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Research Article

Fekete-Szegö problem for generalized bi-subordinate functions of complex order

Sercan Kazımoğlu, Erhan Deniz*

Department of Mathematics, Faculty of Science and Letters, Kafkas University, Campus, 36100, Kars-Turkey

Abstract

In this paper, we obtain the Fekete-Szegö inequality for the generalized bi-subordinate functions of complex order. The various results, which are presented in this paper, would generalize those in related works of several earlier authors.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the class of functions f that are analytic, univalent in $\mathbb{D}$ and are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

The Koebe one-quarter theorem assures that the image of unit disk $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \ (z \in \mathbb{D}) \text{ and } f(f^{-1}(w)) = w, \quad (|w| < r_0, \ r_0 \geq 1/4).$$

Furthermore, the Taylor-Maclaurin series of f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if f is univalent and f^{-1} has univalent analytic continuation, which we denote by g, to the unit disk $\mathbb{D}$. Let σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{D}$. Coefficient problem for bi-univalent functions were recently investigated by several authors [1,4–8,15–17,19,20]. A function $f \in \mathcal{A}$ is said to be subordinate to a function $h \in \mathcal{A}$, denoted by $f \prec h$, if there exists an analytic function $w \in \mathcal{B}_0$, where $\mathcal{B}_0 := \{w : w(0) = 0, \ |w(z)| < 1, \ z \in \mathbb{D}\}$ such that $f(z) = h(w(z))$. We let $\mathcal{S}^*$ consist of starlike functions $f \in \mathcal{A}$, that is, $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ in $\mathbb{D}$ and $\mathcal{C}$ consist of convex functions $f \in \mathcal{A}$, that is, $1 + \operatorname{Re}\{zf''(z)/f'(z)\} > 0$ in $\mathbb{D}$. Ma and

*Corresponding Author.
Email addresses: srcnkzmglu@gmail.com (S. Kazımoğlu), edeniz36@gmail.com (E. Deniz)
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Minda [12] unified various subclasses of starlike and convex functions for which either of the quantity $zf'(z)/f(z)$ or $1+zf''(z)/f'(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the unit disk $\mathbb{D}$ and normalized by $\varphi(0)=1$ and $\varphi'(0)>0$. The class of Ma-Minda starlike functions consists of functions $f\in\mathcal{A}$ satisfying the subordination $zf'(z)/f(z)\prec\varphi(z)$. Similarly, the class of Ma-Minda convex functions consists of functions $f\in\mathcal{A}$ satisfying the subordination $1+zf''(z)/f'(z)\prec\varphi(z)$. Extensions of the above two classes (see [14]) are

$$\mathcal{S}^*(\gamma;\varphi)\equiv\left\{f\in\mathcal{A}:1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\varphi(z),\quad\gamma\in\mathbb{C}\setminus\{0\}\right\}$$

and

$$C(\gamma;\varphi)\equiv\left\{f\in\mathcal{A}:1+\frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)}\right)\prec\varphi(z),\quad\gamma\in\mathbb{C}\setminus\{0\}\right\}.$$

In literature, the functions belonging to these classes are called Ma-Minda starlike and convex of complex order γ $(\gamma\in\mathbb{C}\setminus\{0\})$, respectively. A function f is bi-starlike of Ma-Minda type of complex order γ $(\gamma\in\mathbb{C}\setminus\{0\})$ and bi-convex of Ma-Minda type of complex order γ $(\gamma\in\mathbb{C}\setminus\{0\})$ if both f and g are ,respectively, Ma-Minda starlike and convex of complex order γ $(\gamma\in\mathbb{C}\setminus\{0\})$. The classes consisting of bi-starlike of Ma-Minda type of complex order γ $(\gamma\in\mathbb{C}\setminus\{0\})$ and bi-convex of Ma-Minda type of complex order γ $(\gamma\in\mathbb{C}\setminus\{0\})$ are denoted by $\mathcal{S}^*_\sigma(\gamma;\varphi)$ and $\mathcal{C}_\sigma(\gamma;\varphi)$, respectively. As a special case $\gamma=1$ the classes $\mathcal{S}^*_\sigma(\gamma;\varphi)$ and $\mathcal{C}_\sigma(\gamma;\varphi)$ reduce to bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type functions are denoted by $\mathcal{S}^*_\sigma(\varphi)$ and $\mathcal{C}_\sigma(\varphi)$, respectively.

In this paper, we consider more general class $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$ for $0\le\lambda\le1$, $\gamma\in\mathbb{C}\setminus\{0\}$ which was investigated by Deniz [5] wherein he obtained the bounds for a_2 and a_3. This motivated us to study the Fekete-Szegö inequality to the class $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$. Recently, some authors have investigated the Fekete-Szegö problem for various subclasses of σ (see [3,9,13,21,22]).

2. Coefficient estimates

Throughout this paper φ denotes an analytic univalent function in $\mathbb{D}$ with positive real part and normalized by $\varphi(0)=1$, $\varphi'(0)>0$. Such a function has series expansion of the form

$$\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\ldots\quad(B_1>0)\,. \tag{2.1}$$

Definition 2.1. For $0\le\lambda\le1$ and $\gamma\in\mathbb{C}\setminus\{0\}$, the class $\mathcal{S}(\lambda,\gamma;\varphi)$ consists of functions $f\in\mathcal{A}$ satisfying

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1\right)\prec\varphi(z)\quad(z\in\mathbb{D}).$$

The class $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$ consists of functions $f\in\sigma$ such that $f,g\in\mathcal{S}(\lambda,\gamma;\varphi)$ where g is the analytic continuation of f^{-1} to the unit disk $\mathbb{D}$.

The class $\mathcal{S}(\lambda,\gamma;\varphi)$ was introduced by [18]. Motivated by this class the second author [5] defined and studied the class $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$, which is called the class of generalized bi-subordinate functions of complex order γ and type λ. As special cases of the class $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$, we have $\mathcal{S}_\sigma(0,\gamma;\varphi)\equiv\mathcal{S}^*_\sigma(\gamma;\varphi)$ and $\mathcal{S}_\sigma(1,\gamma;\varphi)\equiv\mathcal{C}_\sigma(\gamma;\varphi)$.

The class $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$ includes many earlier classes, which are mentioned below: $\mathcal{S}_\sigma(0,1;\varphi)\equiv\mathcal{S}^*_\sigma(\varphi)$ and $\mathcal{S}_\sigma(1,1;\varphi)\equiv\mathcal{C}_\sigma(\varphi)$, are classes of Ma-Minda bi-starlike and Ma-Minda bi-convex functions, respectively, introduced and studied in [11].

$\mathcal{S}_\sigma((0,1;(1+Az)/(1+Bz))\equiv\mathcal{S}_\sigma[A,B]$ and $\mathcal{S}_\sigma(1,1;(1+Az)/(1+Bz))\equiv\mathcal{C}_\sigma[A,B]$ $(-1\le B<A\le1)$ are, respectively, the classes of Janowski bi-starlike and bi-convex functions. Additionally, for $0\le\beta<1$, $\mathcal{S}_\sigma[1-2\beta,1]\equiv\mathcal{S}_\sigma(\beta)$ and $\mathcal{C}_\sigma[1-2\beta,1]\equiv\mathcal{C}_\sigma(\beta)$ are, respectively, the classes of bi-starlike and bi-convex functions of order β introduced and studied in [2].

For $0 < \beta \le 1$, $\mathcal{S}_\sigma\left(0, 1; \left(\frac{1+z}{1-z}\right)^\beta\right) \equiv \mathcal{SS}^*_\sigma(\beta)$ and $\mathcal{S}_\sigma\left(1, 1; \left(\frac{1+z}{1-z}\right)^\beta\right) \equiv \mathcal{SC}^*_\sigma(\beta)$ are, respectively, classes of strongly bi-starlike and strongly bi-convex functions of order β introduced and studied in [2].

For $\gamma \in \mathbb{C}\diagdown\{0\}$, $\mathcal{S}_\sigma(0, \gamma; (1+z)/(1-z)) \equiv \mathcal{S}^*_\sigma[\gamma]$ and $\mathcal{S}_\sigma(1, \gamma; (1+z)/(1-z)) \equiv \mathcal{C}_\sigma[\gamma]$ are classes of bi-starlike and bi-convex functions of complex order, respectively.

To prove our next theorems, we shall need the following well-known lemma (see [10]).

Lemma 2.2 ([10]). *Let the function $w \in \mathcal{B}_0$ be given by*

$$w(z) = c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{D}),$$

then for by every complex number s,

$$\left|c_2 - s c_1^2\right| \le 1 + (|s| - 1)\, |c_1|^2 .$$

In the following theorem, we consider functional $\left|a_3 - \mu a_2^2\right|$ for γ nonzero complex number and $\mu \in \mathbb{C}$.

Theorem 2.3. *Let the function f given by (1.1) be in the $\mathcal{S}_\sigma(\lambda, \gamma; \varphi)$. For $\gamma \in \mathbb{C}\diagdown\{0\}$ and $\mu \in \mathbb{C}$, we have*

$$|a_2| \le \frac{|\gamma|\, B_1}{1+\lambda}, \tag{2.2}$$

$$|a_3| \le \frac{|\gamma|\, |B_1|}{4\,(1+2\lambda)} \max\{2, (|s| + |t|)\} \tag{2.3}$$

and

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{B_1|\gamma|}{2(1+2\lambda)} & \text{if } \mathcal{L} \le 2 \\ \frac{B_1|\gamma|}{4(1+2\lambda)}\mathcal{L} & \text{if } \mathcal{L} > 2 \end{cases} \tag{2.4}$$

where $s = \frac{B_2}{B_1} - \frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2}$, $t = \frac{B_2}{B_1}$ *and* $\mathcal{L} = \left|\frac{B_2}{B_1} + (1-\mu)\, \frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2}\right| + \left|\frac{B_2}{B_1}\right|$.

Proof. Since $f \in \mathcal{S}_\sigma(\lambda, \gamma; \varphi)$, there exists two analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = 0 = v(0)$, such that

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right) = \varphi(u(z)) \quad (z \in \mathbb{D}) \tag{2.5}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda)g(w) + \lambda w g'(w)} - 1\right) = \varphi(v(w)). \tag{2.6}$$

Define the functions u and v by

$$u(z) = c_1 z + c_2 z^2 + \cdots \text{ and } v(w) = d_1 w + d_2 w^2 + \cdots . \tag{2.7}$$

Using (2.1) with (2.7), it is evident that

$$\varphi(u(z)) = 1 + (B_1 c_1) z + (B_1 c_2 + B_2 c_1^2) z^2 + \cdots \tag{2.8}$$

and

$$\varphi(v(w)) = 1 + (B_1 d_1) w + (B_1 d_2 + B_2 d_1^2) w^2 + \cdots . \tag{2.9}$$

Also, using (1.1), we get

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right) = 1 + \frac{(1+\lambda)\, a_2}{\gamma} z + \left[\frac{2\,(1+2\lambda)\, a_3 - (1+\lambda)^2\, a_2^2}{\gamma}\right] z^2 + \cdots \tag{2.10}$$

and using (1.2), we get

$$1+\frac{1}{\gamma}\left(\frac{wg'(w)+\lambda w^2g''(w)}{(1-\lambda)g(w)+\lambda wg'(w)}-1\right)$$
$$=1-\frac{(1+\lambda)\,a_2}{\gamma}w\left[\frac{-2\,(1+2\lambda)\,a_3+\left(3+6\lambda-\lambda^2\right)a_2^2}{\gamma}\right]w^2+\cdots. \quad (2.11)$$

Equating coefficients of right sides of equations (2.8) with (2.10) and (2.9) with (2.11) yield

$$\frac{(1+\lambda)\,a_2}{\gamma}=B_1c_1,\quad \frac{2\,(1+2\lambda)\,a_3-(1+\lambda)^2\,a_2^2}{\gamma}=B_1c_2+B_2c_1^2 \quad (2.12)$$

and

$$\frac{-\,(1+\lambda)\,a_2}{\gamma}=B_1d_1,\quad \frac{-2\,(1+2\lambda)\,a_3+\left(3+6\lambda-\lambda^2\right)a_2^2}{\gamma}=B_1d_2+B_2d_1^2 \quad (2.13)$$

so that, on account of (2.12) and (2.13)

$$c_1 = -d_1, \quad (2.14)$$
$$a_2 = \frac{\gamma B_1}{1+\lambda}c_1 \quad (2.15)$$

and

$$a_3=a_2^2+\frac{\gamma}{4\,(1+2\lambda)}\left[B_1c_2+B_2c_1^2-B_1d_2-B_2d_1^2\right]. \quad (2.16)$$

Taking into account (2.14), (2.15), (2.16) and the well known estimate $|c_1|\le 1$ of the Schwarz lemma, we get

$$|a_2|=\left|\frac{\gamma B_1}{1+\lambda}c_1\right|\le\frac{|\gamma|\,B_1}{1+\lambda} \quad (2.17)$$

and from Lemma 2.2,

$$\begin{aligned}
|a_3| &= \left|a_2^2+\frac{\gamma}{4\,(1+2\lambda)}\left[B_1c_2+B_2c_1^2-B_1d_2-B_2d_1^2\right]\right| \\
&= \left|\frac{\gamma^2B_1^2}{(1+\lambda)^2}c_1^2+\frac{\gamma}{4\,(1+2\lambda)}\left[\left(B_1c_2-B_2c_1^2\right)-\left(B_1d_2-B_2d_1^2\right)\right]\right| \\
&= \left|\frac{\gamma B_1}{4\,(1+2\lambda)}\left\{\left[c_2-\left(\frac{B_2}{B_1}-\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right)c_1^2\right]-\left[d_2-\frac{B_2}{B_1}d_1^2\right]\right\}\right| \\
&\le \frac{|\gamma|\,B_1}{4\,(1+2\lambda)}\left\{\left|c_2-\left(\frac{B_2}{B_1}-\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right)c_1^2\right|+\left|d_2-\frac{B_2}{B_1}d_1^2\right|\right\} \\
&\le \frac{|\gamma|\,B_1}{4\,(1+2\lambda)}\left\{1+(|s|-1)\left|c_1^2\right|+1+(|t|-1)\left|c_1^2\right|\right\} \\
&= \frac{|\gamma|\,B_1}{4\,(1+2\lambda)}\left\{2+(|s|+|t|-2)\left|c_1^2\right|\right\}.
\end{aligned}$$

Thus, using $|c_1|\le 1$ we have the desired estimate for $|a_3|$:

$$|a_3|\le\frac{|\gamma|\,|B_1|}{4\,(1+2\lambda)}\max\{2,(|s|+|t|)\},$$

where $s=\frac{B_2}{B_1}-\frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2}$ and $t=\frac{B_2}{B_1}$.

To find an estimate for $\left|a_3-\mu a_2^2\right|$, we express $a_3-\mu a_2^2$ in terms of c_i and d_i. Using the equality (2.16), we have

$$a_3-\mu a_2^2=(1-\mu)\,a_2^2+\frac{\gamma}{4\,(1+2\lambda)}\left[B_1c_2+B_2c_1^2-B_1d_2-B_2d_1^2\right].$$

Therefore from Lemma 2.2, we obtain

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &= \left|(1-\mu)\,a_2^2+\frac{\gamma}{4\,(1+2\lambda)}\left[B_1c_2+B_2c_1^2-B_1d_2-B_2d_1^2\right]\right|\\ &= \left|\frac{\gamma B_1}{4\,(1+2\lambda)}\left\{\left[c_2-\left(\frac{B_2}{B_1}-(1-\mu)\,\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right)c_1^2\right]-\left[d_2-\frac{B_2}{B_1}d_1^2\right]\right\}\right|\\ &\leq \frac{|\gamma|\,B_1}{4\,(1+2\lambda)}\left\{2+\left(\left|\frac{B_2}{B_1}-(1-\mu)\,\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right|+\left|\frac{B_2}{B_1}\right|-2\right)\left|c_1^2\right|\right\}. \quad (2.18)\end{aligned}$$

As a result of this, from $|c_1|\leq 1$ we obtain

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{B_1|\gamma|}{2(1+2\lambda)} & \text{if } \mathcal{L}<2,\\ \frac{B_1|\gamma|}{4(1+2\lambda)}\mathcal{L} & \text{if } \mathcal{L}\geq 2,\end{cases}$$

where $\mathcal{L}=\left|\frac{B_2}{B_1}+(1-\mu)\,\frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2}\right|+\left|\frac{B_2}{B_1}\right|$.
Thus the proof is completed. □

We next consider the cases γ and μ are real.

Theorem 2.4. *Let the function f given by (1.1) be in the $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$. For $\gamma>0$ and $\mu\in\mathbb{R}$, we have*

(1) *If $|B_2|\geq B_1$, then*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{\gamma|B_2|}{2(1+2\lambda)}-(\mu-1)\,\frac{\gamma^2B_1^2}{(1+\lambda)^2} & \text{if } \mu\leq 1\\ \frac{\gamma|B_2|}{2(1+2\lambda)}+(\mu-1)\,\frac{\gamma^2B_1^2}{(1+\lambda)^2} & \text{if } \mu>1\end{cases}.$$

(2) *If $|B_2|<B_1$, then*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{\gamma|B_2|}{2(1+2\lambda)}-(\mu-1)\,\frac{\gamma^2B_1^2}{(1+\lambda)^2} & \text{if } \mu\leq 1-\mathcal{F}\\ \frac{\gamma B_1}{2(1+2\lambda)} & \text{if } 1-\mathcal{F}<\mu<1+\mathcal{F}\\ \frac{\gamma|B_2|}{2(1+2\lambda)}+(\mu-1)\,\frac{\gamma^2B_1^2}{(1+\lambda)^2} & \text{if } \mu\geq 1+\mathcal{F}\end{cases}$$

where $\mathcal{F}=\frac{(1+\lambda)^2(B_1-|B_2|)}{2\gamma B_1^2(1+2\lambda)}$.

Proof. Using (2.18) and Lemma 2.2, we obtain

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &= \left|\frac{\gamma B_1}{4\,(1+2\lambda)}\left\{\left[c_2-\left(\frac{B_2}{B_1}-(1-\mu)\,\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right)c_1^2\right]-\left[d_2-\frac{B_2}{B_1}d_1^2\right]\right\}\right|\\ &\leq \frac{\gamma B_1}{4\,(1+2\lambda)}\left\{2+\left(\left|\frac{B_2}{B_1}-(1-\mu)\,\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right|+\left|\frac{B_2}{B_1}\right|-2\right)\left|c_1^2\right|\right\}\\ &\leq \frac{\gamma B_1}{2\,(1+2\lambda)}+\left\{\frac{\gamma\,(|B_2|-B_1)}{2\,(1+2\lambda)}+|\mu-1|\,\frac{\gamma^2B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right|. \quad (2.19)\end{aligned}$$

Now, the proof will be presented in two cases:

Firstly, we consider the case $|B_2| \geq B_1$.
If $\mu \leq 1$, then using (2.19) and $|c_1| \leq 1$, we obtain

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\} \\ &= \frac{\gamma |B_2|}{2(1+2\lambda)} - (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}.\end{aligned}$$

If $\mu > 1$, then using (2.19) and $|c_1| \leq 1$, we obtain

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\} \\ &= \frac{\gamma |B_2|}{2(1+2\lambda)} + (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}.\end{aligned}$$

Finally, we consider the case $|B_2| < B_1$. By using (2.19) and $|c_1| \leq 1$, we obtain the following results according to the cases of μ and $\mathcal{F}$.
For $\mu \leq 1 - \mathcal{F}$, we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\} \\ &= \frac{\gamma |B_2|}{2(1+2\lambda)} - (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2},\end{aligned}$$

and for $1 - \mathcal{F} < \mu \leq 1$, we yield

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)}.\end{aligned}$$

Similarly for $1 < \mu < 1 + \mathcal{F}$, we obtain

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)}.\end{aligned}$$

Finally for $\mu \geq 1 + \mathcal{F}$, we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\}\left|c_1^2\right| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{\frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}\right\} \\ &= \frac{\gamma |B_2|}{2(1+2\lambda)} + (\mu-1)\frac{\gamma^2 B_1^2}{(1+\lambda)^2}.\end{aligned}$$

Thus the proof is completed. □

Finally, we consider the cases of γ nonzero complex number and $\mu \in \mathbb{R}$.

Theorem 2.5. *Let the function f given by (1.1) be in the $\mathcal{S}_\sigma(\lambda,\gamma;\varphi)$. For $\gamma \in \mathbb{C}\diagdown\{0\}$ and $\mu \in \mathbb{R}$, we have*

(1) *If* $\frac{(1+|\sin\theta|)|B_2|}{2B_1} \geq 1$, *then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(1-\mu-\Re\left(k_1\right)\right)+\frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)} & \text{if } \mu \leq 1-\Re\left(k_1\right) \\ \frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)}-\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(1-\mu-\Re\left(k_1\right)\right) & \text{if } \mu > 1-\Re\left(k_1\right) \end{cases}.$$

(2) *If* $\frac{(1+|\sin\theta|)|B_2|}{2B_1} < 1$, *then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(1-\mu-\Re\left(k_1\right)\right)+\frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)} & \text{if } \mu \leq 1-\Re\left(k_1\right)+\mathcal{N} \\ \frac{|\gamma|B_1}{2(1+2\lambda)} & \text{if } 1-\Re\left(k_1\right)+\mathcal{N}<\mu<1-\Re\left(k_1\right)-\mathcal{N} \\ \frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)}-\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left(1-\mu-\Re\left(k_1\right)\right) & \text{if } \mu \geq 1-\Re\left(k_1\right)-\mathcal{N} \end{cases}$$

where $k_1=\frac{B_2(1+\lambda)^2 e^{i\theta}}{4B_1^2|\gamma|(1+2\lambda)}$, $|\gamma|=\gamma e^{i\theta}$ *and* $\mathcal{N}=\frac{(1+\lambda)^2[|B_2|(1+|\sin\theta|)-2B_1]}{4B_1^2|\gamma|(1+2\lambda)}$.

Proof. Let $f \in \mathcal{S}_\sigma(\lambda,\gamma;\varphi)$. By using (2.18) and Lemma 2.2, then we obtain

$$\begin{aligned} \left|a_3-\mu a_2^2\right| &\leq \frac{|\gamma|\, B_1}{4\,(1+2\lambda)}\left\{2+\left(\left|\frac{B_2}{B_1}-(1-\mu)\,\frac{4\gamma B_1\,(1+2\lambda)}{(1+\lambda)^2}\right|+\left|\frac{B_2}{B_1}\right|-2\right)\left|c_1^2\right|\right\} \\ &= \frac{|\gamma|\, B_1}{2\,(1+2\lambda)}+\frac{|\gamma|^2\, B_1^2}{(1+\lambda)^2} \\ &\quad \times\left[\left|(1-\mu)-\frac{B_2\,(1+\lambda)^2}{4B_1^2\gamma\,(1+2\lambda)}\right|+\frac{(|B_2|-2B_1)\,(1+\lambda)^2}{4B_1^2\,|\gamma|\,(1+2\lambda)}\right]\left|c_1^2\right|. \end{aligned}$$

Taking $|\gamma|=\gamma e^{i\theta}$, $k_1=\frac{B_2(1+\lambda)^2 e^{i\theta}}{4B_1^2|\gamma|(1+2\lambda)}$ and $l_1=\frac{(|B_2|-2B_1)(1+\lambda)^2}{4B_1^2|\gamma|(1+2\lambda)}$, for B_1, $B_2 \in \mathbb{R}$ and $B_1>0$, we rewrite

$$\begin{aligned} \left|a_3-\mu a_2^2\right| &\leq \frac{|\gamma|\, B_1}{2\,(1+2\lambda)}+\frac{|\gamma|^2\, B_1^2}{(1+\lambda)^2}\left(|1-\mu-k_1|+l_1\right)\left|c_1^2\right| \qquad (2.20) \\ &= \frac{|\gamma|\, B_1}{2\,(1+2\lambda)}+\frac{|\gamma|^2\, B_1^2}{(1+\lambda)^2}\left(|1-\mu-\Re\left(k_1\right)-i\left(\mathbf{Im}(k_1)\right|+l_1\right)\left|c_1^2\right| \\ &\leq \frac{|\gamma|\, B_1}{2\,(1+2\lambda)}+\frac{|\gamma|^2\, B_1^2}{(1+\lambda)^2}\left(|1-\mu-\Re\left(k_1\right)|+\frac{|B_2|\,(1+\lambda)^2\,|\sin\theta|}{4B_1^2\,|\gamma|\,(1+2\lambda)}+l_1\right)\left|c_1^2\right| \\ &= \frac{|\gamma|\, B_1}{2\,(1+2\lambda)}+\left[\frac{|\gamma|^2\, B_1^2}{(1+\lambda)^2}\,|1-\mu-\Re\left(k_1\right)|+\frac{|\gamma|\,[|B_2|\,(1+|\sin\theta|)-2B_1]}{4\,(1+2\lambda)}\right]\left|c_1^2\right|. \end{aligned}$$

Firstly, we consider the case $\frac{(1+|\sin\theta|)|B_2|}{2B_1} \geq 1$.
Let $\mu \leq 1-\Re\left(k_1\right)$. Then from (2.20) and $|c_1| \leq 1$, we obtain

$$\begin{aligned} \left|a_3-\mu a_2^2\right| &\leq \frac{|\gamma|\, B_1}{2\,(1+2\lambda)}+\left[\frac{|\gamma|^2\, B_1^2}{(1+\lambda)^2}\,|1-\mu-\Re\left(k_1\right)|+\frac{|\gamma|\,[|B_2|\,(1+|\sin\theta|)-2B_1]}{4\,(1+2\lambda)}\right]\left|c_1^2\right| \\ &\leq \frac{|\gamma|\, B_1}{2\,(1+2\lambda)}+\frac{|\gamma|^2\, B_1^2}{(1+\lambda)^2}\left(1-\mu-\Re\left(k_1\right)\right)+\frac{|\gamma|\,[|B_2|\,(1+|\sin\theta|)-2B_1]}{4\,(1+2\lambda)} \\ &= \frac{|\gamma|^2\, B_1^2}{(1+\lambda)^2}\left(1-\mu-\Re\left(k_1\right)\right)+\frac{|\gamma|\,|B_2|\,(1+|\sin\theta|)}{4\,(1+2\lambda)}. \end{aligned}$$

Let $\mu > 1 - \Re(k_1)$. Then from (2.20) and $|c_1| \leq 1$, we yield

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left|1-\mu-\Re(k_1)\right| + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|)-2B_1\right]}{4(1+2\lambda)}\right]\left|c_1^2\right| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}(\mu+\Re(k_1)-1) + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|)-2B_1\right]}{4(1+2\lambda)} \\ &= \frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}(1-\mu-\Re(k_1)).\end{aligned}$$

Finally, we want to consider the case with $\frac{(1+|\sin\theta|)|B_2|}{2B_1} < 1$. By using (2.20) and $|c_1| \leq 1$, we obtain the following results according to the cases of μ, k_1 and $\mathcal{N}$.
For $\mu \leq 1 - \Re(k_1) + \mathcal{N}$, we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left|1-\mu-\Re(k_1)\right| + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|)-2B_1\right]}{4(1+2\lambda)}\right]\left|c_1^2\right| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}(1-\mu-\Re(k_1)) + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|)-2B_1\right]}{4(1+2\lambda)} \\ &= \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}(1-\mu-\Re(k_1)) + \frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)},\end{aligned}$$

and for $1 - \Re(k_1) + \mathcal{N} < \mu \leq 1 - \Re(k_1)$, we obtain

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left|1-\mu-\Re(k_1)\right| + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|)-2B_1\right]}{4(1+2\lambda)}\right]\left|c_1^2\right| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)}.\end{aligned}$$

Similarly, for $1 - \Re(k_1) < \mu < 1 - \Re(k_1) - \mathcal{N}$, we yield

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left|1-\mu-\Re(k_1)\right| + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|)-2B_1\right]}{4(1+2\lambda)}\right]\left|c_1^2\right| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)},\end{aligned}$$

and finally, for $\mu \geq 1 - \Re(k_1) - \mathcal{N}$, we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}\left|1-\mu-\Re(k_1)\right| + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|)-2B_1\right]}{4(1+2\lambda)}\right]\left|c_1^2\right| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}(\mu+\Re(k_1)-1) + \frac{|\gamma|\left[|B_2|(1+|\sin\theta|)-2B_1\right]}{4(1+2\lambda)} \\ &= \frac{|\gamma||B_2|(1+|\sin\theta|)}{4(1+2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2}(1-\mu-\Re(k_1)).\end{aligned}$$

Thus the proof is completed. □

Taking $\gamma = 1$, $\lambda = 0$ and $\varphi(z) = (1+Az)/(1+Bz)$ $(-1 \leq B < A \leq 1)$ in Theorems 2.3, 2.4 and 2.5, we have the following corollary.

Corollary 2.6. *If $f \in \mathcal{A}$ is given by (1.1) belongs to the class $\mathcal{S}_\sigma[A,B]$, then*

(1) *For $\mu \in \mathbb{C}$,*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{A-B}{2} & \text{if } |B| + |4(1-\mu)(A-B) - B| < 2 \\ \frac{(A-B)}{4}\left[|B| + |4(1-\mu)(A-B) - B|\right] & \text{if } |B| + |4(1-\mu)(A-B) - B| \geq 2 \end{cases}.$$

(2) *For* $\mu \in \mathbb{R}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{|B|(A-B)}{2} - (\mu - 1)\ \ (A-B)^2 & \text{if} \quad \mu \leq 1 - \frac{1-|B|}{2(A-B)} \\ \frac{A-B}{2} & \text{if} \quad 1 - \frac{1-|B|}{2(A-B)} < \mu < 1 + \frac{1-|B|}{2(A-B)} \\ \frac{|B|(A-B)}{2} + (\mu - 1)\ \ (A-B)^2 & \text{if} \quad \mu \geq 1 + \frac{1-|B|}{2(A-B)} \end{cases}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} (A-B)\ \ \left[(A-B)(1-\mu) + \frac{|B|+B}{4}\right] & \text{if} \ \ \mu \leq 1 + \frac{|B|+B-2}{4(A-B)} \\ \frac{A-B}{2} & \text{if} \ \ 1 + \frac{|B|+B-2}{4(A-B)} < \mu < 1 - \frac{|B|-B-2}{4(A-B)} \\ (A-B)\ \ \left[(A-B)(\mu-1) + \frac{|B|-B}{4}\right] & \text{if} \ \ \mu \geq 1 - \frac{|B|-B-2}{4(A-B)} \end{cases}.$$

Taking $\gamma = 1$, $\lambda = 1$ and $\varphi(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ in Theorems 2.3, 2.4 and 2.5, we have the following corollary.

Corollary 2.7. *If $f \in \mathcal{A}$ is given by (1.1) belongs to the class $\mathcal{C}_\sigma[A, B]$, then*

(1) *For* $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{A-B}{6} & \text{if } |B| + |3(1-\mu)(A-B) - B| < 2 \\ \frac{(A-B)}{12}\left[|B| + |3(1-\mu)(A-B) - B|\right] & \text{if } |B| + |3(1-\mu)(A-B) - B| \geq 2 \end{cases}.$$

(2) *For* $\mu \in \mathbb{R}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{|B|(A-B)}{6} - (\mu - 1)\ \frac{(A-B)^2}{4} & \text{if} \quad \mu \leq 1 - \frac{2(1-|B|)}{3(A-B)} \\ \frac{A-B}{6} & \text{if} \quad 1 - \frac{2(1-|B|)}{3(A-B)} < \mu < 1 + \frac{2(1-|B|)}{3(A-B)} \\ \frac{|B|(A-B)}{6} + (\mu - 1)\ \frac{(A-B)^2}{4} & \text{if} \quad \mu \geq 1 + \frac{2(1-|B|)}{3(A-B)} \end{cases}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{A-B}{12}\ \left[3(A-B)(1-\mu) + |B| + B\right] & \text{if} \ \ \mu \leq 1 + \frac{2|B|+2B-1}{6(A-B)} \\ \frac{A-B}{6} & \text{if} \ \ 1 + \frac{2|B|+2B-1}{6(A-B)} < \mu < 1 - \frac{2|B|-2B-1}{6(A-B)} \\ \frac{A-B}{12}\ \left[3(A-B)(\mu-1) + |B| - B\right] & \text{if} \ \ \mu \geq 1 - \frac{2|B|-2B-1}{6(A-B)} \end{cases}.$$

Taking $\gamma \in \mathbb{C}\backslash\{0\}$, $\lambda = 0$ and $\varphi(z) = (1 + z)/(1 - z)$ in Theorems 2.3, 2.4 and 2.5, then we have the following corollary.

Corollary 2.8. *If $f \in \mathcal{A}$ is given by (1.1) belongs to the class $\mathcal{S}^*_\sigma[\gamma]$, then*

(i) *For* $\gamma \in \mathbb{C}\backslash\{0\}$ *and* $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} |\gamma| & \text{if} \quad |1 + (1-\mu)8\gamma| < 1 \\ \frac{|\gamma|}{2}\ \left[|1 + (1-\mu)8\gamma| + 1\right] & \text{if} \quad |1 + (1-\mu)8\gamma| \geq 1 \end{cases}.$$

(ii) *For* $\gamma > 0$ *and* $\mu \in \mathbb{R}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \gamma - 4(\mu - 1)\gamma^2 & \text{if} \quad \mu \leq 1 \\ \gamma + 4(\mu - 1)\gamma^2 & \text{if} \quad \mu > 1 \end{cases}.$$

(iii) *For* $\gamma \in \mathbb{C}\backslash\{0\}$ *and* $\mu \in \mathbb{R}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 4|\gamma|^2(1-\mu) + \frac{|\gamma|(1+|\sin\theta| - \cos\theta)}{2} & \text{if} \ \ \mu \leq 1 + \chi_1(\gamma, \theta) \\ |\gamma| & \text{if} \ \ 1 + \chi_1(\gamma, \theta) < \mu < 1 - \chi_2(\gamma, \theta) \\ \frac{|\gamma|(1+|\sin\theta| - \cos\theta)}{2} - 4|\gamma|^2(1-\mu) & \text{if} \ \ \mu \geq 1 - \chi_2(\gamma, \theta) \end{cases}.$$

where $\chi_1(\gamma, \theta) = \frac{(|\sin\theta| - \cos\theta - 1)}{8|\gamma|}$ *and* $\chi_2(\gamma, \theta) = \frac{(|\sin\theta| + \cos\theta - 1)}{8|\gamma|}$.

Taking $\gamma \in \mathbb{C}\setminus\{0\}$, $\lambda = 1$ and $\varphi(z) = (1+z)/(1-z)$ in Theorems 2.3, 2.4 and 2.5, we obtain the following corollary.

Corollary 2.9. *If $f \in \mathcal{A}$ is given by (1.1) belongs to the class $\mathcal{C}_\sigma[\gamma]$, then*

(i) *For* $\gamma \in \mathbb{C}\setminus\{0\}$ *and* $\mu \in \mathbb{C}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{|\gamma|}{3} & \text{if } \ |1 + (1-\mu)\,6\gamma| < 1 \\ \frac{|\gamma|}{2}\,\left[|1 + (1-\mu)\,6\gamma| + 1\right] & \text{if } \ |1 + (1-\mu)\,6\gamma| \geq 1 \end{cases}.$$

(ii) *For* $\gamma > 0$ *and* $\mu \in \mathbb{R}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{\gamma}{3} - (\mu-1)\,\gamma^2 & \text{if } \ \mu \leq 1 \\ \frac{\gamma}{3} + (\mu-1)\,\gamma^2 & \text{if } \ \mu > 1 \end{cases}.$$

(iii) *For* $\gamma \in \mathbb{C}\setminus\{0\}$ *and* $\mu \in \mathbb{R}$,

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} |\gamma|^2\,(1-\mu) + \frac{|\gamma|(1+|\sin\theta| - \cos\theta)}{6} & \text{if } \ \mu \leq 1 + \varphi_1(\gamma,\theta) \\ \frac{|\gamma|}{3} & \text{if } \ 1 + \varphi_1(\gamma,\theta) < \mu < 1 - \varphi_2(\gamma,\theta) \\ \frac{|\gamma|(1+|\sin\theta| - \cos\theta)}{6} - |\gamma|^2\,(1-\mu) & \text{if } \ \mu \geq 1 - \varphi_2(\gamma,\theta) \end{cases}$$

where $\varphi_1(\gamma,\theta) = \frac{(|\sin\theta| - \cos\theta - 1)}{6|\gamma|}$ *and* $\varphi_2(\gamma,\theta) = \frac{(|\sin\theta| + \cos\theta - 1)}{6|\gamma|}$.

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