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RECOGNITION OF COMPLEX POLYNOMIAL BÉZIER CURVES UNDER SIMILARITY TRANSFORMATIONS

İdris ÖREN and Muhsin İNCESU[2]

[1]Department of Mathematics, Faculty of Science, Karadeniz Technical University, 61080,Trabzon, TURKEY.
[2]Education Faculty, Muş Alparslan University, 49100, Muş, TURKEY

Abstract. In this paper, similarity groups in the complex plane $\mathbb{C}$, polynomial curves and complex Bézier curves in $\mathbb{C}$ are introduced. Global similarity invariants of polynomial curves and complex Bézier curves in $\mathbb{C}$ are given in terms of complex functions. The problem of similarity of two polynomial curves in $\mathbb{C}$ are solved. Moreover, in case two polynomial curve (complex Bézier curve) are similar for the similarity group, a general form of all similarity transformations, carrying one curve into the other curve, are obtained.

1. Introduction

The invariance is a very important tool in areas data registration, object recognition, computer aided design applications. In computer aided applications, the iterative closest point(ICP) algorithm is an accurate and efficient method for rigid registration problem and curve matching. The aim of registration or object recognition is to find the corresponding relationship between two point sets(or two curves) and compute the transformation which aligns two point sets(or two curves)(see [1–4]) Generally, Euclidean invariant features are used in above mentioned methods and a representation of polynomial curve or Bézier curve in the complex plane $\mathbb{C}$ are a useful method to investigate of their global invariants. (see [5, 7–10, 16]) In [16], taking customary rational Bézier curves in complex plane, complex rational Bézier curves are investigated. For Bézier curves, rational curves and implicit algebraic curves, detecting whether two plane curves are similar by an orientation preserving similarity transformation is important. (see [11–19]).

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oren@ktu.edu.tr-Corresponding author; m.incesu@alparslan.edu.tr
0000-0003-2716-3945; 0000-0003-2515-9627.

This paper presents the similarity conditions of two point sets and the similarity conditions of two polynomial paths(two complex Bézier curve) in the complex plane $\mathbb{C}$.

The polynomial curve $Z(u), W(u), u \in [0,1]$ in defined in terms of monomial complex control points $p_j, q_j \in \mathbb{C}$ as

$Z(u) = \sum_{j=0}^{m} p_j u^j$ and $W(u) = \sum_{j=0}^{m} p_j u^j$, resp.

The complex Bézier curves $Z(u), W(u), u \in [0,1]$ in defined in terms of degree m Bernstein polynomials $B_j^m(u)$ and complex control points $z_j, w_j \in \mathbb{C}$ as

$Z(u) = \sum_{j=0}^{m} z_j B_j^m(u)$ and $W(u) = \sum_{j=0}^{m} w_j B_j^m(u)$.

Let $GM(\mathbb{C}^*)$ be the group of all similarities of $\mathbb{C}$, $GM^+(\mathbb{C}^*)$ be the group of all orientation-preserving similarities of $\mathbb{C}$. The group of all linear similarities of $\mathbb{C}$ is denoted by $M(\mathbb{C}^*)$. The group of all orientation-preserving linear similarities of $\mathbb{C}$ is denoted by $M^+(\mathbb{C}^*)$.

The problem of similarity of two polynomial curves(or two complex Bézier curves) $Z(u), W(u)$ for the groups $GM(\mathbb{C}^*)$ and $GM^+(\mathbb{C}^*)$ is reduced to the problem of similarity of two polynomial curves(or two complex Bézier curves) $Z(u), W(u)$ for the groups $M(\mathbb{C}^*)$ and $M^+(\mathbb{C}^*)$, resp. Moreover, since a complex Bézier curve can be define in terms of complex control points, these problems of similarity of two complex Bézier curves is reduced to the problem of similarity of sets of complex control points for these groups. Similarly, same problem can given for polynomial curves. Otherwise, the problem of similarity of sets of complex control points for the above mentioned groups can be applied to the point set rigid registration problem.

For the groups of Euclidean motions $M(n)$ and Euclidean rigid motions $M^+(n)$ in the n-dimensional Euclidean space, the problems of equivalence two Bézier curves of degree m and its global invariants are investigated in [15]. In [9], similar problem in this paper is solved for the groups $M(2)$ and $M^+(2)$. For orientation-preserving similarity group $Sim^+(n)$ in similarity geometry, *local* differential invariants, existence and rigidity theorems for a regular curve are obtained in [20]. For only similarity group $Sim(2)$ and linear similarity group $LSim(2)$, the problems of equivalence two Bézier curves of degree m are investigated in [18]. For orthogonal group $O(2)$, special orthogonal group $O^+(2)$, linear similarity group $LSim(2)$ and orientation linear similarity group $LSim^+(2)$, the conditions of the global G-equivalence of two regular paths are given in [10, 21].

So the paper contains solutions of problems of global similarity of complex Bézier curves and polynomial curves for the above mentioned groups without using differential invariants of a complex Bézier curve and a polynomial curve. In order to make this paper more self contained from a mathematical points of view, the structure of the present paper is the following. In Sect.2, relations between complex plane and two-dimensional Euclidean space and definitions of similarity groups in terms of complex numbers are introduced. In Sect.3, global invariants of a polynomial curve and a complex Bézier curve are given. For above mentioned similarity groups, the problem of similarity of two complex Bézier curves are given. In Sect.4, conditions of similarity for two m-uples complex number sets and a general form of all similarity transformations, carrying one set into the other set, are obtained. In Sect.5, conditions of similarity for two complex Bézier curves and a general form of all similarity transformations, carrying one curve into the other curve, are obtained.

2. Similarity groups in the complex plane

Let $\mathbb{C}$ be the field of complex numbers. The product of two complex numbers z_1 and z_2 has the form

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1) \tag{2.1}$$

Consider the complex number $z = a + ib$ in the matrix form $z = \begin{pmatrix} a \\ b \end{pmatrix}$.

Then, the equality (2.1) has the following form

$$z_1 z_2 = \begin{pmatrix} a_1a_2 - b_1b_2 \\ a_1b_2 + a_2b_1 \end{pmatrix} = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}. \tag{2.2}$$

Here we denote by L_z the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for all $z = a + ib \in \mathbb{C}$. Then $L_z : \mathbb{C} \to \mathbb{C}$ is a mapping and the equality (2.2) has the form, $\forall z_1, z_2 \in \mathbb{C}$,

$$z_1 z_2 = L_{z_1} z_2. \tag{2.3}$$

The field $\mathbb{C}$ can be used to represents $\mathbb{R}^2$ with the inner product $< z_1, z_2 >= a_1a_2 + b_1b_2, \forall z_1 = a_1 + ib_1, z_2 = a_2 + ib_2 \in \mathbb{C}$. Here, the quadratic form on $\mathbb{R}^2$ is $< z_1, z_1 >= |z_1|^2, \forall z_1 \in \mathbb{C}$. The conjugate of z_1, denoted by $\overline{z_1}$, is defined as $\overline{z_1} = a_1 - ib_1$. Clearly, from definition we have $z_1 + \overline{z_1} = 2a_1$, $z_1\overline{z_1} = |z_1|^2$, $|z_1| = |\overline{z_1}|$ and $< \overline{z_1}, \overline{z_2} >=< z_1, z_2 >$. For $|z_1| \neq 0$, the inverse of z_1 is defined as $\frac{1}{z_1} = \frac{\overline{z_1}}{|z_1|^2}$. Moreover, let $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then we have $\overline{z_1} = \Lambda z_1$.

For $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$, the determinant of matrix $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ will be denoted by $[z_1\ z_2]$.

Then we put $Re(\overline{z_1}z_2) =< z_1, z_2 >$ and $Im(\overline{z_1}z_2) = [z_1\ z_2]$.

For $z_1, z_2 \in \mathbb{C}$, in the case $z_1\overline{z_1} \neq 0$, the element $\frac{z_2}{z_1}$ exists and the following equality hold:

$$L_{\frac{z_2}{z_1}} = \begin{pmatrix} Re(\frac{z_2}{z_1}) & -Im(\frac{z_2}{z_1}) \\ Im(\frac{z_2}{z_1}) & Re(\frac{z_2}{z_1}) \end{pmatrix}. \tag{2.4}$$

Put $\mathbb{C}^* = \{z \in \mathbb{C} | z \neq 0\}$, $S(\mathbb{C}^*) = \{z \in \mathbb{C} | z\overline{z} = 1\}$, $M^+(\mathbb{C}^*) = \{L_z | z \in \mathbb{C}^*\}$ and $MS(\mathbb{C}^*) = \{L_z | z \in S(\mathbb{C}^*)\}$.

It is easy to see that $\mathbb{C}^*$ is a group and $S(\mathbb{C}^*)$ is a subgroup of $\mathbb{C}^*$.

We denote the set $M^{-}(\mathbb{C}^*) = \left\{ L_z\Lambda | \Lambda = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, L_z \in M^{+}(\mathbb{C}^*) \right\}$.

Let $M^{+}(\mathbb{C}^*)$ and $M^{-}(\mathbb{C}^*)$ be sets generated by all orientation-preserving and orientation-reversing linear similarities of $\mathbb{R}^2$, resp. Clearly, $M^{+}(\mathbb{C}^*) \cap M^{-}(\mathbb{C}^*) = \varnothing$. The set $M(\mathbb{C}^*)$ of all linear similarities of $\mathbb{R}^2$ can be written in the form $M(\mathbb{C}^*) = M^{+}(\mathbb{C}^*) \cup M^{-}(\mathbb{C}^*)$.

The following theorem is known from [23, p.229].

Theorem 1. (i) $GM^{+}(\mathbb{C}^*) = \{F : \mathbb{C} \to \mathbb{C} | F(v) = L_z v + b, L_z \in M^{+}(\mathbb{C}^*), \forall v \in \mathbb{C}, b \in \mathbb{C}\}$.

(ii) $GM^{-}(\mathbb{C}^*) = \{F : \mathbb{C} \to \mathbb{C} | F(v) = (L_z\Lambda)v + b, L_z \in M^{+}(\mathbb{C}^*), \forall v \in \mathbb{C}, b \in \mathbb{C}\}$.

(iii) $GM(\mathbb{C}^*) = GM^{+}(\mathbb{C}^*) \cup GM^{+}(\mathbb{C}^*)$.

Remark 1. *For the essential notations of the group of all similarity transformations and the group of all orientation-preserving similarity transformations, see some references [10, 18, 20].*

3. On invariant functions of an complex Bézier curve and the theorem on reduction

Let G be a group $GM^{+}(\mathbb{C}^*)$ or $GM(\mathbb{C}^*)$.

Definition 1. *A function $f(z_0, z_1, \ldots, z_m)$ of complex numbers $z_0, z_1, \ldots, z_m$ in $\mathbb{C}$ will be called G-invariant if $f(Fz_0, Fz_1, \ldots, Fz_m) = f(z_0, z_1, \ldots, z_m)$ for all $F \in G$.*

Example 1. *Let z_0, z_1 be two complex number and $z_0 \neq 0$. The function $f(z_0, z_1) = Re(\frac{z_1}{z_0})$ is $M(\mathbb{C}^*)$-invariant. Really, let $L_z \in M(\mathbb{C}^*)$. Then by the equality (2.3), we have $L_z w = zw, \forall z, w \in \mathbb{C}$. We consider $L_z \frac{z_1}{z_0}$. Then, we obtain $L_z \frac{z_1}{z_0} = \frac{zz_1}{zz_0} = \frac{z_1}{z_0}$. Hence, we obtain that $Re(L_z \frac{z_1}{z_0}) = Re(\frac{z_1}{z_0})$. So, $Re(\frac{z_1}{z_0})$ is $M(\mathbb{C}^*)$-invariant.*

Similarly, the function $f(z_0, z_1) = Im(\frac{z_1}{z_0})$ is $M^{+}(\mathbb{C}^*)$-invariant.

Example 2. *Let z_0, z_1, z_2 be three complex number and $z_0 \neq z_1$. The function $f(z_0, z_1, z_2) = Re(\frac{z_2 - z_0}{z_1 - z_0})$ is $GM(\mathbb{C}^*)$-invariant. Really, let $F \in GM^{+}(\mathbb{C}^*)$. Then by Theorem 1, we have $F(v) = L_z v + w, \forall z \in \mathbb{C}^*$ and $v, w \in \mathbb{C}$. We consider $\frac{F(z_2 - z_0)}{F(z_1 - z_0)}$. Using above the equality, we have $\frac{F(z_2 - z_0)}{F(z_1 - z_0)} = \frac{L_z(z_2 - z_0)}{L_z(z_1 - z_0)}) = \frac{z_2 - z_0}{z_1 - z_0}$. By above example, we obtain $Re(\frac{z_2 - z_0}{z_1 - z_0})$ is $GM(\mathbb{C}^*)$-invariant. Similarly, the function $f(z_0, z_1, z_2) = Im(\frac{z_2 - z_0}{z_1 - z_0})$ is $GM^{+}(\mathbb{C}^*)$-invariant.*

A Bézier curve in $\mathbb{C}$ is a parametric curve(or U-path, where $U = [0, 1]$) whose complex points $Z(u)$ are defined by $Z(u) = \sum_{j=0}^{m} z_j B_j^m(u)$, where $z_j \in \mathbb{C}$ and $B_j^m(u)$ is the Bernstein basis polynomials.

A polynomial curve in $\mathbb{C}$ is a parametric curve (or U-path, where $U = [0, 1]$) whose complex points $Z(u)$ are defined by $Z(u) = \sum_{j=0}^{m} p_j(u)$, where $p_j \in \mathbb{C}$ is monomial complex control points(for more details, see [7, 8, 16, 22])

By lemma in [22, p.166], all polynomial curves can be represented in Bézier curve form.

Definition 2. *A G-invariant function $f(z_0, z_1, \ldots, z_m)$ of control complex points $z_0, z_1, \ldots, z_m$ of a Bézier curve $Z(u) = \sum_{j=0}^{m} z_j B_j^m(u)$ will be called a control G-invariant of $Z(u)$. A G-invariant function $f(p_0, p_1, \ldots, p_m)$ of monomial control complex points $p_0, p_1, \ldots, p_m$ of a polynomial curve $Z(u) = \sum_{j=0}^{m} p_j u^j$ will be called a monomial G-invariant of $Z(u)$.*

Now we define similarity of two Bézier curves of degree m and similarity of two m-uples of complex points in $\mathbb{C}$.

Definition 3. *Bézier curves $Z(u)$ and $W(u)$ in $\mathbb{C}$ will be called G -similar if there exists $F \in G$ such that $W(u) = FZ(u)$ for all $u \in [0, 1]$.*

Definition 4. *m-uples $\{z_1, z_2, \ldots, z_m\}$ and $\{w_1, w_2, \ldots, w_m\}$ of complex numbers in $\mathbb{C}$ are called G-similar if there is $F \in G$ such that $w_j = Fz_j$ for all $j = 1, 2, \ldots, m$.*

Since Bézier curves can be introduced by control points, the following two theorems means that the problem of G-similarity of Bézier curves reduce to the problem of G-similarity of two m-uples complex numbers.

Remark 2. *Throughout paper, we consider the curves in forms $Z(u) = \sum_{j=0}^{m} z_j B_j^m(u) = \sum_{j=0}^{m} p_j u^j$ and $W(u) = \sum_{j=0}^{m} w_j B_j^m(u) = \sum_{j=0}^{m} q_j u^j$ in $\mathbb{C}$ of degree m, where $m \geq 1$. Moreover, $Z'(u)$ and $W'(u)$ are their first derivatives.*

Theorem 2. *Let $Z(u)$ and $W(u)$ be Bézier curves. Then the following statements are equivalent:*

- *(i) $Z(u)$ and $W(u)$ are $GM^+(\mathbb{C}^*)$-similar.*
- *(ii) $Z'(u)$ and $W'(u)$ are $M^+(\mathbb{C}^*)$-similar.*
- *(iii) m-uples $\{z_0, z_1, \ldots, z_m\}$ and $\{w_0, w_1, \ldots, w_m\}$ are $GM^+(\mathbb{C}^*)$-similar.*
- *(iv) m-uples $\{z_1 - z_0, z_2 - z_0, \ldots, z_m - z_0\}$ and $\{w_1 - w_0, w_2 - w_0, \ldots, w_m - w_0\}$ are $M^+(\mathbb{C}^*)$-similar.*
- *(v) $\{p_1, p_2, \ldots, p_m\}$ and $\{q_1, q_2, \ldots, q_m\}$ are $M^+(\mathbb{C}^*)$-similar.*

Proof. Proof is similar to proof of Theorem 2 in [15] and Theorem 4.1 in [9]. □

Theorem 3. *Let $Z(u)$ and $W(u)$ be Bézier curves. Then the following statements are equivalent:*

- *(i) $Z(u)$ and $W(u)$ are $GM(\mathbb{C}^*)$-similar.*
- *(ii) $Z'(u)$ and $W'(u)$ are $M(\mathbb{C}^*)$-similar.*
- *(iii) m-uples $\{z_0, z_1, \ldots, z_m\}$ and $\{w_0, w_1, \ldots, w_m\}$ are $GM(\mathbb{C}^*)$-similar.*
- *(iv) m-uples $\{z_1 - z_0, z_2 - z_0, \ldots, z_m - z_0\}$ and $\{w_1 - w_0, w_2 - w_0, \ldots, w_m - w_0\}$ are $M(\mathbb{C}^*)$-similar.*
- *(v) $\{p_1, p_2, \ldots, p_m\}$ and $\{q_1, q_2, \ldots, q_m\}$ are $M(\mathbb{C}^*)$-similar.*

Proof. Proof is similar to proof of Theorem 1 in [15] and Theorem 4.1 in [9]. □

Remark 3. (1)

(*i*) *Let* $\{z_1, z_2, \ldots, z_m\}$ *and* $\{w_1, w_2, \ldots, w_m\}$ *in* $\mathbb{C}$ *be two m-uples such that* $z_k \neq 0$ *and* $w_k = 0$. *Then* $\{z_1, z_2, \ldots, z_m\}$ *and* $\{w_1, w_2, \ldots, w_m\}$ *are not G-similar. In the case* $z_k = w_k = 0$, *we obtain the problem of G- similarity of these* $m-1$*-uples* $\{z_1,\ z_2,\ \ldots, z_{k-1}, z_{k+1}, \ldots, z_m\}$ *and* $\{w_1,\ w_2,\ \ldots,\ w_{k-1},\ w_{k+1},\ \ldots,\ w_m\}$, *Therefore, we put* $z_k \neq 0$ *and* $w_k \neq 0$ *for* $k \in \{1, 2, \ldots, m\}$.

(*ii*) *Let* z_1 *and* w_1 *in* $\mathbb{C}$ *be two complex number such that* $z_1 \neq 0$ *and* $w_1 \neq 0$. *Then there always is an element* L_z *in G such that* $w_1 = L_z z_1$. *Therefore, we put* $m > 1$ *for m-uples* $\{z_1, z_2, \ldots, z_m\}$ *and* $\{w_1, w_2, \ldots, w_m\}$ *in* $\mathbb{C}$.

4. Conditions of similarity for two m-uple complex number sets

Theorem 4. *Let* $\{z_1, z_2, \ldots, z_m\}$ *and* $\{w_1, w_2, \ldots, w_m\}$ *be two m-uples in* $\mathbb{C}$ *such that* $z_k \neq 0$ *and* $w_k \neq 0$, *where* $k \in \{1, 2, \ldots, m\}$. *Then these m-uples are* $M^+(\mathbb{C}^*)$*-similar if and only if*

$$\begin{cases} Re(\dfrac{z_i}{z_k}) = Re(\dfrac{w_i}{w_k}) \\ Im(\dfrac{z_i}{z_k}) = Im(\dfrac{w_i}{w_k}) \end{cases} \tag{4.1}$$

for all $i = 1, 2, \ldots, k-1, k+1, \ldots, m$.
Furthermore, there is the unique $L_z \in M^+(\mathbb{C}^*)$ *such that* $w_i = L_z z_i$ *for all* $i = 1, 2, \ldots, m$, *where the matrix* L_z *can be written as*

$$L_z = \begin{pmatrix} Re(\frac{w_k}{z_k}) & -Im(\frac{w_k}{z_k}) \\ Im(\frac{w_k}{z_k}) & Re(\frac{w_k}{z_k}) \end{pmatrix}. \tag{4.2}$$

Proof. $\Rightarrow$: Assume that $\{z_1, z_2, \ldots, z_m\}$ and $\{w_1, w_2, \ldots, w_m\}$ in $\mathbb{C}$ are $M^+(\mathbb{C}^*)$)-similar. Since the functions $Re(\frac{z_i}{z_k})$ and $Im(\frac{z_i}{z_k})$ are $M^+(\mathbb{C}^*)$)-invariant, we obtain that the equalities (4.1) hold.
$\Leftarrow$: Assume that the equalities (4.1) hold. By the equality (4.1), we have

$$\frac{z_i}{z_k} = \frac{w_i}{w_k} \tag{4.3}$$

for all $i = 1, 2, \ldots, k-1, k+1, \ldots, m$. Consider the element $z = \frac{w_k}{z_k} \in \mathbb{C}$. By the equality (4.3), we have $w_i = w_k \frac{w_i}{w_k} = w_k \frac{z_i}{z_k} = \frac{w_k}{z_k} z_i$ for all $i = 1, 2, \ldots, k-1, k+1, \ldots, m$. So, by the equality (2.3),we have $w_i = L_z z_i$ for all $i = 1, 2, \ldots, k-1, k+1, \ldots, m$. Clearly $L_z \in M^+(\mathbb{C}^*)$. For uniqueness, assume that $L_v \in M^+(\mathbb{C}^*)$ exists such that $w_i = L_v z_i$ for all $i = 1, 2, \ldots, m$. Then, by this equality and the equality (2.3), we have $v \in M^+(\mathbb{C}^*)$ such that $w_i = v z_i$ for all $i = 1, 2, \ldots, m$. Since $z_k \neq 0$, the equality $w_k = v z_k$ implies that $v = \frac{w_k}{z_k} = z$. Hence the uniqueness of L_z is proved. Moreover, using the equality (2.3), the element $z = \frac{w_k}{z_k}$ can be written as the matrix L_z, where L_z has the form (4.2). □

Denote by $rank\,\{z_1, z_2, \ldots, z_m\}$ the rank of the m-uple $\{z_1, z_2, \ldots, z_m\}$ in $\mathbb{C}$. It is easy to see that $rank\,\{z_1, z_2, \ldots, z_m\}$ is $M(\mathbb{C}^*)$-invariant.

Theorem 5. *Let $\{z_1, z_2, \dots, z_m\}$ and $\{w_1, w_2, \dots, w_m\}$ be two m-uples in $\mathbb{C}$ such that $z_k \neq 0$, $w_k \neq 0$ for $k \in \{1, 2, \dots, m\}$ and $rank\,\{z_1, z_2, \dots, z_m\} = rank\,\{w_1, w_2, \dots, w_m\} = 1$. Then these m-uples are $M(\mathbb{C}^*)$-similar if and only if*

$$Re(\frac{z_i}{z_k}) = Re(\frac{w_i}{w_k}) \tag{4.4}$$

for all $i = 1, 2, \dots, k-1, k+1, \dots, m$.
Furthermore, there is the unique $L_z \in M(\mathbb{C}^*)$ such that $w_i = L_z z_i$ for all $i = 1, 2, \dots, m$, where the matrix L_z can be written as the form (4.2).

Proof. $\Rightarrow$: The proof is similar to the proof of Theorem 4.
$\Leftarrow$: Assume that the equality (4.4) holds.
Since $rank\,\{z_1, z_2, \dots, z_m\} = rank\,\{w_1, w_2, \dots, w_m\} = 1$, we have $Im(\frac{z_i}{z_k}) = Im(\frac{w_i}{w_k}) = 0$ for all $i = 1, 2, \dots, k-1, k+1, \dots, m$. Hence the equalities (4.1) in Theorem 4 hold. Using Theorem 4, we have $w_i = L_z z_i$ for all $i = 1, 2, \dots, m$ and the matrix L_z has the form (4.2). □

Let m-uple $\{z_1, z_2, \dots, z_m\}$ in $\mathbb{C}$. In the case $rank\,\{z_1, z_2, \dots, z_m\} = 2$, denote by
$ind\,\{z_1, z_2, \dots, z_m\}$ the smallest of p, $1 \leq p \leq m$, such that $z_p \neq \lambda z_k$ for all $\lambda \in \mathbb{R}$ and $z_k \neq 0$.

Theorem 6. *Let $\{z_1, z_2, \dots, z_m\}$ and $\{w_1, w_2, \dots, w_m\}$ be two m-uples in $\mathbb{C}$ such that $z_k \neq 0$, $w_k \neq 0$, $rank\,\{z_1, z_2, \dots, z_m\} = rank\,\{w_1, w_2, \dots, w_m\} = 2$ and $ind\,\{z_1, z_2, \dots, z_m\} = ind\,\{w_1, w_2, \dots, w_m\} = l$ for $k, l \in \{1, 2, \dots, m\}$ and $k \neq l$. Then these m-uples are $M(\mathbb{C}^*)$-similar if and only if*

$$\begin{cases} Re(\dfrac{z_i}{z_k}) = Re(\dfrac{w_i}{w_k}) \\ \left[Im(\dfrac{z_l}{z_k})\right]^2 = \left[Im(\dfrac{w_l}{w_k})\right]^2 \\ \dfrac{Im(\frac{z_i}{z_k})}{Im(\frac{z_l}{z_k})} = \dfrac{Im(\frac{w_i}{w_k})}{Im(\frac{w_l}{w_k})} \end{cases} \tag{4.5}$$

for all $i = 1, 2, \dots, m$, $i \neq k$.
Furthermore, there is the unique $L_z \in M(\mathbb{C}^*)$ such that $w_i = L_z z_i$ for all $i = 1, 2, \dots, m$. Then there exist two statements:

(i) *In the case $Im(\frac{z_l}{z_k}) = Im(\frac{w_l}{w_k})$, the element $L_z \in M^+(\mathbb{C}^*)$ and it can be represented by* (4.2).
(ii) *In the case $Im(\frac{z_l}{z_k}) = -Im(\frac{w_l}{w_k})$, the element $L_z\Lambda \in M^-(\mathbb{C}^*)$ and it can be written as*

$$L_z\Lambda = \begin{pmatrix} Re(\frac{w_k}{\overline{z}_k}) & -Im(\frac{w_k}{\overline{z}_k}) \\ Im(\frac{w_k}{\overline{z}_k}) & Re(\frac{w_k}{\overline{z}_k}) \end{pmatrix}. \tag{4.6}$$

Proof. $\Rightarrow$: Let m-uples $\{z_1, z_2, \ldots, z_m\}$ and $\{w_1, w_2, \ldots, w_m\}$ are $M(\mathbb{C}^*)$-similar. Since the functions $Re(\frac{z_i}{z_k})$, $\left[Im(\frac{z_l}{z_k})\right]^2$ and $\frac{Im(\frac{z_i}{z_k})}{Im(\frac{z_l}{z_k})}$ are $M(\mathbb{C}^*)$-invariant, we obtain that the equalities (4.6).

$\Leftarrow$: Assume that the equality (4.6) holds. Using the conditions $z_k \neq 0$, $w_k \neq 0$ and $ind\{z_1, z_2, \ldots, z_m\} = ind\{w_1, w_2, \ldots, w_m\} = l$ for $k, l \in \{1, 2, \ldots, m\}, k \neq l$ and the equality $\left[Im(\frac{z_l}{z_k})\right]^2 = \left[Im(\frac{w_l}{w_k})\right]^2$, we have the equality $\left[Im(\frac{z_l}{z_k})\right] = \left[Im(\frac{w_l}{w_k})\right]$ or $\left[Im(\frac{z_l}{z_k})\right] = -\left[Im(\frac{w_l}{w_k})\right]$. Moreover, since $rank\{z_1, z_2, \ldots, z_m\} = rank\{w_1, w_2, \ldots, w_m\} = 2$ and
$ind\{z_1, z_2, \ldots, z_m\} = ind\{w_1, w_2, \ldots, w_m\} = l$, we have $Im(\frac{z_l}{z_k}) \neq 0$.

- (i) Assume that $Im(\frac{z_l}{z_k}) = Im(\frac{w_l}{w_k})$. Then, using this equality and the equality $\frac{Im(\frac{z_i}{z_k})}{Im(\frac{z_l}{z_k})} = \frac{Im(\frac{w_i}{w_k})}{Im(\frac{w_l}{w_k})}$, we have $Im(\frac{z_i}{z_k}) = Im(\frac{w_i}{w_k})$ for all $i \neq k$. So the equalities (4.1) hold. Then by Theorem 4, there is the unique $L_z \in M^+(\mathbb{C}^*)$ such that $w_i = L_z z_i$ for all $i = 1, 2, \ldots, m$. The element L_z has the form (4.2).
- (ii) Assume that $Im(\frac{z_l}{z_k}) = -Im(\frac{w_l}{w_k})$. Then, using this equality and the equality $\frac{Im(\frac{z_i}{z_k})}{Im(\frac{z_l}{z_k})} = \frac{Im(\frac{w_i}{w_k})}{Im(\frac{w_l}{w_k})}$, we have $Im(\frac{z_i}{z_k}) = -Im(\frac{w_i}{w_k})$ for all $i \neq k$. Hence, we obtain $Im(\frac{z_i}{z_k}) = -Im(\frac{\overline{z_i}}{\overline{z_k}})$. Then by this equality and the equality $Re(\frac{z_i}{z_k}) = Re(\frac{\overline{z_i}}{\overline{z_k}})$, we have $Re(\frac{w_i}{w_k}) = Re(\frac{\overline{z_i}}{\overline{z_k}})$ and $Im(\frac{w_i}{w_k}) = Im(\frac{\overline{z_i}}{\overline{z_k}})$. In this case, by Theorem 4, there is the unique $L_z \in M^+(\mathbb{C}^*)$ such that $w_i = L_z\overline{z_i} = L_z(\Lambda z_i) = (L_z\Lambda)z_i$ for all $i = 1, 2, \ldots, m$. Then the element $L_z\Lambda$ has the form (4.6).

□

5. Conditions of similarity for two complex Bézier curves and its applications

Using Theorem 2 and Theorem 4, the following corollary obtain.

Corollary 1. *Let $Z(u) = \sum_{j=0}^{m} p_j u^j$ and $W(u) = \sum_{j=0}^{m} q_j u^j$ be two polynomial curves in $\mathbb{C}$ of degree $m > 1$. Then $Z(u)$ and $W(u)$ are $GM^+(C^*)$-similar if and only if*

$$\begin{cases} Re(\dfrac{p_i}{p_m}) = Re(\dfrac{q_i}{q_m}) \\ Im(\dfrac{p_i}{p_m}) = Im(\dfrac{q_i}{q_m}) \end{cases} \tag{5.1}$$

for all $i = 1, 2, \ldots, m-1$.
Furthermore, there is the unique $F \in GM^+(\mathbb{C}^*)$ such that $q_i = Fp_i = L_z p_i + b$ for all $i = 0, 1, 2, \ldots, m$, where the matrix $L_z \in M^+(\mathbb{C}^*)$ and the constant $b \in \mathbb{C}$ can

be written as

$$L_z = \begin{pmatrix} Re(\frac{q_m}{p_m}) & -Im(\frac{q_m}{p_m}) \\ Im(\frac{q_m}{p_m}) & Re(\frac{q_m}{p_m}) \end{pmatrix} \tag{5.2}$$

and

$$b = q_0 - L_z p_0. \tag{5.3}$$

Example 3. *Consider two polynomial curves* $Z(u) = (2 + 2u, 3 - 8u + 11u^2)$ *and* $W(u) = (-10 + 44u - 55u^2, 20 - 6u + 22u^2)$ *with complex monomial control points* $p_0 = 2 + 3i, p_1 = 2 - 8i, p_2 = 11i$ *and* $q_0 = -10 + 20i, q_1 = 44 - 6i, q_2 = -55 + 22i$ *in* $\mathbb{C}$*, resp. It is easy to see that the equalities in* (5.1) *hold for the curves* $Z(u)$ *and* $W(u)$*. Then by Theorem* 1*,* $Z(u)$ *and* $W(u)$ *are* $GM^+(C^*)$*-similar and* $L_z = 2+5i$ *and* $b = 1 + 4i$*.*

Using Theorem 3 and Theorem 6, the following corollary obtain.

Corollary 2. *Let* $Z(u) = \sum_{j=0}^{m} p_j u^j$ *and* $W(u) = \sum_{j=0}^{m} q_j u^j$ *be two polynomial curves in* $\mathbb{C}$ *of degree* $m > 1$*. Let Then* $Z(u)$ *and* $W(u)$ *are* $GM(C^*)$*-similar if and only if*

$$\begin{cases} Re(\dfrac{p_i}{p_m}) = Re(\dfrac{q_i}{q_m}) \\ \left[Im(\dfrac{p_l}{p_m})\right]^2 = \left[Im(\dfrac{q_l}{q_m})\right]^2 \\ \dfrac{Im(\frac{p_j}{p_m})}{Im(\frac{p_l}{p_m})} = \dfrac{Im(\frac{q_j}{q_m})}{Im(\frac{q_l}{q_m})} \end{cases} \tag{5.4}$$

for all $i = 1, 2, \ldots, m - 1$ *and for all* $j = 1, 2, \ldots, l - 1, l + 1, \ldots, m - 1$*, where* $ind\{p_1, p_2, \ldots, p_m\} = ind\{q_1, q_2, \ldots, q_m\} = l$ *for* $l \in \{1, 2, \ldots, m - 1\}$*.*
Furthermore, there is the unique $F \in GM(\mathbb{C}^*)$ *such that* $q_i = Fp_i$ *for all* $i = 0, 1, 2, \ldots, m$*. There are the following two cases:*

- (*i*) *In the case* $Im(\frac{p_l}{p_m}) = Im(\frac{q_l}{q_m})$*,* F *has the form* $Fp_i = L_z p_i + b_1$ *for all* $i = 0, 1, 2, \ldots, m$*, where the element* $L_z \in M^+(\mathbb{C}^*)$ *and the constant* $b_1 \in \mathbb{C}$ *can be written as* (5.2) *and* (5.3)*, resp.*
- (*ii*) *In the case* $Im(\frac{p_l}{p_m}) = -Im(\frac{q_l}{q_m})$*,* F *has the form* $Fp_i = L_z \Lambda p_i + b_2$ *for all* $i = 0, 1, 2, \ldots, m$*, where the element* $L_z \in M^+(\mathbb{C}^*)$ *and and the constant* $b_2 \in \mathbb{C}$ *can be written as*

$$L_z = \begin{pmatrix} Re(\frac{q_m}{\overline{p_m}}) & -Im(\frac{q_m}{\overline{p_m}}) \\ Im(\frac{q_m}{\overline{p_m}}) & Re(\frac{q_m}{\overline{p_m}}) \end{pmatrix} \tag{5.5}$$

and

$$b_2 = q_0 - L_z \Lambda p_0. \tag{5.6}$$

Example 4. *Consider two polynomial curves* $Z(u) = (2 + 2u, 3 - 8u + 11u^2)$ *and* $W(u) = (20 - 36u + 55u^2, 8 + 26u - 22u^2)$ *with complex monomial control points*

$p_0 = 2+3i, p_1 = 2-8i, p_2 = 11i$ and $q_0 = 20+8i, q_1 = -36+26i, q_2 = 55-22i$ in $\mathbb{C}$, resp. It is easy to see that the equalities in (5.4) hold for the curves $Z(u)$ and $W(u)$. Then by Theorem 2, $Z(u)$ and $W(u)$ are $GM(C^*)$-similar and $L_z = 2+5i$ and $b = 1+4i$. But $Z(u)$ and $W(u)$ are not $GM^+(C^*)$-similar.

Using Theorem 2 and Theorem 4, the following corollary obtain.

Corollary 3. *Let $Z(u) = \sum_{j=0}^{m} z_j B_j^m$ and $W(u) = \sum_{j=0}^{m} w_j B_j^m$ be two Bézier curves in $\mathbb{C}$ of degree $m > 1$ such that $z_m - z_0 \neq 0$ and $w_m - w_0 \neq 0$. Then $Z(u)$ and $W(u)$ are $GM^+(C^*)$-similar if and only if*

$$\begin{cases} Re(\dfrac{z_i - z_0}{z_m - z_0}) = Re(\dfrac{w_i - w_0}{w_m - w_0}) \\ Im(\dfrac{z_i - z_0}{z_m - z_0}) = Im(\dfrac{w_i - w_0}{w_m - w_0}) \end{cases} \tag{5.7}$$

for all $i = 1, 2, \ldots, m-1$.
Furthermore, there is the unique $F \in GM^+(\mathbb{C}^*)$ such that $w_i = Fz_i = L_z z_i + b$ for all $i = 0, 1, 2, \ldots, m$, where the matrix $L_z \in M^+(\mathbb{C}^*)$ and the constant $b \in \mathbb{C}$ can be written as

$$L_z = \begin{pmatrix} Re(\frac{w_m - w_0}{z_m - z_0}) & -Im(\frac{w_m - w_0}{z_m - z_0}) \\ Im(\frac{w_m - w_0}{z_m - z_0}) & Re(\frac{w_m - w_0}{z_m - z_0}) \end{pmatrix} \tag{5.8}$$

and

$$b = w_0 - L_z z_0. \tag{5.9}$$

Example 5. *Consider two complex Bézier curves $Z(u) = \sum_{j=0}^{2} z_j B_j^m$ and $W(u) = \sum_{j=0}^{2} w_j B_j^m$ with complex control points $z_0 = 2+3i, z_1 = 3-i, z_2 = 4+6i$ and $w_0 = -10+20i, w_1 = 12+17i, w_2 = -21+36i$ in $\mathbb{C}$, resp. It is easy to see that the equalities in (5.7) hold for the curves $Z(u)$ and $W(u)$. Then by Theorem 3, $Z(u)$ and $W(u)$ are $GM^+(C^*)$-similar and $L_z = 2+5i$ and $b = 1+4i$.*

Using Theorem 3 and Theorem 6, the following corollary obtain.

Corollary 4. *Let $Z(u) = \sum_{j=0}^{m} z_j B_j^m$ and $W(u) = \sum_{j=0}^{m} w_j B_j^m$ be two Bézier curves in $\mathbb{C}$ of degree $m > 1$ such that $z_m - z_0 \neq 0$ and $w_m - w_0 \neq 0$. Let Then $Z(u)$ and $W(u)$ are $GM(C^*)$-similar if and only if*

$$\begin{cases} Re(\dfrac{z_i - z_0}{z_m - z_0}) = Re(\dfrac{w_i - w_0}{w_m - w_0}) \\ \left[Im(\dfrac{z_l - z_0}{z_m - z_0})\right]^2 = \left[Im(\dfrac{w_l - w_0}{w_m - w_0})\right]^2 \\ \dfrac{Im(\frac{z_j - z_0}{z_m - z_0})}{Im(\frac{z_l - z_0}{z_m - z_0})} = \dfrac{Im(\frac{w_j - w_0}{w_m - w_0})}{Im(\frac{w_l - w_0}{w_m - w_0})} \end{cases} \tag{5.10}$$

for all $i = 1, 2, \ldots, m-1$ and for all $j = 1, 2, \ldots, l-1, l+1, \ldots, m-1$, where $ind\{z_1 - z_0, z_2 - z_0, \ldots, z_m - z_0\} = ind\{w_1 - w_0, w_2 - w_0, \ldots, w_m - w_0\} = l$ for

$l \in \{1, 2, \ldots, m-1\}$.
Furthermore, there is the unique $F \in GM(\mathbb{C}^*)$ *such that* $w_i = Fz_i$ *for all* $i = 0, 1, 2, \ldots, m$. *There are the following two cases:*

(*i*) *In the case* $Im(\frac{z_l - z_0}{z_m - z_0}) = Im(\frac{w_l - w_0}{w_m - w_0})$, *F has the form* $Fz_i = L_z z_i + b_1$ *for all* $i = 0, 1, 2, \ldots, m$, *where the element* $L_z \in M^+(\mathbb{C}^*)$ *and the constant* $b_1 \in \mathbb{C}$ *can be written as* (5.8) *and* (5.9), *resp.*

(*ii*) *In the case* $Im(\frac{z_l - z_0}{z_m - z_0}) = -Im(\frac{w_l - w_0}{w_m - w_0})$, *F has the form* $Fz_i = L_z \Lambda z_i + b_2$ *for all* $i = 0, 1, 2, \ldots, m$, *where the element* $L_z \in M^+(\mathbb{C}^*)$ *and and the constant* $b_2 \in \mathbb{C}$ *can be written as*

$$L_z = \begin{pmatrix} Re(\frac{w_m - w_0}{z_m - z_0}) & -Im(\frac{w_m - w_0}{z_m - z_0}) \\ Im(\frac{w_m - w_0}{z_m - z_0}) & Re(\frac{w_m - w_0}{z_m - z_0}) \end{pmatrix} \tag{5.11}$$

and

$$b_2 = w_0 - L_z \Lambda z_0. \tag{5.12}$$

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