Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (5) (2020), 1706 – 1717
DOI : 10.15672/hujms.573548

Research Article

Finite groups with given weakly τ_σ-quasinormal subgroups

Muhammad Tanveer Hussain[1], Chenchen Cao*[2], Li Zhang[3]

[1] *School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, P.R. China*
[2] *School of Mathematics and Statistics, Ningbo University, Ningbo, 315211, P.R. China*
[3] *Department of Mathematics and Physics, Anhui Jianzhu University, Hefei, 230022, P.R. China*

Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$ and G a finite group. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $i \in I$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \emptyset$. Let $\tau_{\mathcal{H}}(A) = \{\sigma_i \in \sigma(G) \backslash \sigma(A) \mid \sigma(A) \cap \sigma(H^G) \neq \emptyset$ for a Hall σ_i-subgroup $H \in \mathcal{H}\}$. A subgroup A of G is said to be *τ_σ-permutable* or *τ_σ-quasinormal* in G with respect to $\mathcal{H}$ if $AH^x = H^xA$ for all $x \in G$ and $H \in \mathcal{H}$ such that $\sigma(H) \subseteq \tau_{\mathcal{H}}(A)$, and *$\tau_\sigma$-permutable* or *$\tau_\sigma$-quasinormal* in G if A is *τ_σ-permutable* in G with respect to some complete Hall σ-set of G. We say that a subgroup A of G is *weakly τ_σ-quasinormal* in G if G has a σ-subnormal subgroup T such that $AT = G$ and $A \cap T \leq A_{\tau_\sigma G}$, where $A_{\tau_\sigma G}$ is the subgroup generated by all those subgroups of A which are τ_σ-quasinormal in G. We study the structure of G being based on the assumption that some subgroups of G are weakly τ_σ-quasinormal in G.

Mathematics Subject Classification (2010). 20D10, 20D15, 20D20, 20D35

Keywords. finite groups, σ-permutable subgroup, τ_σ-quasinormal subgroup, weakly τ_σ-quasinormal subgroup, supersoluble group

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P}\backslash\pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of all primes $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. Let $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$.

Following [18, 20, 34–36], a set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma(G)$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup for every $\sigma_i \in \sigma(G)$. G is said to be *σ-full* if G possesses a complete Hall σ-set; *σ-primary* if $|\sigma(G)| \leq 1$; *σ-nilpotent* if G has a complete

*Corresponding Author.
Email addresses: mthussain@mail.ustc.edu.cn (M.T. Hussain), cccao@mail.ustc.edu.cn (C. Cao), zhang12@mail.ustc.edu.cn (L. Zhang)
Received: 03.06.2019; Accepted: 17.12.2019

Hall σ-set $\mathcal{H} = \{H_1, \cdots, H_t\}$ such that $G = H_1 \times \cdots \times H_t$; *$\sigma$-soluble* if every chief factor of G is σ-primary; *σ-full group of Sylow type* if every subgroup of G is a D_{σ_i}-group for all $\sigma_i \in \sigma(G)$. Π is always supposed to be a non-empty subset of the set σ and $\Pi' = \sigma\backslash\Pi$. n is said to be a Π-*number* if $\pi(n) \subseteq \bigcup_{\sigma_i\in\Pi}\sigma_i$. A subgroup A of G is said to be Π-*subgroup* of G if $|A|$ is a Π-number; *σ-subnormal* in G if there exists a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \cdots, n$.

Let $\mathcal{L}$ be some non-empty set of subgroups of G and $K \leq G$. A subgroup A of G is called *$\mathcal{L}$-permutable* if $AH = HA$ for all $H \in \mathcal{L}$; *$\mathcal{L}^K$-permutable* if $AH^x = H^xA$ for all $H \in \mathcal{L}$ and all $x \in K$. In particular, a subgroup A of G is *σ-permutable* in G if G has a complete Hall σ-set $\mathcal{H}$ such that A is $\mathcal{L}^G$-permutable (see [34]).

It is well known that permutable subgroups and supplemented subgroups play an important role in the theory of finite groups. Recall that a subgroup A of G is said to be *σ-semipermutable* in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$ with $\sigma(A) \cap \sigma(H) = \emptyset$ (see [19]). Let $\tau_{\mathcal{H}}(A) = \{\sigma_i \in \sigma(G)\backslash\sigma(A) \mid \sigma(A) \cap \sigma(H^G) \neq \emptyset$ for a Hall σ_i-subgroup $H \in \mathcal{H}\}$. A subgroup A of G is said to be *τ_σ-permutable* or *τ_σ-quasinormal* in G with respect to $\mathcal{H}$ if $AH^x = H^xA$ for all $x \in G$ and $H \in \mathcal{H}$ such that $\sigma(H) \subseteq \tau_{\mathcal{H}}(A)$ (see [6]), and *τ_σ-permutable* or *τ_σ-quasinormal* in G if A is *τ_σ-permutable* in G with respect to some complete Hall σ-set $\mathcal{H}$ of G (see [6]). A subgroup A of G is said to be *c-normal* in G if G has a normal subgroup T such that $G = AT$ and $A \cap T \leq A_G$, where A_G is the maximal normal subgroup of G contained in A (see [38]). A subgroup A of G is said to be *weakly σ-permutable* in G if G has a σ-subnormal subgroup T such that $G = AT$ and $A \cap T \leq A_{\sigma G}$, where $A_{\sigma G}$ is the subgroup of A generated by all those subgroups of A which are σ-permutable in G (see [42]). By using the above subgroups and supplemented subgroups, the researchers have obtained a series of interesting results (see, for example, [4, 6, 8, 10, 14, 19, 26, 27, 31, 34, 38, 42]). Now, we consider the following new generalized supplemented subgroup.

Definition 1.1. We say that a subgroup A of G is said to be weakly τ_σ-quasinormal in G if G has a σ-subnormal subgroup T such that $AT = G$ and $A \cap T \leq A_{\tau_\sigma G}$, where $A_{\tau_\sigma G}$ is the subgroup generated by all those subgroups of A which are τ_σ-quasinormal in G.

In the classical case when $\sigma = \{\{2\}, \{3\}, \cdots\}$, σ-permutable subgroup, σ-semipermutable subgroup, τ_σ-quasinormal subgroup, weakly σ-permutable subgroup and weakly τ_σ-quasinormal subgroup are also called *S-permutable* subgroup [4, 10], *S-semipermutable* subgroup [14], *τ-quasinormal* subgroup [27], *weakly s-permutable* subgroup [31] and *weakly τ-quasinormal* subgroup [26], respectively. It is clear that every σ-permutable subgroup, every σ-semipermutable subgroup, every τ_σ-quasinormal subgroup and every weakly σ-permutable subgroup are weakly τ_σ-quasinormal.

Remark 1.2. In the case when $\sigma = \{\{2\}, \{3\}, \cdots\}$, [26, Examples 1.2 and 1.3] show that weakly τ_σ-quasinormal subgroups of G are not necessarily τ_σ-quasinormal, c-normal and weakly σ-permutable in G.

In this paper, we study the properties of weakly τ_σ-quasinormal subgroups and use them to determine the structure of finite groups. We obtain the following results.

Theorem 1.3. *Let G be a σ-full group of Sylow type and $\mathcal{H}$ a complete Hall σ-set of G such that every member of $\mathcal{H}$ is supersoluble. If every maximal subgroup of any non-cyclic $H \in \mathcal{H}$ is weakly τ_σ-quasinormal in G, then G is supersoluble.*

Recall that a normal subgroup E of G is called hypercyclically embedded in G (see [30, p.217]) if every chief factor of G below E is cyclic. Hypercyclically embedded subgroups play an important role in the theory of soluble groups (see [4, 14, 30, 41]) and the condition under which a normal subgroup is hypercyclically embedded in G were found by many authors (see books [4, 14, 30, 41] and the recent papers [15, 20, 23, 32, 33, 42]).

Suppose that G has a complete Hall σ-set $\mathcal{H} = \{H_1, \cdots, H_t\}$. Following [20], for any subgroup H (resp. normal subgroup N) of G we write $H \cap \mathcal{H}$ (resp. $\mathcal{H}N/N$) to denote the set $\{H \cap H_1, \cdots, H \cap H_t\}$ (resp. $\{H_1N/N, \cdots, H_tN/N\}$).

Theorem 1.4. *Let G be a σ-full group of Sylow type, $\mathcal{H}$ a complete Hall σ-set of G such that every member of $\mathcal{H}$ is nilpotent, and E a normal subgroup of G. If every maximal subgroup of any non-cyclic $H \in E \cap \mathcal{H}$ is weakly τ_σ-quasinormal in G, then E is hypercyclically embedded in G.*

Theorem 1.5. *Let G be a σ-full group of Sylow type, $\mathcal{H}$ a complete Hall σ-set of G such that every member of $\mathcal{H}$ is supersoluble and E a normal subgroup of G. If every cyclic subgroup H of any non-cyclic $T \in E \cap \mathcal{H}$ of prime order and order 4 (if the Sylow 2-subgroup of E is non-abelian and $H \nleq Z_\infty(G)$) is weakly τ_σ-quasinormal in G, then E is hypercyclically embedded in G.*

We shall give the proofs of Theorems 1.3-1.5 in section 3. In section 4, we consider some applications of our results.

All unexplained terminologies and notations are standard, as in [4, 11, 14].

2. Preliminaries

We use $\mathfrak{S}_\sigma$ to denote the class of all σ-soluble groups and $F_\sigma(G)$ to denote the product of all normal σ-nilpotent subgroups of G.

Lemma 2.1 (see [34, Lemma 2.1])**.** *The class $\mathfrak{S}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of a σ-soluble group by a σ-soluble group is a σ-soluble group.*

Lemma 2.2 (see [17, Lemma 2.6(i)])**.** *$F_\sigma(G)$ is σ-nilpotent.*

Following [18, 34], we use $O^{\Pi}(G)$ to denote the subgroup of G generated by all its Π'-subgroups. Instead of $O^{\{\sigma_i\}}(G)$ we write $O^{\sigma_i}(G)$. We use $O_{\Pi}(G)$ to denote the subgroup of G generated by all its normal Π-subgroups. Instead of $O_{\{\sigma_i\}}(G)$ (resp. $O_{\{\sigma_i\}'}(G)$) we write $O_{\sigma_i}(G)$ (resp. $O_{\sigma_i'}(G)$).

Lemma 2.3 (see [34, Lemma 2.6] and [18, Lemma 2.1])**.** *Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G.*

(1) *If A is a Π-group, then $A \leq O_\Pi(G)$.*
(2) *AN/N is σ-subnormal in G/N.*
(3) *$A \cap K$ is σ-subnormal in K.*
(4) *If $|G : A|$ is a Π-number, then $O^\Pi(A) = O^\Pi(G)$.*

Lemma 2.4 (see [34, Lemma 2.8])**.** *Let H, K and N be subgroups of a σ-full group G. Let $\mathcal{H} = \{H_1, \cdots, H_t\}$ be a complete Hall σ-set of G and $\mathcal{L} = \mathcal{H}^K$. Suppose that H is $\mathcal{L}$-permutable and N is normal in G. Then HN/N is $\mathcal{L}^*$-permutable, where $\mathcal{L}^* = \{H_1N/N, \cdots, H_tN/N\}^{KN/N}$. In particular, if H is σ-permutable in G, then HN/N is σ-permutable in G/N.*

Lemma 2.5 (see [34, Theorem C])**.** *Let G be a σ-full group of Sylow type. Then the set of all σ-permutable subgroups of G forms a sublattice of the lattice of all σ-subnormal subgroups of G.*

Lemma 2.6 (see [34, Lemma 3.1])**.** *Let H be a σ_1-subgroup of a σ-full group G. Then H is σ-permutable in G if and only if $O^{\sigma_1}(G) \leq N_G(H)$.*

Lemma 2.7 (see [21, VI, 4.10])**.** *Assume that A and B are two subgroups of G and $G \neq AB$. If $AB^g = B^gA$ holds for any $g \in G$, then either A or B is contained in a proper normal subgroup of G.*

Before continuing, we give some facts about τ_σ-quasinormal and weakly τ_σ-quasinormal subgroups of G.

Lemma 2.8 (see [6, Lemma 2.6]). *Suppose that G has a complete Hall σ-set $\mathcal{H} = \{H_1, \cdots, H_t\}$ such that the subgroups H and K of G are τ_σ-quasinormal in G with respect to $\mathcal{H}$. Let R be a normal subgroup of G and $H \le L \le G$. Then:*

(1) *$\mathcal{H}_0 = \{H_1R/R, \cdots, H_tR/R\}$ is a complete Hall σ-set of G/R. Moreover, if $\sigma(H) = \sigma(HR/R)$, then HR/R is τ_σ-quasinormal in G/N with respect to $\mathcal{H}_0$.*

(2) *If $HK = KH$ and $\sigma(H \cap K) = \sigma(H) = \sigma(K)$, then $H \cap K$ is τ_σ-quasinormal in G with respect to $\mathcal{H}$.*

(3) *If for some i we have $H \le O_{\sigma_i}(G)$, then H is σ-quasinormal in G.*

(4) *If $\mathcal{H}$ reduces into L, then H is τ_σ-quasinormal in L with respect to $L \cap \mathcal{H}$.*

(5) *If G is a σ-full group of Sylow type, then H is τ_σ-quasinormal in L.*

From Lemma 2.8 we directly have the following lemma.

Lemma 2.9. *Let G is a σ-full group of Sylow type and $H \le K$ be subgroups of G. Suppose that $\sigma_i \in \sigma(G)$ for some i.*

(1) *If H is a σ_i-group, then $H_{\tau_\sigma G}$ is τ_σ-quasinormal in G and $H_G \le H_{\tau_\sigma G}$.*

(2) *$H_{\tau_\sigma G} \le H_{\tau_\sigma K}$.*

(3) *If K is a σ_i-group and H is normal in G, then $K_{\tau_\sigma G}/H \le (K/H)_{\tau_\sigma (G/H)}$.*

(4) *If H is normal in G and E is a σ_i-subgroup of G such that $(|H|, |E|) = 1$, then $E_{\tau_\sigma G}H/H \le (EH/H)_{\tau_\sigma (G/H)}$.*

Lemma 2.10. *Let G be a σ-full group of Sylow type and $\mathcal{H} = \{H_1, \cdots, H_t\}$ a complete Hall σ-set of G. Suppose that $H \le K \le G$ and $\sigma_i \in \sigma(G)$ for some i.*

(1) *If H is τ_σ-quasinormal in G, then H is weakly τ_σ-quasinormal in G.*

(2) *Suppose that K is a σ_i-group and H is normal in G. If K is weakly τ_σ-quasinormal in G, then K/H is weakly τ_σ-quasinormal in G/H.*

(3) *If H is weakly τ_σ-quasinormal in G, then H is weakly τ_σ-quasinormal in K.*

(4) *Suppose that H is normal in G and E is a σ_i-subgroup of G such that $(|H|, |E|) = 1$. If E is weakly τ_σ-quasinormal in G, then EH/H is weakly τ_σ-quasinormal in G/H.*

Proof. (1) This is obvious.

(2) Assume that for some σ-subnormal subgroup T of G, we have $KT = G$ and $T \cap K \le K_{\tau_\sigma G}$. Then by Lemma 2.3(2), TH/H is σ-subnormal in G/H. Besides, we have that $(TH/H)(K/H) = G/H$ and $(TH/H) \cap (K/H) = (TH \cap K)/H = (T \cap K)H/H \le K_{\tau_\sigma G}H/H = K_{\tau_\sigma G}/H \le (K/H)_{\tau_\sigma (G/H)}$ by Lemma 2.9(1)(3). This shows that K/H is weakly τ_σ-quasinormal in G/H.

(3) Let T be a σ-subnormal subgroup of G such that $HT = G$ and $T \cap H \le H_{\tau_\sigma G}$. Then $K = K \cap HT = H(K \cap T)$. By Lemma 2.3(3), we have that $K \cap T$ is σ-subnormal in K. Moreover, we have that $(K \cap T) \cap H \le H_{\tau_\sigma G} \le H_{\tau_\sigma K}$ by Lemma 2.9(2). Therefore, H is weakly τ_σ-quasinormal in K.

(4) Assume that for some σ-subnormal subgroup T of G we have $ET = G$ and $T \cap E \le E_{\tau_\sigma G}$. Clearly, $(TH/H)(EH/H) = G/H$. Since $(|H|, |E|) = 1$, we have that

$$\begin{aligned}(|T \cap EH : T \cap E|, &|T \cap EH : T \cap H|)\\ &= (|(T \cap EH)E : E|, |(T \cap EH)H : H|) | (|EH : E|, |EH : H|)\\ &= 1.\end{aligned}$$

Hence by [11, Ch. A, 1.6(b)], $T \cap EH = (T \cap E)(T \cap H)$. It follows from Lemma 2.9(4) that $(TH/H) \cap (EH/H) = (TH \cap EH)/H = (T \cap EH)H/H = (T \cap E)H/H \le E_{\tau_\sigma G}H/H \le (EH/H)_{\tau_\sigma (G/H)}$. Besides, since TH/H is σ-subnormal in G/H by Lemma 2.3(2), we obtain that EH/H is weakly τ_σ-quasinormal in G/H. □

Let P be a p-group. If P is not a non-abelian 2-group, then we use $\Omega(P)$ to denote $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

The following lemma is a corollary of [16, Lemma 4.4] and [9, Lemma 2.12].

Lemma 2.11. *Let P be a normal p-subgroup of G and C a Thompson critical subgroup of P (see [12, p.185]). If either $P/\Phi(P)$ is hypercyclically embedded in $G/\Phi(P)$ or $\Omega(C)$ is hypercyclically embedded in G, then P is hypercyclically embedded in G.*

Lemma 2.12 (see [16, Lemma 4.3])**.** *Let C be a Thompson critical subgroup of a p-group P.*

(1) *If p is odd, then the exponent of $\Omega(C)$ is p.*
(2) *If P is a non-abelian 2-group, then the exponent of $\Omega(C)$ is 4.*

Lemma 2.13 (see [33, Theorem C])**.** *Let E be a normal subgroup of G. If $F^*(E)$ is hypercyclically embedded in G, then E is hypercyclically embedded in G.*

In this Lemma, $F^*(E)$ is the generalized Fitting subgroup of E, that is, the largest normal quasinilpotent subgroup of E (see [22, Chapter X]).

Recall that a class of groups $\mathfrak{F}$ is said to be a *formation* provided that (i) if $G \in \mathfrak{F}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{F}$, and (ii) $G/(M \cap N) \in \mathfrak{F}$ for any normal subgroups M, N of G with $G/M \in \mathfrak{F}$ and $G/N \in \mathfrak{F}$. A formation $\mathfrak{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$.

Lemma 2.14 (see [31, Lemma 2.16] or [14, Theorem 1.2.7(b)])**.** *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

3. Proofs of Theorems 1.3-1.5

The following Proposition is the main stage in the proof of Theorem 1.3 and Theorem 1.4.

Proposition 3.1. *Let G be a σ-full group of Sylow type and $\mathcal{H} = \{H_1, \cdots, H_t\}$ be a complete Hall σ-set of G such that H_i is a supersoluble σ_i-group for all $i \in \{1, \cdots, t\}$, and let the smallest prime p of $\pi(G)$ belongs to σ_j. If every maximal subgroup of H_j is weakly τ_σ-quasinormal in G, then G is soluble.*

Proof. Assume that this is false and let (G, H_j) be a counterexample with minimal $|G| + |H_j|$. Without loss of generality, we may assume that $j = 1$. Then $p = 2 \in \pi(H_1)$ by the Feit-Thompson theorem.

(1) *G is not σ-soluble, and so $|\sigma(G)| > 1$.*

Assume that G is σ-soluble. Then every chief factor H/K of G is σ-primary, that is, H/K is a σ_i-group for some i. But since H_i is supersoluble, H/K is an elementary abelian group. It follows that G is soluble. This contradiction shows that (1) holds.

(2) $O_{\sigma_1}(G) = 1$.

Assume that $O_{\sigma_1}(G) \neq 1$. Let $N = O_{\sigma_1}(G)$. If $N = H_1$, then G/N is soluble by the Feit-Thompson theorem, and so G is σ-soluble, contrary to Claim (1). Hence $N \neq H_1$, so H_1/N is a non-identity Hall σ_1-subgroup of G/N. Let M/N be a maximal subgroup of H_1/N. Then M is a maximal subgroup of H_1. By the hypothesis and Lemma 2.10(2), M/N is weakly τ_σ-quasinormal in G/N. This shows that the hypothesis holds for $(G/N, H_1/N)$. Hence G/N is soluble by the choice of (G, H_1). Consequently, G is σ-soluble by Lemma 2.1, which contradicts Claim (1). Hence we have (2).

(3) $O_{\sigma_1'}(G) = 1$.

Assume that $K = O_{\sigma_1'}(G) \neq 1$. Then H_1K/K is a Hall σ_1-subgroup of G/K. Let W/K be a maximal subgroup of H_1K/K. Then $W = (H_1 \cap W)K$ is a maximal subgroup of H_1K. If $H_1 \cap W$ is not a maximal subgroup of H_1, then there exists a subgroup E of H_1

such that $H_1 \cap W < E < H_1$. Since $(|H_1|, |K|) = 1$, $W < EK < H_1K$. This contradiction shows that $H_1 \cap W$ is a maximal subgroup of H_1. By the hypothesis and Lemma 2.10(4), W/K is weakly τ_σ-quasinormal in G/K. This shows that $(G/K, H_1K/K)$ satisfies the hypothesis, so G/K is soluble by the choice of (G, H_1). But since K is soluble by the Feit-Thompson theorem, it follows that G is soluble. This contradiction shows that (3) holds.

(4) *Let R be a minimal normal subgroup of G. Then R is not σ-soluble, $G = RH_1$ and G/R is soluble.*

Assume that R is σ-soluble. Then R is a σ_i-subgroup of G for some i. It follows that $R \leq O_{\sigma_1}(G)$ or $R \leq O_{\sigma_1'}(G)$, which contradicts Claim (2) or (3). Hence R is not σ-soluble. By the hypothesis and Lemma 2.10(3), it is easy to see that (RH_1, H_1) satisfies the hypothesis. If $RH_1 < G$, then RH_1 is soluble by the choice of G. It follows that R is soluble, and so R is σ-soluble, a contradiction. Hence, $G = RH_1$. Consequently, $G/R = H_1R/R \cong H_1/(H_1 \cap R)$ is soluble since H_1 is supersoluble.

(5) *R is the unique minimal normal subgroup of G and $F_\sigma(G) = 1$.*

This directly follows from Claim (4) and Lemma 2.2.

(6) $R \cap H_1 \nleq \Phi(H_1)$.

Assume that $R \cap H_1 \leq \Phi(H_1)$. Then by [21, IV, Theorem 4.6], there exists a normal subgroup M of R such that R/M is a σ_1-group and $|R \cap H_1| \mid |R/M|$. It follows that $O^{\sigma_1}(R) \leq M$. Since $O^{\sigma_1}(R)$ *char* $R \trianglelefteq G$, we have $O^{\sigma_1}(R) \trianglelefteq G$, so $O^{\sigma_1}(R) = 1$ or R by Claim (5). If $O^{\sigma_1}(R) = 1$, then $R \leq H_1$, which contradicts Claim (4). Hence $O^{\sigma_1}(R) = R$, and therefore $M = R$. Moreover, since $|R \cap H_1| \mid |R/M|$, we obtain that $R \cap H_1 = 1$. But, clearly, $R \cap H_1$ is a Hall σ_1-subgroup of R. Thus R is a σ_1'-subgroup, so $R \leq O_{\sigma_1'}(G) = 1$, a contradiction. Hence (6) holds.

(7) *Final contradiction.*

By Claim (6), H_1 has a maximal subgroup L such that $H_1 = (R \cap H_1)L$. By the hypothesis, there exists a σ-subnormal subgroup T of G such that $G = LT$ and $L \cap T \leq L_{\tau_\sigma G}$. Since $|G : T| = |LT : T| = |L : L \cap T|$ is a σ_1-number, we obtain that $O^{\sigma_1}(T) = O^{\sigma_1}(G)$ by Lemma 2.3(4). As $t > 1$, $O^{\sigma_1}(G) > 1$. It follows from Claim (5) that $R \leq O^{\sigma_1}(G) = O^{\sigma_1}(T) \leq T_G \leq T$. Hence $L \cap R \leq L \cap T \leq L_{\tau_\sigma G}$, and so $L \cap R = L_{\tau_\sigma G} \cap R$. Let R_j be any Hall σ_j-subgroup of R with $j \neq 1$. Then R_j is also a Hall σ_j-subgroup of G by Claim (4). It follows from Claim (3) that $L_{\tau_\sigma G}R_j = R_jL_{\tau_\sigma G}$. Hence

$$R_j(L \cap R) = R_j(L_{\tau_\sigma G} \cap R) = R_jL_{\tau_\sigma G} \cap R = L_{\tau_\sigma G}R_j \cap R = (L_{\tau_\sigma G} \cap R)R_j = (L \cap R)R_j,$$

which implies that $L \cap R$ is τ_σ-quasinormal in R. Clearly, we can see that $(L \cap R)R_j$ is a proper subgroup of R. Applying Lemma 2.7, we can assume that M is a proper normal subgroup of R such that either $L \cap R \leq M$ or $R_j \leq M$. If $R_j \leq M$, then $R_j = 1$ since R is the minimal normal subgroup of G by Claim (5)(see [21, I, Theorem 9.12(b)]). Hence R is a σ_1-group, a contradiction. If $L \cap R \leq M$, then $L \cap R \leq L \cap M$. It follows that

$$|R/M|_{\sigma_1} = |R|_{\sigma_1}/|M|_{\sigma_1} = |H_1 \cap R : H_1 \cap M| \;\Big|\; |H_1 \cap R : L \cap M| \;\Big|\; |H_1 \cap R : L \cap R|,$$

But as H_1 is supersoluble and L is a maximal subgroup of H_1, we have that $|H_1 \cap R : L \cap R| = |H_1 : L| = q$, where $q \in \sigma_1$ is a prime. This shows that $|R/M|_{\sigma_1}|q$. Note that $2||R|$ by Claim (4), we have that $2||R/M|$ by [21, I, Theorem 9.12(b)]. This implies that $q = 2$. Hence R/M is 2-nilpotent by [21, IV, Theorem 2.8], and so it is soluble. Again by [21, I, Theorem 9.12(b)], we obtain that R is soluble, a contradiction. This final contradiction completes the proof. □

Proof of Theorem 1.3. Let $\mathcal{H} = \{H_1, \cdots, H_t\}$ be a complete Hall σ-set of G. We can assume without loss of generality that H_i is a supersoluble σ_i-group for all $i \in \{1, \cdots, t\}$. Assume that this is false and let G be a counterexample of minimal order. Then:

(1) *G is soluble.*

By the Feit-Thompson theorem, we may assume that $2||G|$. Without loss of generality, we may assume that $2 \in \pi(H_1)$. If H_1 is cyclic, then G has a cyclic Sylow 2-subgroup. Hence G is 2-nilpotent by [21, IV, Theorem 2.8] and so G is soluble. If H_1 is non-cyclic, then G is soluble by Proposition 3.1.

(2) *Let R be a minimal normal subgroup of G. Then G/R is supersoluble.*

It is clear that $\overline{\mathcal{H}} = \{H_1R/R, H_2R/R, \cdots, H_tR/R\}$ is a complete Hall σ-set of G/R and $H_iR/R \cong H_i/H_i \cap R$ is supersoluble. By Claim (1), R is an elementary abelian p-group for some prime p. Without loss of generality, we may assume that $R \leq H_1$. Assume that H_1/R is non-cyclic. Then H_1 is non-cyclic. Let M/R be a maximal subgroup of H_1/R. Then M is a maximal subgroup of H_1. By the hypothesis and Lemma 2.10(2), M/R is weakly τ_σ-quasinormal in G/R. Now let M_i/R be a maximal subgroup of H_iR/R, where $i \neq 1$, and suppose that H_iR/R is non-cyclic. Then $M_i = (H_i \cap M_i)R$ is a maximal subgroup of H_iR. With the same discussion as Claim (3) in the proof of Proposition 3.1, we have that $H_i \cap M_i$ is a maximal subgroup of H_i. Then by the hypothesis and Lemma 2.10(4), M_i/R is weakly τ_σ-quasinormal in G/R. This shows that the hypothesis holds for G/R. The choice of G implies that G/R is supersoluble.

(3) *R is the unique minimal normal subgroup of G, $\Phi(G) = 1$, $C_G(R) = R = F(G) = O_p(G)$, R is an elementary abelian p-group for some prime p and $|R| > p$.*

This directly follows from Claims (1), (2) and [11, Chapter A, Theorem 15.2].

Without loss of generality, we may assume that $p \in \pi(H_1)$. Then $R \leq H_1$.

(4) *Final contradiction.*

Since $\Phi(G) = 1$, $R \nleq \Phi(H_1)$ by [21, III, Lemma 3.3]. Hence there exists a maximal subgroup K of H_1 such that $H_1 = RK$. Let $E = R \cap K$. By Claim (3), we have that $E \trianglelefteq H_1$. Since H_1 is supersoluble, $|R : E| = |RK : K| = |H_1 : K|$ is a prime. Hence E is a maximal subgroup of R, and so $E \neq 1$ by Claim (3). Since R is not cyclic by Claim (3) and $R \leq H_1$, H_1 is non-cyclic. Then by the hypothesis, there exists a σ-subnormal subgroup T of G such that $G = KT$ and $K \cap T \leq K_{\tau_\sigma G}$. Since $|G : T|$ is a σ_1-number, we have that $O^{\sigma_1}(T) = O^{\sigma_1}(G)$ by Lemma 2.3(4). If $O^{\sigma_1}(G) = 1$, then $G = H_1$. Hence $E \trianglelefteq G$, which contradicts the minimality of R. Hence $O^{\sigma_1}(G) \neq 1$, and so $R \leq O^{\sigma_1}(T) \leq T$ by Claim (3). It follows that $K \cap R \leq K \cap T \leq K_{\tau_\sigma G}$, and so $K \cap R = K_{\tau_\sigma G} \cap R$. Let H_j be any Hall σ_j-subgroup of G with $j \neq 1$. In view of Claim (3) and Lemma 2.9(1), we have that $K_{\tau_\sigma G}H_j = H_jK_{\tau_\sigma G}$. Hence $E = K \cap R = K_{\tau_\sigma G} \cap R = K_{\tau_\sigma G}H_j \cap R \trianglelefteq K_{\tau_\sigma G}H_j$. Moreover, since $E \trianglelefteq H_1$ by above, we obtain that $E \trianglelefteq G$. By the minimality of R, we have that $E = 1$, which contradicts with Claim (3). This final contradiction completes the proof. □

Proof of Theorem 1.4. Let $\mathcal{H} = \{H_1, \cdots, H_t\}$ be a complete Hall σ-set of G. We can assume without loss of generality that H_i is a nilpotent σ_i-group for all $i \in \{1, \cdots, t\}$. Assume that this is false and (G, E) be a counterexample with minimal $|G| + |E|$. Then:

(1) *E is supersoluble.*

It is clear that $E \cap \mathcal{H}$ is a complete Hall σ-set of E, $H_i \cap E$ is nilpotent and E is a σ-full group of Sylow type. By Lemma 2.10(3) and Theorem 1.3, we get that E is supersoluble.

(2) *Let R be a minimal normal subgroup of G contained in E. Then R is an elementary abelian p-group for some prime p, E/R is hypercyclically embedded in G/R and R is non-cyclic.*

By Claim (1), R is an elementary abelian p-group for some prime p. Without loss of generality, we may assume that $R \leq H_1$. Clearly, $\mathcal{H}R/R$ is a complete Hall σ-set of G/R and $H_iR/R \cong H_i/(H_i \cap R)$ is nilpotent. Assume that $(H_1/R) \cap (E/R)$ is non-cyclic. Then $H_1 \cap E$ is non-cyclic. Let M/R be a maximal subgroup of $(H_1/R) \cap (E/R)$. Then M is a maximal subgroup of $H_1 \cap E$. Hence M/R is weakly τ_σ-quasinormal in G/R by the hypothesis and Lemma 2.10(2). Now assume that M_i/R is a maximal subgroup of some non-cyclic $(H_iR/R) \cap (E/R)$, where $i \neq 1$. Then $H_iR \cap E$ is non-cyclic and

$M_i = (H_i \cap M_i)R$ is a maximal subgroup of $H_iR \cap E$. With the same discussion as Claim (3) in the proof of the Proposition 3.1, we have that $H_i \cap M_i$ is a maximal subgroup of $H_i \cap E$. Then by the hypothesis and Lemma 2.10(4), M_i/R is weakly τ_σ-quasinormal in G/R. This shows that $(G/R, E/R)$ satisfies the hypothesis. Hence E/R is hypercyclically embedded in G/R by the choice of (G, E). It is also clear that R is non-cyclic. Hence (2) holds.

(3) *R is the unique minimal normal subgroup of G contained in E.*

Let L be a minimal normal subgroup of G contained in E such that $L \neq R$. Then E/L is also hypercyclically embedded in G/L by Claim (2). It follows that RL/L is hypercyclically embedded in G/L. Then $|R| = p$ for $RL/L \cong R$, contrary to Claim (2). Hence we have (3).

Without loss of generality, we may assume that $p \in \pi(H_1)$.

(4) *E is a p-group, and so $E \leq H_1$.*

Let Q be a Sylow q-subgroup of E, where q is the largest prime belongs of $\pi(E)$. Since E is supersoluble by Claim (1), we obtain that $Q \; char E \trianglelefteq G$ and so $Q \trianglelefteq G$. Hence $R \leq Q$, $p = q$ and $F(E) = Q$ is a Sylow p-subgroup of E by Claim (3). It follows from [13, Theorem 1.8.18] that $C_E(Q) \leq Q$. Moreover, since $Q \leq H_1 \cap E$ and H_1 is nilpotent, we obtain that $Q = H_1 \cap E$. Hence $H_1 \cap Q = Q = H_1 \cap E$ and $H_i \cap Q = 1$ for all $i \in \{2, \cdots, t\}$. This implies the hypothesis holds for (G, Q). Assume that $Q < E$. Then Q is hypercyclically embedded in G by the choice of (G, E). It follows that R is hypercyclically embedded in G, and so R is cyclic by Claim (3), contrary to Claim (2). Hence $E = Q$ is a p-group, and so $E \leq H_1$.

(5) *$\Phi(E) = 1$, so E is an elementary abelian p-group.*

Assume that $\Phi(E) \neq 1$. Then $R \leq \Phi(E)$ by Claim (3). Hence $E/\Phi(E)$ is hypercyclically embedded in $G/\Phi(E)$ by Claim (2) and [14, Chapter 1, Theorem 2.6(d)]. It follows from Claim (4) and Lemma 2.11 that E is hypercyclically embedded in G. This contradiction shows that (5) holds.

(6) *Final contradiction.*

Let R_1 be a maximal subgroup of R such that $R_1 \trianglelefteq H_1$. Then $|R_1| > 1$ by Claim (3). By Claim (5), there exists a complement S of R in E (maybe $S = 1$). Let $V = R_1S$. Then, clearly, $R_1 = R \cap V$ and V is a maximal subgroup of E. By the hypothesis and Claims $(2)-(5)$, there exists a σ-subnormal subgroup T of G such that $G = VT$ and $V \cap T \leq V_{\tau_\sigma G}$. In view of Claim (4) and Lemma 2.8(3), we have that $V_{\tau_\sigma G}$ is σ-quasinormal subgroup of G. With the same discussion as Claim (6) in the proof of [42, Theorem 1.13], we have that $R_1 = 1$. This contradiction completes the proof. □

In order to prove Theorem 1.5, we first prove the following:

Lemma 3.2. *Let G be a σ-full group of Sylow type, $\mathcal{H}$ a complete Hall σ-set of G such that every member of $\mathcal{H}$ is supersoluble and P a normal p-subgroup of G. If every cyclic subgroup H of P of prime order and order 4 (if P is a non-abelian 2-group and $H \not\leq Z_\infty(G)$) is weakly τ_σ-quasinormal in G, then P is hypercyclically embedded in G.*

Proof. Let $\mathcal{H} = \{H_1, \cdots, H_t\}$ be a complete Hall σ-set of G. We can assume without loss of generality that H_i is a supersoluble σ_i-group for all $i \in \{1, \cdots, t\}$. Assume that this is false and let (G, P) be a counterexample with minimal $|G| + |P|$. Without loss of generality, we may assume that $P \leq H_1$.

(1) *Let P/N be a chief factor of G. Then N is hypercyclically embedded in G. Hence N is the unique normal subgroup of G such that P/N is a chief factor of G and $|P/N| > p$.*

It is clear that (G, N) satisfies the hypothesis. Hence N is hypercyclically embedded in G by the choice of (G, P). Assume that G has another normal subgroup $R \neq N$ of G such that P/R is a chief factor of G. Then R is also hypercyclically embedded in G. It follows that $P/N = RN/N$ is hypercyclically embedded in G/N. Hence P is hypercyclically embedded in G. This contradiction shows that N is the unique normal subgroup such that P/N is a chief factor of G. It is also clear that $|P/N| > p$.

(2) *The exponent of P is p or 4 (if P is a non-abelian 2-group).*

Let C be a Thompson critical subgroup of P (see [12, p.185]). If $\Omega(C) < P$, then $\Omega(C) \leq N$ is hypercyclically embedded in G by Claim (1). Hence by Lemma 2.11, P is hypercyclically embedded in G, a contradiction. Hence $\Omega(C) = P$, so by Lemma 2.12, the exponent of P is p or 4 (if P is a non-abelian 2-group).

(3) *Final contradiction.*

Since H_1/N is supersoluble and $|P/N| > p$, H_1/N has a minimal normal subgroup L/N such that $1 \neq L/N < P/N$ and L/N is cyclic. Let $x \in L\backslash N$ and $H = \langle x \rangle$. Then $L = HN$ and $|H| = p$ or 4 (if P is a non-abelian 2-group) by Claim (2). If $H \leq Z_\infty(G)$, then $L/N = HN/N \leq Z_\infty(G)N/N \leq Z_\infty(G/N)$ by [14, Chapter 1, Theorem 2.6(d)]. So $Z_\infty(G/N) \cap P/N \neq 1$. Hence $P/N \leq Z_\infty(G/N)$ since P/N is a chief factor of G. It follows from Claim (1) that P is hypercyclically embedded in G. This contradiction shows that $H \nleq Z_\infty(G)$. Then by the hypothesis, there exists a σ-subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{\tau_{\sigma G}}$. With a similar argument as Claim (6) in the proof of Theorem 1.4, we have that $H_{\tau_{\sigma G}}$ is σ-quasinormal in G. In view of Claim (3) in the proof of [8, Lemma 3.2], we obtain that $L/N \trianglelefteq G/N$. This contradiction completes the proof. □

Proof of Theorem 1.5. Let $\mathcal{H} = \{H_1, \cdots, H_t\}$ be a complete Hall σ-set of G. We can assume without loss of generality that H_i is a supersoluble σ_i-group for all $i \in \{1, \cdots, t\}$. Assume that this is false and let (G, E) be a counterexample with minimal $|G| + |E|$. Let P be a Sylow p-subgroup of E, where p is the smallest prime containing in $\pi(E)$. Without loss of generality, we may assume that $P \leq H_1 \cap E$.

(1) *$H_1 \cap E$ is non-cyclic.*

Assume that $H_1 \cap E$ is cyclic. Then P is cyclic. By [21, Chapter IV, Theorem 2.8], E is p-nilpotent. Let $E_{p'}$ be a normal Hall p'-subgroup of E. Then $E_{p'} \trianglelefteq G$. If $E_{p'} = 1$, then E is cyclic, so E is hypercyclically embedded in G, a contradiction. Hence $E_{p'} \neq 1$. Clearly, $H_i \cap E_{p'} = H_i \cap E$ for $i = 2, \cdots, t$. This shows the hypothesis holds for $(G, E_{p'})$, so $E_{p'}$ is hypercyclically embedded in G by the choice of (G, E). But as $E/E_{p'} \cong P$ is cyclic, it follows that E is hypercyclically embedded in G. This contradiction shows that (1) holds.

(2) *If $E = P$, then E is hypercyclically embedded in G.*

This directly follows from Lemma 3.2 and Claim (1).

(3) *E is not p-nilpotent.*

Assume that E is p-nilpotent. Let $E_{p'}$ be a normal Hall p'-subgroup of E. Then $E_{p'} \trianglelefteq G$. By Claim (2), $E_{p'} \neq 1$. Clearly, $\mathcal{H}E_{p'}/E_{p'}$ is a complete Hall σ-set of $G/E_{p'}$ and $H_iE_{p'}/E_{p'} \cong H_i/H_i \cap E_{p'}$ is supersoluble.

We claim that the hypothesis holds for $(G/E_{p'}, E/E_{p'})$. In fact, $H_iE_{p'}/E_{p'} \cap E/E_{p'} = 1$ for $i = 2, \cdots, t$ and $H_1E_{p'}/E_{p'} \cap E/E_{p'} = E/E_{p'}$. It is trivial when $E/E_{p'}$ is cyclic. We may therefore, assume that $E/E_{p'}$ is non-cyclic. Let $H/E_{p'}$ be a cyclic subgroup of $E/E_{p'}$ of order p or 4 (if the Sylow 2-subgroup of $E/E_{p'}$ is non-abelian and $H/E_{p'} \nleq Z_\infty(G/E_{p'})$). Then by Schur-Zassenhaus theorem, $H = E_{p'} \rtimes L$ and without loss of generality, we may assume that $L \leq E \cap H_1$. Note that if $L \leq Z_\infty(G)$, then $H/E_{p'} = LE_{p'}/E_{p'} \leq Z_\infty(G)E_{p'}/E_{p'} \leq Z_\infty(G/E_{p'})$ by [14, Chapter 1, Theorem 2.6(d)]. Hence L is of order p or 4 (if the Sylow 2-subgroup of E is non-abelian and $L \nleq Z_\infty(G)$). Then by Lemma 2.10(4), we see that the hypothesis holds for $(G/E_{p'}, E/E_{p'})$. Hence $E/E_{p'}$ is hypercyclically embedded in $G/E_{p'}$ by the choice of (G, E). On the other hand, it is clear that the hypothesis holds for $(G, E_{p'})$, so $E_{p'}$ is hypercyclically embedded in G by the choice of (G, E). Therefore E is hypercyclically embedded in G, a contradiction. Hence we have (3).

(4) *Final contradiction.*

By Claim (3), [21, Chapter IV, Theorem 5.4] and [13, Theorem 3.4.11], E has a p-closed Schmidt subgroup $S = P_1 \rtimes Q$, where P_1 is a Sylow p-subgroup of S of exponent p or 4 (if P_1 is non-abelian 2-group), Q is a Sylow q-subgroup of S for some prime $q \neq p$, $P_1/\Phi(P_1)$ is an S-chief factor, $Z_\infty(S) = \Phi(S)$ and $\Phi(S) \cap P_1 = \Phi(P_1)$.

We claim that $|P_1 : \Phi(P_1)| = p$. If $q \in \pi(H_1)$, then S is a σ_1-group, and so $S \leq H_1^g$ for some $g \in G$ since G is a σ-full group of Sylow type. Since H_1 is supersoluble and $P_1/\Phi(P_1)$ is an S-chief factor, $|P_1 : \Phi(P_1)| = p$. Now we consider that $q \notin \pi(H_1)$. Assume that there exists a minimal subgroup $D/\Phi(P_1)$ of $P_1/\Phi(P_1)$ such that $D/\Phi(P_1)$ is not σ-quasinormal in $S/\Phi(P_1)$. Let $x \in D \backslash \Phi(P_1)$ and $U = \langle x \rangle$. Then $D = U\Phi(P_1)$ and $|U| = p$ or 4 (if P_1 is non-abelian 2-group). If $U \leq Z_\infty(G)$, then $U \leq Z_\infty(S) \cap P_1 = \Phi(S) \cap P_1 = \Phi(P_1)$, a contradiction. Hence $U \nleq Z_\infty(G)$. Then by the hypothesis and Lemma 2.10(3), U is weakly τ_σ-quasinormal in S. Hence there exists a σ-subnormal subgroup T of S such that $S = UT$ and $U \cap T \leq U_{\tau_\sigma S}$. Let $U_{\tau_\sigma S} = \langle U_1, \cdots, U_t \rangle$, where $U_1, \cdots, U_t$ are all non-identity τ_σ-quasinormal subgroups of S contained in U. Lemma 2.8(3) implies that U_i is σ-quasinormal in S since $U_i \leq P_1 \trianglelefteq S$. Then by Lemma 2.5, $U_{\tau_\sigma S}$ is σ-quasinormal in S. Arguing as for Claim (2) in the proof [8, Proposition 3.1], we have that $|P_1 : \Phi(P_1)| = p$. Hence P_1 is cyclic of exponent p. This implies that P_1 is a group of order p. Since $N_S(P_1)/C_S(P_1) \lesssim Aut(P_1)$ is a group of order $p-1$ and p is the smallest prime containing in $\pi(E)$, it follows that $N_S(P_1) = C_S(P_1) = S$. Thus $Q \trianglelefteq S$. This contradiction completes the proof. □

4. Some applications of our results

By Theorems 1.4 and 1.5, we may obtain the following results.

Corollary 4.1. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that G is a σ-full group of Sylow type and $\mathcal{H}$ a complete Hall σ-set of G such that every member of $\mathcal{H}$ is nilpotent. If every maximal subgroup of any non-cyclic $H \in E \cap \mathcal{H}$ is weakly τ_σ-quasinormal in G, then $G \in \mathfrak{F}$.*

Corollary 4.2. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that G is a σ-full group of Sylow type and $\mathcal{H}$ a complete Hall σ-set of G such that every member of $\mathcal{H}$ is supersoluble. If every cyclic subgroup H of any non-cyclic $T \in E \cap \mathcal{H}$ of prime order and order 4 (if the Sylow 2-subgroup of E is non-abelian and $H \nleq Z_\infty(G)$) is weakly τ_σ-quasinormal in G, then $G \in \mathfrak{F}$.*

Corollary 4.3. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that G is a σ-full group of Sylow type and $\mathcal{H}$ a complete Hall σ-set of G such that every member of $\mathcal{H}$ is nilpotent. If every maximal subgroup of any non-cyclic $H \in F^*(E) \cap \mathcal{H}$ is weakly τ_σ-quasinormal in G, then $G \in \mathfrak{F}$.*

Proof. By the hypothesis and Theorem 1.4, we obtain that $F^*(E)$ is hypercyclically embedded in G. Then E is hypercyclically embedded in G by Lemma 2.13. Hence by Lemma 2.14, $G \in \mathfrak{F}$. □

A similar argument as in the proof of Corollary 4.3, we can get the following corollary from Theorem 1.5.

Corollary 4.4. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. Suppose that G is a σ-full group of Sylow type and $\mathcal{H}$ a complete Hall σ-set of G such that every member of $\mathcal{H}$ is supersoluble. If every cyclic subgroup H of any non-cyclic $T \in F^*(E) \cap \mathcal{H}$ of prime order and order 4 (if* *the Sylow 2-subgroup of E is non-abelian and $H \nleq Z_\infty(G)$) is weakly τ_σ-quasinormal in G, then $G \in \mathfrak{F}$.*

Theorems 1.3-1.5 and Corollaries 4.1-4.4 cover lots of known results, in particular, [7, Theorem 3], [37, Theorems 1 and 2], [42, Theorems 1.5 and 1.13, Corollaries 1.6 and 1.14, and Proposition 4.1], [8, Theorems 1.2 and 1.10], [21, Chap. VI, Theorem 10.3], [28, Corollary 3.4], [31, Theorem 1.4], [38, Theorem 4.1], [3, Theroems 3.2 and 4.1, and Corollary 4.4], [2, Theroems 1.3 and 1.4], [39, Theorem 1 and Corollary 1], [29, Theorem 3.5], [24, Theorem 2], [40, Theorem 3.1], [25, Theorem 3.4], [1, Theorem 3.1], [5, Theorems 2 and 5].

Acknowledgment. Research was supported by the NNSF of China (11771409), and Wu Wen-Tsun Key Laboratory of Mathematics of Chinese Academy of Sciences and Anhui Initiative in Quantum Information Technologies (AHY150200). The third author was supported by the PHD Research Startup Foundation of Anhui Jianzhu University (2017QD20).

References

[1] M. Asaad, *On the solvability of finite groups*, Arch. Math. **51**, 289–293, 1988.

[2] M. Asaad, *On maximal subgroups of Sylow subgroups of finite groups*, Comm. Algebra **26**, 3647–3652, 1998.

[3] M. Asaad, M. Ramadan and A. Shaalan, *Influence of π-quasinormality on maximal subgroups of Sylow subgroups of Fitting subgroup of a finite group*, Arch. Math. **56**, 521–527, 1991.

[4] A. Ballester-Bolinches, R. Esteban-Romero and M. Asaad, *Products of Finite Groups*, Walter de Gruyter, Berlin, 2010.

[5] A. Ballester-Bolinches and M.C. Pedraza-Aguilera, *On minimal subgroups of finite groups*, Acta Math. Hungar. **73**, 335–342, 1996.

[6] J.C. Beidleman and A.N. Skiba, *On τ_σ-quasinormal subgroups of finite groups*, J. Group Theory, **20**, 955–969, 2017.

[7] J. Buckley, *Finite groups whose minimal subgroups are normal*, Math. Z. **116**, 15–17, 1970.

[8] C. Cao, Z. Wu and W. Guo, *Finite groups with weakly σ-permutable subgroups*, Siberian Math. J. **59**, 157–165, 2018.

[9] X. Chen, W. Guo and A.N. Skiba, *Some conditions under which a finite group belongs to a Baer-local formation*, Comm. Algebra, **42**, 4188–4203, 2014.

[10] W.E. Deskins, *On quasinormal subgroups of finite groups*, Math. Z. **82**, 125–132, 1963.

[11] K. Doerk and T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin-New York, 1992.

[12] D. Gorenstein, *Finite Groups*, Harper and Row Publishers, New York-Evanston-London, 1968.

[13] W. Guo, *The Theory of Classes of Groups*, Science Press-Kluwer Academic Publishers, Dordrecht-Boston-London, 2000.

[14] W. Guo, *Structure Theory for Canonical Classes of Finite Groups*, Springer, Heidelberg-New York-Dordrecht-London, 2015.

[15] W. Guo and A.N. Skiba, *Finite groups with permutable complete Wielandt sets of subgroups*, J. Group Theory **18**, 191–200, 2015.

[16] W. Guo and A.N. Skiba, *Finite groups with generalized Ore supplement conditions for primary subgroups*, J. Algebra, **432**, 205–227, 2015.

[17] W. Guo and A.N. Skiba, *Groups with maximal subgroups of Sylow subgroups σ-permutably embedded*, J. Group Theory, **20**, 169–183, 2017.

[18] W. Guo and A.N. Skiba, *On Π-quasinormal subgroups of finite groups*, Monatsh. Math. **185**, 443–453, 2018.

[19] W. Guo and A.N. Skiba, *On σ-semipermutable subgroups of finite groups*, Acta Math. Sin. **34**, 1379–1390, 2018.

[20] W. Guo, C. Cao, A.N. Skiba and D.A. Sinitsa, *Finite groups with $\mathcal{H}$-permutable subgroups*, Commun. Math. Stat. **5**, 83–92, 2017.

[21] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[22] B. Huppert and N. Blackburn, *Finite groups III*, Springer-verlag, Berlin-Heidelberg-New York, 1982.

[23] B. Li, *On Π-property and Π-normality of subgroups of finite groups*, J. Algebra, **334**, 321–337, 2011.

[24] D. Li and X. Guo, *The influence of c-normality of subgroups on the structure of finite groups II*, Commun. Algebra, **26**, 1913–1922, 1998.

[25] Y. Li, Y. Wang and H. Wei, *The influence of π-quasinormality of some subgroups of a finite groups*, Arch. Math. **81**, 245–252, 2003.

[26] V.O. Lukyanenko and A.N. Skiba, *On weakly τ-quasinormal subgroups of finite groups*, Acta Math. Hungar. **125**, 237–248, 2009.

[27] V.O. Lukyanenko and A.N. Skiba, *Finite groups in which τ-quasinormality is a transitive relation*, Rend. Sem. Mat. Univ. Padova, **124**, 231–246, 2010.

[28] L. Miao, *On weakly s-permutable subgroups of finite groups*, Bull. Barz. Math. Soc. **41** (2), 223–235, 2010.

[29] M. Ramadan, *Influence of normality on maximal subgroups of Sylow subgroups of a finite group*, Acta Math. Hungar. **59**, 107–110, 1992.

[30] R. Schmidt, *Subgroup Lattices of Groups*, Walter de Gruyter, Berlin, 1994.

[31] A.N. Skiba, *On weakly s-permutable subgroups of finite groups*, J. Algebra, **315**, 192–209, 2007.

[32] A.N. Skiba, *On two questions of L. A. Shemetkov concerning hypercyclically embedded subgroups of finite groups*, J. Group Theory, **13**, 841–850, 2010.

[33] A.N. Skiba, *A characterization of the hypercyclically embedded subgroups of finite group*, J. Pure Appl. Algebra, **215**, 257–261, 2011.

[34] A.N. Skiba, *On σ-subnormal and σ-permutable subgroups of finite groups*, J. Algebra, **436**, 1–16, 2015.

[35] A.N. Skiba, *On some results in the theory of finite partially soluble groups*, Commun. Math. Stat. **4**, 281–309, 2016.

[36] A.N. Skiba, *Some characterizations of finite σ-soluble $P\sigma T$-groups*, J. Algebra, **495**, 114–129, 2018.

[37] S. Srinivasan, *Two sufficient conditions for supersolvability of finite groups*, Israel J. Math. **35**, 210–214, 1980.

[38] Y. Wang, *c-normality of groups and its properties*, J. Algebra, **180**, 954–965, 1996.

[39] H. Wei, *On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups*, Comm. Algebra, **29**, 2193–2200, 2001.

[40] H. Wei, Y. Wang and Y. Li, *On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups II*, Comm. Algebra, **31**, 4807–4816, 2003.

[41] M. Weinstein et al., *Between Nilpotent and Solvable*, Polygonal Publishing House, Washington, 1982.

[42] C. Zhang, Z. Wu and W. Guo, *On weakly σ-permutable subgroups of finite groups*, Publ. Math. Debrecen, **91**, 489–502, 2017.