) replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^t H\,(t,s)\,\eta(s)\left(\sum_{i=1}^n q_i(s)\psi^\beta\big(h_i(s)\big)\frac{A^\beta\big(\tau_2^{-1}(h_i(s)),t_1\big)}{A^\beta\,(s,t_1)}\right)\Delta s = \infty$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^t \frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{\eta^\beta\,(s)}\Delta s < \infty,$$

then equation (1.1) is oscillatory.

Theorem 2.3. *Assume that conditions (i)-(iii$_a$) hold and $h_i(t) \ge \tau_2(t)$ for $i = 1, 2, ..., n$. If there exists a positive function $\eta \in C_{rd}^1\,([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that*

$$\limsup_{t\to\infty}\int_T^t \left(\eta(s)\left(\sum_{i=1}^n q_i(s)\psi^\beta\big(h_i(s)\big)\right) - \frac{r(s)\big(\eta^\Delta(s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^\beta(s)}\right)\Delta s = \infty, \qquad (2.22)$$

for $T > t_1$ with $t_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large, then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$ and $x(h_i(t)) > 0$ for $t \geq t_1$ and $i = 1, 2, ..., n$. Proceeding as in the proof of Theorem 2.1, we again arrive at (2.11) for $t \geq t_2$. Using the fact that $\tau_2(t)$ is strictly increasing and noting that $h_i(t) \geq \tau_2(t)$, we have

$$\tau_2^{-1}(h_i(t)) \geq t$$

for $i = 1, 2, ..., n$. So, from the fact that y is increasing, we obtain

$$\frac{y\left(\tau_2^{-1}(h_i(t))\right)}{y(t)} \geq 1, \qquad i = 1, 2, ..., n. \tag{2.23}$$

Using (2.23) in (2.11) yields

$$\begin{aligned}
\omega^\Delta(t) &\leq \eta^\Delta(t) \frac{r\left(\sigma(t)\right)\left(y^\Delta\left(\sigma(t)\right)\right)^\beta}{y^\beta\left(\sigma(t)\right)} - \eta(t) \frac{r\left(\sigma(t)\right)\left(y^\Delta\left(\sigma(t)\right)\right)^\beta \left(y^\beta(t)\right)^\Delta}{y^\beta(t) y^\beta\left(\sigma(t)\right)} \\
&\quad - \eta(t)\left(\sum_{i=1}^{n} q_i(t)\psi^\beta\left(h_i(t)\right)\right) \quad \text{for } t \geq t_2. \tag{2.24}
\end{aligned}$$

The rest of proof is similar to that of Theorem 2.1, and so we omit the details. $\square$

Theorem 2.4. *Assume that conditions* (i)-(iii$_\mathfrak{a}$) *hold and* $h_i(t) \geq \tau_2(t)$ *for* $i = 1, 2, ..., n$. *Suppose also that there exist functions* $\eta \in C_{rd}^1\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$ *and* $H, h \in C_{rd}\left(D, \mathbb{R}\right)$ *with* H *belongs to the class* $\mathcal{P}$ *such that*

$$\limsup_{t \to \infty} \frac{1}{H(t, t_*)} \int_{t_*}^t \left[H(t, s)\eta(s)\left(\sum_{i=1}^{n} q_i(s)\psi^\beta\left(h_i(s)\right)\right) - \frac{r(s)\left(h_+(t, s)\right)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^\beta(s)} \right] \Delta s = \infty, \tag{2.25}$$

where $t_* > t_1$ *for sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$. *Then every solution of equation* (1.1) *is oscillatory.*

Proof. The proof follows from (2.23), (2.24) and Theorem 2.2, and so we omit the details. $\square$

3. OSCILLATION RESULTS WHEN (iii$_\mathfrak{b}$) HOLDS

In this section, we establish some new criteria for the oscillation of (1.1) in the cases where $h_i(t) \leq \tau_1(t)$ and $h_i(t) \geq \tau_1(t)$ for $i = 1, 2, ..., n$, respectively. For notational purposes, we let

$$\begin{aligned}
\phi(t) := \frac{1}{p_1(\tau_1^{-1}(t))} &\left[1 - \frac{1}{p_1(\tau_1^{-1}(\tau_1^{-1}(t)))} \frac{A\left(\tau_1^{-1}(\tau_1^{-1}(t)), t_1\right)}{A\left(\tau_1^{-1}(t), t_1\right)} \right. \\
&\left. - \frac{p_2(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))))} \frac{A\left(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))), t_1\right)}{A\left(\tau_1^{-1}(t), t_1\right)} \right]
\end{aligned}$$

where τ_1^{-1} denotes the inverse function of τ_1, and throughout this section we assume that $\phi(t) > 0$ for all sufficiently large t.

Theorem 3.1. *Suppose that conditions* (i), (ii), (iii$_b$) *hold and* $h_i(t) \leq \tau_1(t)$ *for* $i = 1, 2, ..., n$. *If there exists a positive function* $\eta \in C_{rd}^1\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}\right)$ *such that*

$$\limsup_{t \to \infty} \int_T^t \left(\Psi_2(s) - \frac{r(s)\left(\eta^{\Delta}(s)\right)^{\beta+1}}{(\beta + 1)^{\beta+1}\eta^{\beta}(s)} \right) \Delta s = \infty \tag{3.1}$$

for $T > t_1$ *where* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *sufficiently large and*

$$\Psi_2(t) = \eta(t)\left(\sum_{i=1}^n q_i(t)\phi^{\beta}\left(h_i(t)\right)\frac{A^{\beta}\left(\tau_1^{-1}\left(h_i(t)\right), t_1\right)}{A^{\beta}\left(t, t_1\right)} \right),$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$ and $x(h_i(t)) > 0$ for $t \geq t_1$ and $i = 1, 2, ..., n$. Following a similar procedure to the proof of Theorem 2.1, we see that (2.2), (2.3) hold again, and from the definition of $y(t)$, we obtain

$$x(t) \geq \frac{y(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(t))} - \frac{y(\tau_1^{-1}(\tau_1^{-1}(t)))}{p_1(\tau_1^{-1}(t))p_1(\tau_1^{-1}(\tau_1^{-1}(t)))} - \frac{p_2(\tau_1^{-1}(t))y(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))))}{p_1(\tau_1^{-1}(t))p_1(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))))}. \tag{3.2}$$

Using the fact that τ_1 and τ_2 are strictly increasing, and noting that $\tau_1(t) < t < \tau_2(t)$, we get

$$\tau_1^{-1}(\tau_1^{-1}(t)) > \tau_1^{-1}(t) \tag{3.3}$$

and

$$\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))) > \tau_1^{-1}(t). \tag{3.4}$$

Taking $y(t)/A(t, t_1)$ is nonincreasing on $[t_1, \infty)_{\mathbb{T}}$ into account that, we deduce from (3.3) and (3.4) that

$$\frac{A\left(\tau_1^{-1}(\tau_1^{-1}(t)), t_1\right)y(\tau_1^{-1}(t))}{A\left(\tau_1^{-1}(t), t_1\right)} \geq y(\tau_1^{-1}(\tau_1^{-1}(t))) \quad \text{for } t \geq t_1, \tag{3.5}$$

and

$$\frac{A\left(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))), t_1\right)y(\tau_1^{-1}(t))}{A\left(\tau_1^{-1}(t), t_1\right)} \geq y(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t)))) \quad \text{for } t \geq t_1, \tag{3.6}$$

respectively. Using (3.5) and (3.6) in (3.4) yields

$$x(t) \geq \frac{1}{p_1(\tau_1^{-1}(t))}\left[1 - \frac{1}{p_1(\tau_1^{-1}(\tau_1^{-1}(t)))}\frac{A\left(\tau_1^{-1}(\tau_1^{-1}(t)), t_1\right)}{A\left(\tau_1^{-1}(t), t_1\right)} \right.$$

$$\left. -\frac{p_2(\tau_1^{-1}(t))}{p_1(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))))}\frac{A\big(\tau_1^{-1}(\tau_2(\tau_1^{-1}(t))),t_1\big)}{A\big(\tau_1^{-1}(t),t_1\big)}\right] y(\tau_1^{-1}(t)) \quad (3.7)$$

for $t \geq t_1$. Since $\lim_{t\to\infty} h_i(t) = \infty$, we can choose $t_2 \geq t_1$ such that all $h_i(t) \geq t_1$ for $t \geq t_2$, where $i = 1, 2, ..., n$. Thus, from (3.7) we obtain

$$x(h_i(t)) \geq \phi(h_i(t))y\left(\tau_1^{-1}(h_i(t))\right), \qquad i = 1, 2, ..., n \qquad (3.8)$$

for $t \geq t_2$. Using (3.8) in (1.1) gives

$$\left(r(t)\big(y^\Delta(t)\big)^\beta\right)^\Delta + \sum_{i=1}^{n} q_i(t)\phi^\beta\big(h_i(t)\big)y^\beta\big(\tau_1^{-1}(h_i(t))\big) \leq 0. \qquad (3.9)$$

Define again Riccati substitution ω by (2.10). Then, it follows from (2.10) and (3.9) that

$$
\begin{aligned}
\omega^\Delta(t) \;\leq\; & \eta^\Delta(t)\frac{r\big(\sigma(t)\big)\big(y^\Delta\big(\sigma(t)\big)\big)^\beta}{y^\beta\big(\sigma(t)\big)} - \eta(t)\frac{r\big(\sigma(t)\big)\big(y^\Delta\big(\sigma(t)\big)\big)^\beta\big(y^\beta(t)\big)^\Delta}{y^\beta(t)y^\beta\big(\sigma(t)\big)} \\
& -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\phi^\beta\big(h_i(t)\big)\frac{y^\beta\big(\tau_1^{-1}(h_i(t))\big)}{y^\beta(t)}\right) \quad \text{for } t \geq t_2. \qquad (3.10)
\end{aligned}
$$

Using the fact $y(t)/A(t,t_1)$ is nonincreasing, and noting that $h_i(t) \leq \tau_1(t)$ implies $\tau_1^{-1}(h_i(t)) \leq t$, we obtain

$$\frac{y\big(\tau_1^{-1}(h_i(t))\big)}{y(t)} \geq \frac{A\big(\tau_1^{-1}(h_i(t)),t_1\big)}{A\big(t,t_1\big)}. \qquad (3.11)$$

for $i = 1, 2, ..., n$. Substituting (3.11) into (3.10) gives

$$
\begin{aligned}
\omega^\Delta(t) \;\leq\; & \eta^\Delta(t)\frac{r\big(\sigma(t)\big)\big(y^\Delta\big(\sigma(t)\big)\big)^\beta}{y^\beta\big(\sigma(t)\big)} - \eta(t)\frac{r\big(\sigma(t)\big)\big(y^\Delta\big(\sigma(t)\big)\big)^\beta\big(y^\beta(t)\big)^\Delta}{y^\beta(t)y^\beta\big(\sigma(t)\big)} \\
& -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\phi^\beta\big(h_i(t)\big)\frac{A^\beta\big(\tau_1^{-1}(h_i(t)),t_1\big)}{A^\beta(t,t_1)}\right) \quad \text{for } t \geq t_2. \quad (3.12)
\end{aligned}
$$

The remainder of the proof is similar to that of Theorem 2.1, and so the details are omitted. $\qquad\square$

Theorem 3.2. *Suppose that conditions* (i), (ii), (iii$_\flat$) *hold and $h_i(t) \leq \tau_1(t)$ for $i = 1, 2, ..., n$. Suppose also that there exist functions $\eta \in C_{rd}^1\left([t_0, \infty)_\mathbb{T}, \mathbb{R}^+\right)$ and $H, h \in C_{rd}\left(D, \mathbb{R}\right)$ with H belongs to the class $\mathcal{P}$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t}\left[H(t,s)\,\Psi_2(s) - \frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^\beta(s)}\right]\Delta s = \infty, \qquad (3.13)$$

where $\Psi_2(t)$ is as in Theorem 3.1 and $t_* > t_1$ for sufficiently large $t_1 \in [t_0, \infty)_\mathbb{T}$. Then every solution of equation (1.1) is oscillatory.

Proof. The proof follows from (3.11), (3.12) and Theorem 2.2, and so we omit the details. □

From Theorem 3.3, we immediately have the following oscillation criterion.

Corollary 3.1. *Suppose that all conditions of Theorem 3.2 are satisfied with (3.13) replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_*)} \int_{t_*}^{t} H(t,s)\,\eta(s) \left(\sum_{i=1}^{n} q_i(s)\phi^{\beta}\big(h_i(s)\big) \frac{A^{\beta}\big(\tau_1^{-1}(h_i(s)),t_1\big)}{A^{\beta}(s,t_1)} \right) \Delta s = \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_*)} \int_{t_*}^{t} \frac{r(s)\big(h_+(t,s)\big)^{\beta+1}}{\eta^{\beta}(s)} \Delta s < \infty,$$

then equation (1.1) is oscillatory.

Theorem 3.3. *Assume that conditions* (i), (ii), (iii$_\flat$) *hold and* $h_i(t) \geq \tau_1(t)$ *for* $i = 1, 2, ..., n$. *If there exists a positive function* $\eta \in C_{rd}^1\left([t_0,\infty)_{\mathbb{T}}, \mathbb{R}\right)$ *such that*

$$\limsup_{t\to\infty} \int_{T}^{t} \left(\eta(s)\left(\sum_{i=1}^{n} q_i(s)\phi^{\beta}\big(h_i(s)\big) \right) - \frac{r(s)\big(\eta^{\Delta}(s)\big)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^{\beta}(s)} \right) \Delta s = \infty, \quad (3.14)$$

for $T > t_1$ *with* $t_1 \in [t_0,\infty)_{\mathbb{T}}$ *sufficiently large, then equation (1.1) is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau_1(t)) > 0$, $x(\tau_2(t)) > 0$ and $x(h_i(t)) > 0$ for $t \geq t_1$ and $i = 1, 2, ..., n$. Proceeding as in the proof of Theorem 3.1, we again arrive at (3.10) for $t \geq t_2$. Using the fact that $\tau_1(t)$ is strictly increasing and noting that $h_i(t) \geq \tau_1(t)$, we have

$$\tau_1^{-1}(h_i(t)) \geq t$$

for $i = 1, 2, ..., n$. So, from the fact that y is increasing, we obtain

$$\frac{y\big(\tau_1^{-1}(h_i(t))\big)}{y(t)} \geq 1, \qquad i = 1, 2, ..., n. \quad (3.15)$$

Using (3.15) in (3.10) yields

$$\begin{aligned}
\omega^{\Delta}(t) \quad \leq \quad & \eta^{\Delta}(t) \frac{r(\sigma(t))\big(y^{\Delta}(\sigma(t))\big)^{\beta}}{y^{\beta}(\sigma(t))} - \eta(t) \frac{r(\sigma(t))\big(y^{\Delta}(\sigma(t))\big)^{\beta}\big(y^{\beta}(t)\big)^{\Delta}}{y^{\beta}(t)y^{\beta}(\sigma(t))} \\
& -\eta(t)\left(\sum_{i=1}^{n} q_i(t)\phi^{\beta}\big(h_i(t)\big) \right) \quad \text{for } t \geq t_2. \quad (3.16)
\end{aligned}$$

The rest of proof is similar to the proof of Theorem 2.1, and so the details are omitted. □

Theorem 3.4. *Suppose that conditions* (i), (ii), (iii$_\flat$) *hold and* $h_i(t) \geq \tau_1(t)$ *for* $i = 1, 2, ..., n$. *Suppose also that there exist functions* $\eta \in C^1_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$ *and* $H, h \in C_{rd}\left(D, \mathbb{R}\right)$ *with* H *belongs to the class* $\mathcal{P}$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t, t_*)} \int_{t_*}^t \left[H(t,s)\eta(s)\left(\sum_{i=1}^n q_i(s)\phi^\beta\left(h_i(s)\right)\right) - \frac{r(s)\left(h_+(t,s)\right)^{\beta+1}}{(\beta+1)^{\beta+1}\eta^\beta(s)} \right] \Delta s = \infty,$$
$$\tag{3.17}$$

where $t_* > t_1$ *for sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$. *Then every solution of equation* (1.1) *is oscillatory.*

Proof. The proof follows from (3.15), (3.16) and Theorem 2.2, and so we omit the details. $\qquad\square$

4. Examples and Remarks

Example 4.1. Consider the neutral differential equation

$$\left[\left(\left(x(t) + x(\frac{t}{2}) + tx(2t)\right)'\right)^3\right]^{'4} x^3(t-1) + t^5 x^3(2t-1) = 0, \tag{4.1}$$

for $t \geq 13$. Here we have $\mathbb{T} = \mathbb{R}$, $\beta = 3$, $n = 2$, $r(t) = p_1(t) = 1$, $p_2(t) = t$, $q_1(t) = t^4$, $q_2(t) = t^5$, $\tau_1(t) = t/2$, $\tau_2(t) = 2t$, $h_1(t) = t-1$ and $h(t) = 2t-1$. It is clear that conditions (i)-(iii$_\mathfrak{a}$) hold, $\tau_2(t) > h_2(t) > t > h_1(t)$ and

$$\psi(t) = \frac{1}{t/2}\left[1 - \frac{1}{t/4} - \frac{1}{t/8}\right] = \frac{2t-24}{t^2} > 0, \quad \text{for } t \geq 13. \tag{4.2}$$

On the other hand, we see that

$$A(t, t_1) = \int_{t_1}^t \frac{1}{r^{1/\beta}(s)} ds = \int_{13}^t ds = t - 13.$$

With $\eta(t) = t$, condition (2.1) is satisfied due to

$$\limsup_{t\to\infty} \int_T^t \left[s^5 \left(\frac{2s-26}{(s-1)^2}\right)^3 \left(\frac{s-27}{2s-26}\right)^3 + s^6 \left(\frac{4s-26}{(2s-1)^2}\right)^3 \left(\frac{s-27/2}{s-13}\right)^3 \right] ds = \infty$$

and

$$\limsup_{t\to\infty} \int_T^t \frac{1}{4^4 s^3} ds < \infty.$$

where $T > 13$. Hence, by Theorem 2.1, every solution of (4.1) is oscillatory.

Example 4.2. Let $\mathbb{T} = \mathbb{Z}$ and consider the neutral difference equation

$$\Delta\left[\frac{1}{t^{1/3}}\left(\Delta\left(x(t) + 3x(t-3) + 7x(t+1)\right)\right)^{1/3}\right] + \sum_{i=1}^n t^i x^{1/3}(t+i+1) = 0, \tag{4.3}$$

for $t \geq 2$. Here we have $\beta = 1/3$, $r(t) = 1/t^{1/3}$, $p_1(t) = 3$, $p_2(t) = 7$, $\tau_1(t) = t - 3$, $\tau_2(t) = t + 1$, $q_i(t) = t^i$ and $h_i(t) = t + i + 1$ for $i = 1, 2, ..., n$. It is clear that conditions (i)-(iii$_a$) hold, $t < \tau_2(t) < h_i(t)$ for $i = 1, 2, ..., n$ and

$$\psi(t) = \frac{1}{7}\left[1 - \frac{1}{7} - \frac{3}{7}\right] = \frac{3}{49} > 0. \tag{4.4}$$

It is easy to see that condition (2.22) holds with $\eta(t) = c > 0$. So, by Theorem 2.3, all solutions of equation (4.3) are oscillatory.

Example 4.3. Consider the second order neutral dynamic equation

$$\left(\left(\left(x(t) + 5tx(\frac{t}{2}) + tx(4t)\right)^\Delta\right)^3\right)^\Delta + t^2 x^3\left(\frac{t}{4}\right) + t^3 x^3\left(\frac{t}{8}\right) = 0, \tag{4.5}$$

for $t \geq 2$ and $\mathbb{T} := \overline{2^{\mathbb{Z}}} = \{2^k : k \in \mathbb{Z}\} \cup \{0\}$. Here we have $\beta = 3$, $n = 2$, $r(t) = 1$, $p_1(t) = 5t$, $p_2(t) = t$, $\tau_1(t) = t/2$, $\tau_2(t) = 4t$, $q_1(t) = t^2$, $q_2(t) = t^3$, $h_1(t) = t/4$ and $h_2(t) = t/8$. It is clear that conditions (i), (ii), (iii$_b$) hold and $\tau_1(t) \geq h_i(t)$ for $i = 1, 2$. Also, it obvious that $A(t, t_1) = t - 2$ for any time scale $\mathbb{T}$, and

$$\phi(t) = \frac{1}{10t}\left[1 - \frac{1}{20t}\frac{4t - 2}{2t - 2} - \frac{2t}{80t}\frac{16t - 2}{2t - 2}\right] = \frac{32t^2 - 43t + 2}{400t^3 - 400t^2} > 0 \tag{4.6}$$

for $t \geq 2$. Then, with $\eta(t) = t^2$, we see that condition (3.1) holds for $T > 2$. Hence, by Theorem 3.1, every solution of second order mixed neutral q-difference equation (4.5) is oscillatory.

Example 4.4. Consider the neutral differential equation

$$\left(x(t) + tx(t - 2\pi) + x(t + \pi)\right)'' + 2x\left(t - \frac{\pi}{2}\right) + tx(t + 2\pi) = 0, \quad t \geq 5. \tag{4.7}$$

Here we have $\mathbb{T} = \mathbb{R}$, $\beta = 1$, $n = 2$, $r(t) = p_2(t) = 1$, $p_1(t) = t$, $\tau_1(t) = t - 2\pi$, $\tau_2(t) = t + \pi$, $q_1(t) = 2$, $q_2(t) = t$, $h_1(t) = t - \frac{\pi}{2}$ and $h_2(t) = t + 2\pi$. It is clear that conditions (i), (ii), (iii$_b$) hold, and $\tau_1(t) < h_1(t) < h_2(t)$. On the other hand, we see that $A(t, t_1) = t - 5$, and $\phi(t) > 0$. With $\eta(t) = 1$, it is easy to see that condition (3.14) holds. Hence, by Theorem 3.3, every solution of equation (4.7) is oscillatory. In fact, $x(t) = \sin t$ is such a solution.

Remark 4.1. Note that oscillation results presented in [6, 10, 12, 16, 19, 23, 28, 30, 36, 37] fail to apply to the equations (4.1), (4.3), (4.5) and (4.7).

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