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Research Article

Fekete-Szegö Problem for Certain Subclass of Analytic Functions with Complex Order Defined by $q-$Analogue of Ruscheweyh Operator

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ABSTRACT. In this paper, we study Fekete-Szegö problem for certain subclass of analytic functions with complex order in the open unit disk by applying the $q-$analogue of Ruscheweyh operator in conjunction with the principle of subordination between analytic functions.

Keywords: Analytic functions, univalent functions, $q-$derivative operator, $q-$analogue of Ruscheweyh operator, Fekete-Szego problem, subordination.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written as $f \prec g$ in $\mathbb{U}$ or $f(z) \prec g(z)\ (z \in \mathbb{U})$, if there exists a Schwarz function ω, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1\ (z \in \mathbb{U})$ such that $f(z) = g(\omega(z))\ (z \in \mathbb{U})$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence holds (see [12] and [7]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For function $f \in \mathcal{A}$ given by (1.1) and $0 < q < 1$, the $q-$derivative of a function f is defined by (see [10, 9] and [6])

$$D_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z} & , z \neq 0 \\ f'(0) & , z = 0 \end{cases} \tag{1.2}$$

provided that $f'(0)$ exists and $D_q^2 f(z) = D_q(D_q f(z))$. We note from (1.2) that

$$\lim_{q \to 1^-} D_q f(z) = f'(z) \quad \textit{and} \quad \lim_{q \to 1^-} D_q^2 f(z) = f''(z).$$

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It is readily deduced from (1.1) and (1.2) that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \ a_k z^{k-1}, \tag{1.3}$$

where

$$[k]_q = \frac{q^k - 1}{q - 1}. \tag{1.4}$$

Aldweby and Darus [1] defined $q-$analogue of Ruscheweyh operator $\mathcal{R}_q^\delta : \mathcal{A} \to \mathcal{A}$ as follows:

$$\mathcal{R}_q^\delta f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\delta-1]_q!}{[\delta]_q! \, [k-1]_q!} a_k z^k \quad (\delta \geq -1),$$

where $[i]_q!$ is given by

$$[i]_q! = \begin{cases} [i]_q [i-1]_q \ldots [1]_q & , i \in \mathbb{N} = \{1, 2, 3, \ldots\} \\ 1 & , i = 0 \end{cases}.$$

We note that

$$\mathcal{R}_q^0 f(z) = f(z) \quad and \quad \mathcal{R}_q^1 f(z) = z D_q f(z).$$

From the definition of $\mathcal{R}_q^\delta$ we observe that if $q \to 1^-$, we have

$$\lim_{q \to 1} \mathcal{R}_q^\delta f(z) = \mathcal{R}^\delta f(z) = z + \sum_{k=2}^{\infty} \frac{(k+\delta-1)!}{\delta! \, (k-1)!} a_k z^k,$$

where $\mathcal{R}^\delta$ is Ruscheweyh differential operator defined by Ruscheweyh [16].

It is easy to check that

$$z D_q \left(\mathcal{R}_q^\delta f(z) \right) = \left(1 + \frac{[\delta]_q}{q^\delta} \right) \mathcal{R}_q^{\delta+1} f(z) - \frac{[\delta]_q}{q^\delta} \mathcal{R}_q^\delta f(z). \tag{1.5}$$

If $q \to 1^-$, the equality (1.5) implies

$$z \left(\mathcal{R}^\delta f(z) \right)' = (1+\delta) \mathcal{R}^{\delta+1} f(z) - \delta \mathcal{R}^\delta f(z)$$

which is the well known recurrence formula for Ruscheweyh differential operator.

By making use of the $q-$analogue of Ruscheweyh operator $\mathcal{R}_q^\delta$ and the principle of subordination, we now introduce the following subclass of analytic functions of complex order.

Definition 1.1. *Let $\mathcal{P}$ be the class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $\Re\phi(z) > 0$ for $z \in \mathbb{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_{q,b}^\delta(\gamma, \phi)$ if it satisfies the following subordination condition:*

$$1 + \frac{1}{b} \left[\frac{(1-\gamma) z D_q \mathcal{R}_q^\delta f(z) + \gamma z D_q (z D_q \mathcal{R}_q^\delta f(z))}{(1-\gamma) \mathcal{R}_q^\delta f(z) + \gamma z D_q \mathcal{R}_q^\delta f(z)} - 1 \right] \prec \phi(z) \ \ (b \in \mathbb{C}^*). \tag{1.6}$$

We note that:

(i) $\lim_{q \to 1^-} \mathcal{K}_{q,b}^0(\gamma, \phi) = \mathcal{K}_b(\gamma, \phi) \ (b \in \mathbb{C}^*)$

$$= \left\{ f \in \mathcal{A} : 1 + \frac{1}{b} \left[\frac{z f'(z) + \gamma z^2 f''(z)}{(1-\gamma) f(z) + \gamma z f'(z)} - 1 \right] \prec \phi(z) \right\},$$

(ii) $\mathcal{K}^{0}_{q,(1-\alpha)e^{-i\theta}\cos\theta}(0,\phi) = \mathcal{S}^{\theta}_{q}(\alpha;\phi)\ \left(|\theta| \le \frac{\pi}{2}, 0 \le \alpha < 1\right)$

$$= \left\{ f \in \mathcal{A} : \frac{e^{i\theta}\frac{zD_q f(z)}{f(z)} - \alpha\cos\theta - i\sin\theta}{(1-\alpha)\cos\theta} \prec \phi(z) \right\},$$

(iii) $\mathcal{K}^{0}_{q,(1-\alpha)e^{-i\theta}\cos\theta}(1,\phi) = \mathcal{C}^{\theta}_{q}(\alpha;\phi)\ \left(|\theta| \le \frac{\pi}{2}, 0 \le \alpha < 1\right)$

$$= \left\{ f \in \mathcal{A} : \frac{e^{i\theta}\frac{D_q(zD_q f(z))}{D_q f(z)} - \alpha\cos\theta - i\sin\theta}{(1-\alpha)\cos\theta} \prec \phi(z) \right\},$$

(iv) $\mathcal{K}^{\delta}_{q,1}(0,\phi) = \mathcal{S}^{\delta}_{q}(\phi)$ and $\mathcal{K}^{\delta}_{q,1}(1,\phi) = \mathcal{C}^{\delta}_{q}(\phi)$ (Alweby and Darus [3]),
(v) $\mathcal{K}^{0}_{q,b}(0,\phi) = \mathcal{S}_{q,b}(\phi)$ and $\mathcal{K}^{0}_{q,b}(1,\phi) = \mathcal{C}_{q,b}(\phi)$ (Seoudy and Aouf [18]),
(vi) $\mathcal{K}^{0}_{q,1}(0,\phi) = \mathcal{S}_{q}(\phi)$ and $\mathcal{K}^{0}_{q,1}(1,\phi) = \mathcal{C}_{q}(\phi)$ (Alweby and Darus [2]),
(vii) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,b}(0,\phi) = \mathcal{S}_{b}(\phi)$ and $\lim_{q\to1^-}\mathcal{K}^{0}_{q,b}(1,\phi) = \mathcal{C}_{b}(\phi)$ (Ravichandran et al. [15]),
(viii) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,1}(0,\phi) = \mathcal{S}^{*}(\phi)$ and $\lim_{q\to1^-}\mathcal{K}^{0}_{q,1}(1,\phi) = \mathcal{C}(\phi)$ (Ma and Minda [11]),
(ix) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,b}\left(0, \frac{1+(1-2\alpha)z}{1-z}\right) = \mathcal{S}^{*}_{\alpha}(b)$ and $\lim_{q\to1^-}\mathcal{K}^{0}_{q,b}\left(1, \frac{1+(1-2\alpha)z}{1-z}\right) = \mathcal{C}_{\alpha}(b)$ $(0 \le \alpha < 1)$ (Frasin [8]),
(x) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,b}\left(0, \frac{1+z}{1-z}\right) = \mathcal{S}^{*}(b)$ (Nasr and Aouf [14]),
(xi) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,b}\left(1, \frac{1+z}{1-z}\right) = \mathcal{C}(b)\ (b \in \mathbb{C}^{*})$ (Nasr and Aouf [13] and Wiatrowski [19]),
(xii) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,1-\alpha}\left(0, \frac{1+z}{1-z}\right) = \mathcal{S}^{*}(\alpha)$ and $\lim_{q\to1^-}\mathcal{K}^{0}_{q,1-\alpha}\left(1, \frac{1+z}{1-z}\right) = \mathcal{C}(\alpha)\ (0 \le \alpha < 1)$ (Robertson [17]),
(xiii) $\lim_{q\to1^-}\mathcal{K}^{0}_{q,be^{-i\theta}\cos\theta}\left(0, \frac{1+z}{1-z}\right) = \mathcal{S}^{\theta}(b)$ and $\lim_{q\to1^-}\mathcal{K}^{0}_{q,be^{-i\theta}\cos\theta}\left(1, \frac{1+z}{1-z}\right) = \mathcal{C}^{\theta}(b)$ $(|\theta| < \frac{\pi}{2})$ (Al-Oboudi and Haidan [4] and Aouf et al. [5]).

In order to establish our main results, we need the following lemma.

Lemma 1.1. [11] *If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is a function with positive real part in $\mathbb{U}$ and μ is a complex number, then*

$$\left|c_2 - \mu c_1^2\right| \le 2\max\{1; |2\mu - 1|\}.$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \ \text{ and } \ p(z) = \frac{1+z}{1-z}.$$

Lemma 1.2. [11] *If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with a positive real part in $\mathbb{U}$, then*

$$\left|c_2 - \nu c_1^2\right| \le \begin{cases} -4\nu + 2 & if \quad \nu \le 0 \\ 2 & if \quad 0 \le \nu \le 1, \\ 4\nu - 2 & if \quad \nu \ge 1 \end{cases}$$

when $\upsilon < 0$ or $\nu > 1$, the equality holds if and only if $p(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1+\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\lambda}{2}\right)\frac{1-z}{1+z} \qquad (0 \le \lambda \le 1)$$

or one of its rotations. If $\nu = 1$*, the equality holds if and only if* p *is the reciprocal of one of the functions such that equality holds in the case of* $\nu = 0$.

Also the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$\left|c_2 - \nu c_1^2\right| + \nu \left|c_1\right|^2 \leq 2 \qquad \left(0 \leq \nu \leq \frac{1}{2}\right)$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu) \left|c_1\right|^2 \leq 2 \qquad \left(\frac{1}{2} \leq \nu \leq 1\right).$$

In the present paper, we obtain the Fekete-Szegö inequalities for the class $\mathcal{K}_{q,b}(\gamma, \phi)$. The motivation of this paper is to generalize previously results. Unless otherwise mentioned, we assume throughout this paper that the function $0 < q < 1$, $b \in \mathbb{C}^*$, $0 \leq \gamma \leq 1$, $\phi \in \mathcal{P}$, $[k]_q$ is given by (1.4) and $z \in \mathbb{U}$.

Theorem 1.1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ *with* $B_1 \neq 0$*. If* f *given by (1.1) belongs to the class* $\mathcal{K}_{q,b}(\gamma, \phi)$*, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|bB_1|}{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q} \max\left\{1; \left|\frac{B_2}{B_1} + \left(1 - \frac{[1+\gamma q(q+1)]\,[\delta+2]_q}{(1+\gamma q)^2\,[\delta+1]_q}\mu\right)\frac{B_1 b}{q}\right|\right\}. \tag{1.7}$$

The result is sharp.

Proof. If $f \in \mathcal{K}_{q,b}^{\delta}(\gamma, \phi)$, then there is a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ such that

$$1 + \frac{1}{b}\left[\frac{(1-\gamma)\, zD_q\mathcal{R}_q^{\delta} f(z) + \gamma zD_q(zD_q\mathcal{R}_q^{\delta} f(z))}{(1-\gamma)\,\mathcal{R}_q^{\delta} f(z) + \gamma zD_q\mathcal{R}_q^{\delta} f(z)} - 1\right] = \phi(\omega(z)). \tag{1.8}$$

Define the function $p(z)$ by

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + \ldots. \tag{1.9}$$

Since ω is a Schwarz function, we see that $\Re p(z) > 0$ and $p(0) = 1$. Therefore,

$$\begin{aligned}
\phi(\omega(z)) &= \phi\left(\frac{p(z)-1}{p(z)+1}\right) \\
&= \phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) z^3 + \ldots\right]\right) \\
&= 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \ldots. \qquad (1.10)
\end{aligned}$$

Now, by substituting (1.10) in (1.8), we have

$$\begin{aligned}
&1 + \frac{1}{b}\left[\frac{(1-\gamma)\, zD_q\mathcal{R}_q^{\delta} f(z) + \gamma zD_q(zD_q\mathcal{R}_q^{\delta} f(z))}{(1-\gamma)\,\mathcal{R}_q^{\delta} f(z) + \gamma zD_q\mathcal{R}_q^{\delta} f(z)} - 1\right] \\
= \; &1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \ldots.
\end{aligned}$$

From the above equation, we obtain

$$\frac{1}{b} q (1+\gamma q) [\delta+1]_q a_2 = \frac{B_1 c_1}{2}$$

and

$$\frac{q}{b}\left(\left[1+\gamma q\left(q+1\right)\right]\left[\delta+2\right]_q\left[\delta+1\right]_q a_3-\left(1+\gamma q\right)^2\left(\left[\delta+1\right]_q\right)^2 a_2^2\right)=\frac{B_1c_2}{2}-\frac{B_1c_1^2}{4}+\frac{B_2c_1^2}{4}$$

or, equivalently,

$$a_2=\frac{B_1c_1b}{2q\left(1+\gamma q\right)\left[\delta+1\right]_q}$$

and

$$a_3=\frac{bB_1}{2\left[1+\gamma q\left(q+1\right)\right]q\left[\delta+2\right]_q\left[\delta+1\right]_q}\left\{c_2-\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1b}{q}\right]c_1^2\right\}.$$

Therefore, we have

$$a_3-\mu a_2^2=\frac{bB_1}{2q\left[1+\gamma q\left(q+1\right)\right]\left[\delta+2\right]_q\left[\delta+1\right]_q}\left\{c_2-\nu c_1^2\right\}, \tag{1.11}$$

where

$$\nu=\frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1b}{q}\left(1-\frac{\left[1+\gamma q\left(q+1\right)\right]\left[\delta+2\right]_q\mu}{\left(1+\gamma q\right)^2\left[\delta+1\right]_q}\right)\right]. \tag{1.12}$$

Our result now follows from Lemma 1.1. The result is sharp for the functions

$$1+\frac{1}{b}\left[\frac{\left(1-\gamma\right)zD_q\mathcal{R}_q^{\delta}f(z)+\gamma zD_q(zD_q\mathcal{R}_q^{\delta}f(z))}{\left(1-\gamma\right)\mathcal{R}_q^{\delta}f\left(z\right)+\gamma zD_q\mathcal{R}_q^{\delta}f(z)}-1\right]=\phi\left(z^2\right)$$

and

$$1+\frac{1}{b}\left[\frac{\left(1-\gamma\right)zD_q\mathcal{R}_q^{\delta}f(z)+\gamma zD_q(zD_q\mathcal{R}_q^{\delta}f(z))}{\left(1-\gamma\right)\mathcal{R}_q^{\delta}f\left(z\right)+\gamma zD_q\mathcal{R}_q^{\delta}f(z)}-1\right]=\phi\left(z\right).$$

This completes the proof of Theorem 1.1. □

Taking $\gamma=0$ and $b=1$ in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 6].

Corollary 1.1. *Let $\phi\left(z\right)=1+B_1z+B_2z^2+...$ with $B_1\neq 0$. If f given by (1.1) belongs to the class $\mathcal{S}_q^{\delta}\left(\phi\right)$, then*

$$\left|a_3-\mu a_2^2\right|\leq\frac{|B_1|}{q[\delta+2]_q[\delta+1]_q}\max\left\{1;\left|\frac{B_2}{B_1}+\left(1-\frac{[\delta+2]_q}{[\delta+1]_q}\mu\right)\frac{B_1}{q}\right|\right\}.$$

The result is sharp.

Taking $\gamma=b=1$ in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 7].

Corollary 1.2. *Let $\phi\left(z\right)=1+B_1z+B_2z^2+...$ with $B_1\neq 0$. If f given by (1.1) belongs to the class $\mathcal{K}_q^{\delta}\left(\phi\right)$, then*

$$\left|a_3-\mu a_2^2\right|\leq\frac{|B_1|}{q[1+q(q+1)][\delta+2]_q[\delta+1]_q}\max\left\{1;\left|\frac{B_2}{B_1}+\left(1-\frac{[1+q(q+1)][\delta+2]_q}{[\delta+1]_q(1+q)^2}\mu\right)\frac{B_1b}{q}\right|\right\}.$$

The result is sharp.

Taking $\gamma=\delta=0$ and $b=1$ in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [2, Theorem 2.1].

Corollary 1.3. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 \neq 0$. If f given by (1.1) belongs to the class $\mathcal{S}_q(\phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|B_1|}{q(q+1)} \max\left\{1; \left|\frac{B_2}{B_1} + (1 - (q+1)\mu)\frac{B_1}{q}\right|\right\}.$$

The result is sharp.

Taking $\gamma = b = 1$ and $\delta = 0$ in Theorem 1.1, we obtain the following corollary which improves the result of Aldweby and Darus [2, Theorem 2.2].

Corollary 1.4. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 \neq 0$. If f given by (1.1) belongs to the class $\mathcal{K}_q(\phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|B_1|}{q(q+1)[1+q(q+1)]} \max\left\{1; \left|\frac{B_2}{B_1} + \left(1 - \frac{[1+q(q+1)]}{(1+q)}\mu\right)\frac{B_1}{q}\right|\right\}.$$

The result is sharp.

Taking $\gamma = \delta = 0$ and $q \to 1^-$ in Theorem 1.1, we obtain the following corollary which improves the result of Ravichandran et al. [15, Theorem 4.1].

Corollary 1.5. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 \neq 0$. If f given by (1.1) belongs to the class $\mathcal{S}_b(\phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|B_1 b|}{2} \max\left\{1; \left|\frac{B_2}{B_1} + (1 - 2\mu) B_1 b\right|\right\}.$$

The result is sharp.

Theorem 1.2. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ with $B_1 > 0$ and $B_2 \geq 0$. Let*

$$\sigma_1 = \frac{(1+\gamma q)^2 [\delta+1]_q \left[bB_1^2 + q(B_2 - B_1)\right]}{[1+\gamma q(q+1)][\delta+2]_q bB_1^2}, \tag{1.13}$$

$$\sigma_2 = \frac{(1+\gamma q)^2 [\delta+1]_q \left[bB_1^2 + q(B_2 + B_1)\right]}{[1+\gamma q(q+1)][\delta+2]_q bB_1^2}, \tag{1.14}$$

$$\sigma_3 = \frac{(1+\gamma q)^2 [\delta+1]_q \left(bB_1^2 + qB_2\right)}{[1+\gamma q(q+1)][\delta+2]_q bB_1^2}. \tag{1.15}$$

If f given by (1.1) belongs to the class $\mathcal{K}_{q,b}^{\delta}(\gamma, \phi)$ with $b > 0$, then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{b}{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q}\left[B_2 + \frac{B_1^2 b}{q}\left(1 - \frac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma)^2[\delta+1]_q}\mu\right)\right] & , \mu \leq \sigma_1 \\ \frac{bB_1}{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q} & , \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{b}{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q}\left[-B_2 - \frac{B_1^2 b}{q}\left(1 - \frac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma)^2[\delta+1]_q}\mu\right)\right] & , \mu \geq \sigma_2 \end{cases}. \tag{1.16}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{q(1+\gamma q)^2[\delta+1]_q}{[1+\gamma q(q+1)][\delta+2]_q B_1^2 b}\left[B_1 - B_2 - \frac{B_1^2 b}{q}\left(1 - \frac{[1+\gamma q(q+1)][\delta+2]_q}{(1+\gamma)^2[\delta+1]_q}\mu\right)\right]|a_2|^2$$

$$\leq \frac{bB_1}{q[1+\gamma q(q+1)][\delta+2]_q[\delta+1]_q}. \tag{1.17}$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \frac{q(1+\gamma q)^2[\delta+1]_q}{[1+\gamma q(q+1)][\delta+2]_q B_1^2 b}\left[B_1 + B_2 + \frac{B_1^2 b}{q}\left(1 - \frac{[1+\gamma q(q+1)][\delta+2]_q \mu}{(1+\gamma)^2[\delta+1]_q}\right)\right]|a_2|^2$$

$$\leq \frac{bB_1}{q\left[1+\gamma q\left(q+1\right)\right]\left[\delta+2\right]_q\left[\delta+1\right]_q}. \tag{1.18}$$

The result is sharp.

Proof. Applying Lemma 1.2 to (1.11) and (1.12), we can obtain our results asserted by Theorem 1.2. □

Taking $\gamma = 0$ and $b = 1$ in Theorem 1.2, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 10].

Corollary 1.6. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $B_1 > 0$ and $B_2 \geq 0$. Let*

$$\begin{aligned} \chi_1 &= \frac{[\delta+1]_q\left[B_1^2+q\left(B_2-B_1\right)\right]}{[\delta+2]_q B_1^2}, \\ \chi_2 &= \frac{[\delta+1]_q\left[B_1^2+q\left(B_2+B_1\right)\right]}{[\delta+2]_q B_1^2}, \\ \chi_3 &= \frac{[\delta+1]_q\left(B_1^2+qB_2\right)}{[\delta+2]_q B_1^2}. \end{aligned}$$

If f given by (1.1) belongs to the class $\mathcal{S}_q^{\delta}(\phi)$, then

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{1}{q[\delta+2]_q[\delta+1]_q}\left[B_2+\frac{B_1^2}{q}\left(1-\frac{[\delta+2]_q}{[\delta+1]_q}\mu\right)\right] & ,\mu \leq \chi_1 \\ \frac{B_1}{q[\delta+2]_q[\delta+1]_q} & ,\chi_1 \leq \mu \leq \chi_2 \\ \frac{1}{q[\delta+2]_q[\delta+1]_q}\left[-B_2-\frac{B_1^2}{q}\left(1-\frac{[\delta+2]_q}{[\delta+1]_q}\mu\right)\right] & ,\mu \geq \chi_2 \end{cases}.$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\left|a_3-\mu a_2^2\right| + \frac{q[\delta+1]_q}{[\delta+2]_q B_1^2}\left[B_1-B_2-\frac{B_1^2}{q}\left(1-\frac{[\delta+2]_q}{[\delta+1]_q}\mu\right)\right]\left|a_2\right|^2 \leq \frac{B_1}{q[\delta+2]_q[\delta+1]_q}.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\left|a_3-\mu a_2^2\right| + \frac{q[\delta+1]_q}{[\delta+2]_q B_1^2}\left[B_1+B_2+\frac{B_1^2}{q}\left(1-\frac{[\delta+2]_q\mu}{[\delta+1]_q}\right)\right]\left|a_2\right|^2 \leq \frac{B_1}{q[\delta+2]_q[\delta+1]_q}.$$

The result is sharp.

Taking $\gamma = b = 1$ in Theorem 1.2, we obtain the following corollary which improves the result of Aldweby and Darus [3, Theorem 11].

Corollary 1.7. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $B_1 > 0$ and $B_2 \geq 0$. Let*

$$\begin{aligned} \varkappa_1 &= \frac{[2]_q^2[\delta+1]_q\left[B_1^2+q\left(B_2-B_1\right)\right]}{[3]_q[\delta+2]_q B_1^2}, \\ \varkappa_2 &= \frac{[2]_q^2[\delta+1]_q\left[B_1^2+q\left(B_2\mu+B_1\right)\right]}{[3]_q[\delta+2]_q B_1^2}, \\ \varkappa_3 &= \frac{[2]_q^2[\delta+1]_q\left(B_1^2+qB_2\right)}{[3]_q[\delta+2]_q B_1^2}. \end{aligned}$$

If f given by (1.1) belongs to the class $\mathcal{K}_q^\delta(\phi)$, then

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{1}{q[3]_q[\delta+2]_q[\delta+1]_q}\left[B_2+\frac{B_1^2}{q}\left(1-\frac{[3]_q[\delta+2]_q}{[2]_q^2[\delta+1]_q}\mu\right)\right] & ,\mu \leq \varkappa_1 \\ \frac{B_1}{q[3]_q[\delta+2]_q[\delta+1]_q} & ,\varkappa_1 \leq \mu \leq \varkappa_2 \\ \frac{1}{q[3]_q[\delta+2]_q[\delta+1]_q}\left[-B_2-\frac{B_1^2}{q}\left(1-\frac{[3]_q[\delta+2]_q}{[2]_q^2[\delta+1]_q}\mu\right)\right] & ,\mu \geq \varkappa_2 \end{cases}.$$

Further, if $\varkappa_1 \leq \mu \leq \varkappa_3$, then

$$\left|a_3-\mu a_2^2\right|+\frac{q[2]_q^2[\delta+1]_q}{[3]_q[\delta+2]_qB_1^2}\left[B_1-B_2-\frac{B_1^2}{q}\left(1-\frac{[3]_q[\delta+2]_q}{[2]_q^2[\delta+1]_q}\mu\right)\right]|a_2|^2 \leq \frac{B_1}{q[3]_q[\delta+2]_q[\delta+1]_q}.$$

If $\varkappa_3 \leq \mu \leq \varkappa_2$, then

$$\left|a_3-\mu a_2^2\right|+\frac{q[2]_q^2[\delta+1]_q}{[3]_q[\delta+2]_q\,B_1^2}\left[B_1+B_2+\frac{B_1^2}{q}\left(1-\frac{[3]_q[\delta+2]_q}{[2]_q^2[\delta+1]_q}\mu\right)\right]|a_2|^2 \leq \frac{B_1}{q[3]_q[\delta+2]_q[\delta+1]_q}.$$

The result is sharp.

Remark 1.1. *Putting $\delta=\gamma=0$ in Theorems 1.1 and 1.2, respectively, we deduce the corresponding results derived by Seoudy and Aouf* [18, Theorems 1 and 3, respectively].

Remark 1.2. *Putting $\delta=0$ and $\gamma=1$ in Theorems 1.1 and 1.2, respectively, we deduce the corresponding results derived by Seoudy and Aouf* [18, Theorems 2 and 4, respectively].

Remark 1.3. *For different choices of the parameters b,δ,q,γ and ϕ in Theorems 1.1 and 1.2, we can deduce some results for the classes $\mathcal{K}_b(\gamma,\phi)$, $\mathcal{S}_q^\theta(\alpha;\phi)$, $\mathcal{C}_q^\theta(\alpha;\phi)$, $\mathcal{S}_q(\phi)$, $\mathcal{C}_q(\phi)$, $\mathcal{S}_b(\phi)$, $\mathcal{C}_b(\phi)$, $\mathcal{S}^*(\phi)$, $\mathcal{C}(\phi)$, $\mathcal{S}_\alpha^*(b)$, $\mathcal{C}_\alpha(b)$, $\mathcal{S}^*(b)$, $\mathcal{C}(b)$, $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$, $\mathcal{S}^\theta(b)$ and $\mathcal{C}^\theta(b)$ which are defined in Section 1.*

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