

On the Configuration Space of Planar Closed Kinematic Chains

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ABSTRACT

A planar kinematic chain consists of n links connected by joints. In this work we investigate the space of configurations, described in terms of joint angles, that guarantee that the kinematic chain is closed. We give explicit formulas expressing the joint angles that guarantee closedness by a new set of parameters. Moreover, it turns out that these parameters are contained in a domain that possesses a simple structure. We expect that the new insight can be applied for several issues such as motion planning for closed kinematic chains or singularity analysis of their configuration spaces. In order to demonstrate practicality of the new method we present numerical examples.

Keywords: configuration space; closed kinematic chain; path planning.

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1. Introduction

In this work we investigate the configuration space of closed planar kinematic chain (CKC) with n links connected by revolute joints in terms of its joint angles. In many fields like robotics computational biology or protein kinematics it is of immense interest to understand the configuration space of a CKC. For instance, in robotics the problem to connect a start, α_s , and goal configuration, α_g naturally appears and thus requires knowledge of the configuration space, which is typically a manifold or variety in the ambient space formed by the robots joint variables. The configuration space is even more complicated if additional constraints like obstacle, link-link avoidance, or limited joint angles are included. Two main strategies, probabilistic and geometric approaches, to investigate configuration spaces have been developed so far.

Probabilistic methods have been successfully applied for constrained motion planning. They are especially important in practical situations with high dimensions that include complex constraints such as obstacle avoiding. Typically these methods are based on the generation of random configurations in ambient joint space followed by a check up if they approximately satisfy the desired constraints. Repeating this procedure results in a discrete version of the configuration space that is very useful in applications. Probabilistic methods have been applied in different situations, which can be found in [1, 4, 13, 6, 7, 18, 17],

Besides the approaches using randomness other works focused on questions about the geometry and topology of the configuration spaces of kinematic chains. Insight about the global geometry of configuration spaces is very important in applications. Early discoveries have been made by [14, 3]. In their fundamental work Kapovitch and Milgram established important results about the geometry, which led to novel path planning algorithms. For instance in [11, 12] it is used that the configuration space of a CKC consists of two connected components when it possess three long links. An application of this result is that path planning can be done easily for this special kind of CKC's. Also for the more difficult case, when CKC's do not have three long links algorithms were derived in [11, 12]. They also developed path planners in the case of p point obstacles in the plane [15]. Another geometric approach was recently recognized by Han, Rudolph and Blumenthal. They

discovered that it is very beneficial to describe the configuration space of CKC by different parameters than the joint angles, see [8, 10, 9]. Their idea is to use the length of diagonals from the positions of revolute joints to the origin O as depicted on the right side of Figure 2. It turns out that for a CKC the length of these diagonals can be computed as solution of a system of linear inequalities, which means that all feasible diagonal lengths can be described by a convex polyhedron that can be handled by methods of linear programming [16]. Given feasible diagonal lengths, several configurations of the CKC can be constructed, since each link of the chain can be flipped over a diagonal. Thus in [8, 10] any configuration can be obtained from a set of diagonals and a vector that represents the choices of flipping, which shows that the configuration space is formed by several copies of the polyhedron given by the system of inequalities. This practically convex structure is very useful for motion planning. In [9, 10] paths between CKC with 1000 links are computed very efficiently.

Contribution of this work: We develop a new method that explicitly computes configurations of a CKC with n links, which are described by its joint angles. Compared to other methods it does not require linear programming to solve a system of linear inequalities like in [8, 10] nor does it rely on probabilistic principles. More precisely, it turns out that a configuration can be computed from new parameters contained in a very simple domain, namely a $n - 3$ dimensional cube. The developed method can be used to easily sample configuration space of a CKC and thus is expected to be useful in practical applications.

Outline of this text: In section 2 we give a mathematical description of a CKC and its configuration space. Then the basic algorithm that explicitly describes how configurations of a CKC can be computed is developed in section 2. In section 3.1 we describe the set of new parameters and show how they can be used to compute a vector of joint angles of a CKC. Finally, we give numerical examples that show the validity of the developed method.

2. Configuration space

To describe the configuration space of a CKC with link lengths a_1, \dots, a_n we introduce Cartesian coordinates in two dimensional Euclidean space. Moreover we place one of the links of the CKC so that it is supported by the positive x -axis and so that one of its ends coincides with the origin. Without loss of generality we can assume that the link a_n of the chain is fixed in the described manner, see Figure 1. In the following, we identify an angle α with its corresponding point on S^1 . Further, for $1 \leq k \leq n$ and a vector of angles $\alpha^k := (\alpha_1, \dots, \alpha_k) \in (S^1)^k$ we denote by

$$f_{a,k}: (S^1)^k \rightarrow \mathbb{R}^2, \quad f_{a,k}(\alpha^k) = \sum_{j=1}^k a_j \begin{pmatrix} \cos(\alpha_j) \\ \sin(\alpha_j) \end{pmatrix}. \quad (2.1)$$

the k -th endpoint map of a kinematic chain, where $a = (a_1, \dots, a_n)$ is the vector of link lengths. We will call α^{n-1} a configuration of the CKC with link lengths a_1, \dots, a_n if it satisfies the closure condition, which means that it is contained in the set

$$C_a^{\text{onv}} = \left\{ \alpha^{n-1} \in (S^1)^{n-1} : f_{a,n-1}(\alpha^{n-1}) = \begin{pmatrix} a_n \\ 0 \end{pmatrix} \right\} = f_{a,n-1}^{-1}(a_n, 0). \quad (2.2)$$

If no restrictions on the endpoint map are imposed α^{n-1} will just be called a configuration of the kinematic chain (KC) with $n - 1$ links.

Furthermore, the analysis in this work uses the simple observation that it is sufficient to understand the space

$$CC_a^{\text{onv}} = \left\{ \beta^{n-1} \in (S^1)^{n-1} : \|f_{a,n-1}(\beta^{n-1})\|_2^2 = a_n^2 \right\}, \quad (2.3)$$

in order to describe C_a , where $\|\cdot\|_2$ denotes the Euclidean norm. From the definition of CC_a^{onv} it is clear that any configuration $\beta^{n-1} \in CC_a^{\text{onv}}$ satisfies that its endpoint

$$f_{a,n-1}(\beta^{n-1}) \in K_{a_n}$$

lies on the circle K_{a_n} that is centred on the origin and has radius a_n . We will say that β^{n-1} is closed up to a rotation and call it a *circular configuration* of a CKC. Clearly, any circular configuration β^{n-1} can be rotated by

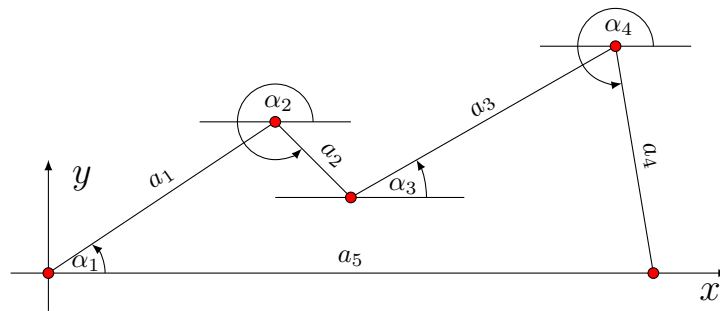


Figure 1. A CKC with $n = 5$ five links. The link a_5 is supported on the positive x -axis and one of its ends coincides with the origin

an angle λ ,

$$\beta^{n-1} + \lambda := (\beta_1 + \lambda, \dots, \beta_{n-1} + \lambda),$$

so that $\beta^{n-1} + \lambda \in C_a$. Thus, if we are able to give an efficient method to compute the set of solutions to the implicit equation

$$\|f_{a,n-1}(\beta^{n-1})\|_2^2 = a_n^2, \quad (2.4)$$

we also obtain configurations in C_a by the following two step algorithm:

- (i) Compute a circular configuration $\beta^{n-1} \in \mathcal{CC}_a^{onv}$
- (ii) Determine λ such that $\alpha^{n-1} = \beta^{n-1} + \lambda \in C_a^{onv}$

Once a circular configuration is obtained step (ii) is a rather simple task. Therefore, in the following we will focus on the solution of step (i). This step is based on the fact that the trigonometric equation (2.4), which in its expanded form is given as

$$\sum_{i=1}^{n-1} a_i^2 + 2 \sum_{i < j}^{n-1} a_i a_j \cos(\beta_i - \beta_j) = a_n^2, \quad (2.5)$$

allows for some kind of backwards substitution, see section 2.2. By the preimage theorem we know that the set of all circular configurations of a CKC with n links satisfying (2.5) is a manifold of dimension $n - 2$, whenever a_n^2 is a regular value of the map $g(\beta^{n-1}) := \|f_{a,n-1}(\beta^{n-1})\|_2^2$. In all other cases the space \mathcal{CC}_a^{onv} may have singular points.

2.1. Mathematical tools and notations

Surprisingly, the trigonometric equation (2.5) can be rearranged into an equation of the same type but with one joint angle less appearing on its left hand side. For the computations we use that a linear combination of sine and cosine functions can be written as

$$a \sin(x) + b \cos(x) = c \sin(x + \varphi(a, b)), \quad (2.6)$$

where $c = \sqrt{a^2 + b^2}$ and $\varphi(a, b) = \text{atan2}(b, a)$ is the function described in Figure 2.

In order to achieve a compact presentation of the results that will follow it is important to introduce abbreviations. For this purpose consider

$$a_{n-1} \sum_{j=1}^{n-2} a_j \cos(\beta_{n-1} - \beta_j) + \sum_{i < j}^{n-2} a_i a_j \cos(\beta_i - \beta_j) = \frac{a_n^2 - \sum_{i=1}^{n-1} a_i^2}{2},$$

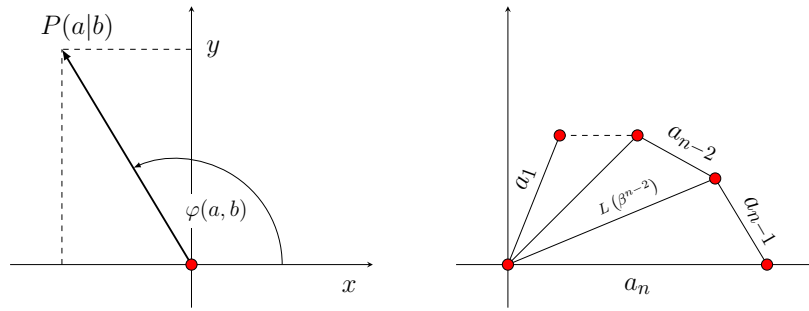


Figure 2. Left: The function atan2 gives the angle between the x -axis and the vector from the origin to $P(a|b)$. Right: A circular configuration with endpoint $(a_n, 0)$. The picture shows anchored diagonals of the CKC

which is an equivalent form of (2.5) that is obtained by fixing an index to be $n - 1$ and rearranging the remaining terms. Finally, using trigonometric summation formulas we arrive at

$$\begin{aligned}
 & a_{n-1} \sin(\beta_{n-1}) \sum_{j=1}^{n-2} a_j \sin(\beta_j) + a_{n-1} \cos(\beta_{n-1}) \sum_{j=1}^{n-2} a_j \cos(\beta_j) + \\
 & \sum_{i < j}^{n-2} a_i a_j \cos(\beta_i - \beta_j) = \frac{a_n^2 - \sum_{i=1}^{n-1} a_i^2}{2}.
 \end{aligned} \tag{2.7}$$

In the last expression addition formula (2.6) can be applied, which motivates the following abbreviations:

For a CKC with link lengths a_1, a_2, \dots, a_n , a circular configuration $\beta^{n-1} = (\beta_1, \dots, \beta_{n-1})$ and for $1 \leq k \leq n - 1$ we denote by $\beta^{n-k} := (\beta_1, \dots, \beta_{n-k})$ and we write $S_{n-k} = \sum_{i=1}^{n-k} a_i^2$ for the sum of the squared link lengths. Furthermore, for β^{n-k} we abbreviate terms

$$\begin{aligned}
 X(\beta^{n-k}) & := \sum_{i < j}^{n-k} a_i a_j \cos(\beta_i - \beta_j), \\
 \Phi(\beta^{n-k}) & := \varphi \left(\sum_{j=1}^{n-k} a_j \cos(\beta_j), \sum_{j=1}^{n-k} a_j \sin(\beta_j) \right),
 \end{aligned}$$

which naturally appear, when applying formula (2.6) for equation (2.7). With this shortened notation an important quantity for a CKC, its diagonal length, which is the distance of $f_{a,n-k}(\beta^{n-k})$ and the origin, can be expressed as

$$L(\beta^{n-k}) := \sqrt{S_{n-k} + 2X(\beta^{n-k})}.$$

Note that we assume $\beta^1 = \beta_1$ and $X(\beta^1) = 0$. Finally, for computations carried out in 2.2 it is convenient to denote by $C^k := (C_3, \dots, C_k)$ a vector with $k - 2$ entries for $3 \leq k \leq n - 1$ and let C_n be the constant given by $C_n := (a_n^2 - S_{n-1}) / 2$, which is the right hand side of equation (2.7).

2.2. Method for the Computation of Circular Configurations

We give a new method to obtain solutions to equation (2.5). In the proof of Theorem 2.1 a procedure is described how solutions are obtained by reducing the length of the CKC step by step. For $k \geq 1$ the solution method involves the choice of a real value C_{n-k} between the roots of a quadratic equation $E_k(C_{n-k+1})$ corresponding to the value C_{n-k+1} , see Definition (A.1). This value C_{n-k} then defines again a quadratic equation $E_k(C_{n-k})$ and a new value C_{n-k+1} is chosen between its roots and so on. In order to guarantee that this procedure is well defined, we need that all quadratic equations connected in this manner have real solutions, which is guaranteed by Lemma A.1 in the appendix.

In the following Theorem we will work with equation

$$\sum_{i < j}^{n-1} a_i a_j \cos(\beta_i - \beta_j) = C_n, \tag{2.8}$$

which is just a rearranged form of (2.5). Also abbreviations introduced in section 2.1 appear quite naturally in the proof of the Theorem.

Theorem 2.1 (Computation of circular configurations). *Circular configurations $\beta^{n-1} \in \mathcal{CC}_a^{onv}$, that is a configuration satisfying the equation (2.8) can be obtained by the following procedure:*

1. Compute a vector $C^{n-1} = (C_3, \dots, C_{n-1})$ with entries satisfying

$$C_{n-k+1}^- \leq C_{n-k} \leq C_{n-k+1}^+, \quad \text{and} \quad C_{n-k}^{\min} \leq C_{n-k} \leq C_{n-k}^{\max} \tag{2.9}$$

for $1 \leq k \leq n - 3$, where

$$C_{n-k+1}^\pm = C_{n-k+1} + a_{n-k}^2 \pm a_{n-k} \sqrt{2C_{n-k+1} + S_{n-k}}. \tag{2.10}$$

and $C_{n-k}^{\min}, C_{n-k}^{\max}$ denote the maximal and minimal values that $X(\delta^{n-k-1})$ can take on for $\delta^{n-1} \in (S^1)^{n-1}$.

2. For $1 \leq k \leq n - 2$ compute β_{n-k} according to the equation (where $C_2 := 0$ and $\beta^1 := \beta_1$)

$$a_{n-k} \sin(\beta_{n-k} + \Phi(\beta^{n-k-1})) = \frac{C_{n-k+1} - C_{n-k}}{\sqrt{S_{n-k-1} + 2C_{n-k}}}, \tag{2.11}$$

whenever the denominator of the term on the right hand side is not zero. Otherwise the angle β_{n-k} can be chosen arbitrarily.

Then $\beta^{n-1} = (\beta_1, \dots, \beta_{n-1}) \in \mathcal{CC}_a^{onv}$.

Proof. Assume β^{n-1} solves equation (2.8), which due to (2.7) and the abbreviations in section 2.1 can be written as

$$a_{n-1} \sin(\beta_{n-1}) \sum_{j=1}^{n-2} a_j \sin(\beta_j) + a_{n-1} \cos(\beta_{n-1}) \sum_{j=1}^{n-2} a_j \cos(\beta_j) + X(\beta^{n-2}) = C_n.$$

The latter equation is a linear combination of $\sin(\beta_{n-1})$ and $\cos(\beta_{n-1})$ and thus can further be simplified to

$$\sqrt{\left(\sum_{j=1}^{n-2} a_j \sin(\beta_j)\right)^2 + \left(\sum_{j=1}^{n-2} a_j \cos(\beta_j)\right)^2} a_{n-1} \sin(\beta_{n-1} + \Phi(\beta^{n-2})) + X(\beta^{n-2}) = C_n.$$

Expanding the squares shows that this is equivalent to

$$a_{n-1} \sin(\beta_{n-1} + \Phi(\beta^{n-2})) \sqrt{S_{n-2} + 2X(\beta^{n-2})} + X(\beta^{n-2}) = C_n,$$

where the square root is just given by the abbreviation $L(\beta^{n-2})$. If $L(\beta^{n-2}) = 0$ we have that $X(\beta^{n-2}) = C_n$ has to be satisfied and β_{n-1} is an arbitrary value. Otherwise, rewriting the latter equation gives

$$a_{n-1} \sin(\beta_{n-1} + \Phi(\beta^{n-2})) = \frac{C_n - X(\beta^{n-2})}{\sqrt{S_{n-2} + 2X(\beta^{n-2})}}. \tag{2.12}$$

Since the latter equation can be solved for β_{n-1} the right hand side satisfies

$$-a_{n-1} \leq \frac{C_n - X(\beta^{n-2})}{\sqrt{S_{n-2} + 2X(\beta^{n-2})}} \leq a_{n-1},$$

which is the case when $X(\beta^{n-2})$ is equal to a number C_{n-1} that is contained within the roots C_n^\pm of the quadratic equation

$$(C_n - C)^2 - a_{n-1}^2 (S_{n-2} + 2C) = 0, \tag{2.13}$$

and when $C^{\min} \leq C_{n-1} \leq C^{\max}$ holds. Note that the roots of the latter equation are real by lemma A.1. Thus we have that β^{n-2} satisfies the equation

$$X(\beta^{n-2}) = \sum_{i < j}^{n-2} a_i a_j \cos(\beta_i - \beta_j) = C_{n-1}, \tag{2.14}$$

which is of the same Type as (2.8). Consequently, the computations from above can be repeated and after k times we end up with

$$a_{n-k} \sin(\beta_{n-k} + \Phi(\beta^{n-k-1})) = \frac{C_{n-k+1} - X(\beta^{n-k-1})}{\sqrt{S_{n-k-1} + 2X(\beta^{n-k-1})}}.$$

Again, if $L(\beta^{n-k}) \neq 0$, the latter equation can be solved for the β_{n-k} and thus $X(\beta^{n-k-1})$ equals a value C_{n-k} that satisfies $C_{n-k}^{\min} \leq C_{n-k} \leq C_{n-k}^{\max}$ and which is contained within the roots C_{n-k+1}^\pm of the equation

$$(C_{n-k+1} - C)^2 - a_{n-k}^2 (S_{n-k-1} + 2C) = 0,$$

given by (2.10). Thus a circular configuration $\beta^{n-1} \in \mathcal{CC}_a^{onv}$ defines values C_{n-k} satisfying the system of inequalities (2.9). Conversely, if we have a solution to the systems of inequalities (2.9) a circular configuration $\beta^{n-1} \in \mathcal{CC}_a^{onv}$ can be defined according to (2.11). \square

The C^{n-3} , which entries C_{n-k} are recursively obtained by the system of inequalities (2.9) form a domain in $n - 3$ dimensional real space. By the proof of the last Theorem it is clear that the parameters C_{n-k} are closely related to the abbreviations $L(\beta^{n-k-1})$ introduced in section 2.1. More precisely, for $\beta \in \mathcal{CC}_a^{onv}$ the term $L(\beta^{n-k})$ is the length of the line segment connecting the origin with the endpoint $f_{a,n-k}(\beta^{n-k})$, which we will call a *diagonal* of a CKC according to [8], see Figure 2. In the appendix A the relation between C_{n-k} and the diagonals of a CKC is explained in more detail. However, the connection between the C_{n-k} and the diagonals of a CKC motivates the following definition.

Definition 2.1 (Domain of Semi-Diagonals). We will denote the set given by

$$\mathcal{SD}_a := \{C^{n-1} \in \mathbb{R}^{n-3} : C_{n-k} \in [C_{n-k+1}^-, C_{n-k+1}^+], 1 \leq k \leq n - 3\} \tag{2.15}$$

as semi-diagonal parameters of a CKC with links a .

According to Theorem 2.1 from any $C^{n-3} \in \mathcal{SD}_a \cap Q_a$, where

$$Q_a = \{C^{n-3} \in \mathbb{R}^{n-3} : C_{n-k}^{\min} \leq C_{n-k} \leq C_{n-k}^{\max}\},$$

circular configurations can be computed by solving (2.11). Note, that $C_{n-k}^{\max} = \sum_{i < j}^{n-k} a_i a_j$ and $C_{n-k}^{\min} = \min_{\beta} X(\beta^{n-k-1})$ can be easily computed. Since the solution of (2.11) for a β_{n-k} is not unique, an element $C^{n-1} \in \mathcal{SD}_a \cap Q_a$ will yield several circular configurations and we will consider all possible configurations that can be obtained from it in section 3.2. In the following section we will further investigate the set \mathcal{SD}_a . It will turn out, that it can be described in a very easy way, after a substitution of variables and thus also leads to an easy description for $\mathcal{SD}_a \cap Q_a$.

3. Further Analysis of Circular Configurations

In this section we will study the domain \mathcal{SD}_a , which naturally appears when we compute circular configurations, in more detail. Moreover, we will have a closer look on the second step of Theorem 2.1, which requires solving equation (2.11) for an angle β_{n-k} .

3.1. The domain of Semi-Diagonals

We investigate the space SD_a of Semi-Diagonals further. We recall that for a CKC with n links a_1, a_2, \dots, a_n the space SD_a is defined by the tuples C^{n-3} , which entries satisfy the system

$$C_{n-k+1}^- \leq C_{n-k} \leq C_{n-k+1}^+ \tag{3.1}$$

of inequalities, where

$$C_{n-k+1}^\pm = C_{n-k+1} + a_{n-k}^2 \pm a_{n-k} \sqrt{2C_{n-k+1} + S_{n-k}}$$

for $1 \leq k \leq n - 3$. We will see that this system can be transformed into another system after a substitution of variables, which can then be easily parametrized by a map that is defined on the $n - 3$ dimensional unit cube $I^{n-3} = [-1, 1]^{n-3}$.

Theorem 3.1. *Let $C^{n-1} \in SD_a$ and define new parameters U_{n-k} by*

$$C_{n-k} = U_{n-k} + C_{n-k+1} + a_{n-k}^2, \tag{3.2}$$

for $1 \leq k \leq n - 3$. Moreover we set $U_n := C_n$ and $U^{n-1} = (U_3, \dots, U_{n-1})$. Then the entries U_{n-k} satisfy the new system of inequalities

$$-a_{n-k} \sqrt{T_k(U_n, \dots, U_{n-k+1})} \leq U_{n-k} \leq a_{n-k} \sqrt{T_k(U_n, \dots, U_{n-k+1})}, \tag{3.3}$$

where

$$T_k(U_n, \dots, U_{n-k+1}) = 2 \sum_{j=1}^{k-1} U_{n-j} + \sum_{j=0}^{k-1} a_{n-j}^2, \quad \text{for } 1 \leq k \leq n - 3. \tag{3.4}$$

Note that the first sum on the right hand side is zero in the case $k = 1$ and thus $t_1 = t_1(U_n) = a_n^2$.

Proof. We will apply (3.2) to the right hand side of system (3.1) only, since the computations for the left hand side are analogous. If we apply (3.2) we obtain

$$U_{n-k} + C_{n-k+1} + a_{n-k}^2 \leq C_{n-k+1} + a_{n-k}^2 \pm a_{n-k} \sqrt{2C_{n-k+1} + S_{n-k}}, \tag{3.5}$$

which is equivalent to

$$U_{n-k} \leq a_{n-k} \sqrt{2C_{n-k+1} + S_{n-k}}.$$

If we apply the substitution (3.2) repeatedly on the right hand side we end up with

$$\begin{aligned} U_{n-k} &\leq a_{n-k} \sqrt{2 \left(U_n + \sum_{j=1}^{k-1} U_{n-j} + \sum_{j=1}^{k-1} a_{n-j}^2 \right) + S_{n-k}} \\ &\leq a_{n-k} \sqrt{2U_n + 2 \sum_{j=1}^{k-1} U_{n-j} + \sum_{j=1}^{k-1} a_{n-j}^2 + S_{n-1}} \\ &\leq a_{n-k} \sqrt{2 \sum_{j=1}^{k-1} U_{n-j} + \sum_{j=0}^{k-1} a_{n-j}^2}, \end{aligned}$$

where the expression under the square root is the abbreviation $T_k(U_n, \dots, U_{n-k+1})$ in (3.4). □

We will denote the space of parameters U^{n-1} introduced in Theorem 3.1 by SDU_a . It is clear that any $U^{n-1} \in SDU_a$ yields a unique $C^{n-1} \in SD_a$ and vice versa, since (3.2) defines an affine map from $A_a: SDU_a \rightarrow SD_a$. We will further investigate the domain SDU_a defined by the system of inequalities (3.3). It turns out that SDU_a is the image of a map that is defined on the unit cube, which clearly is a nice representation for the parameters U_{n-k} .

Theorem 3.2. Let I^{n-3} be the $n - 3$ dimensional unit cube and define the k -tuple $s^k := (s_1, \dots, s_k)$ for $1 \leq k \leq n - 3$. Then

$$P^a : I^{n-3} \longrightarrow \mathcal{SDU}_a, \quad s^{n-3} \mapsto \Phi^a (s^{n-3}),$$

where the components of P^a are given by

$$\begin{aligned} P_3^a (s^{n-3}) &= s_{n-3} a_3 \sqrt{T_{n-3} (U_n, P_{n-1}^a (s^1), \dots, P_4^a (s^{n-4}))} \\ &\vdots \\ P_{n-2}^a (s^2) &= s_2 a_{n-2} \sqrt{T_2 (U_n, P_{n-1}^a (s^1))} \\ P_{n-1}^a (s^1) &= s_1 a_{n-1} \underbrace{\sqrt{T_1 (U_n)}}_{a_n} \end{aligned}$$

maps the unit cube onto the space \mathcal{SDU}_a .

Proof. We have to show that $U_{n-k} := P_k^a (s^{n-k})$ satisfy the system of inequalities (3.3), which are equivalent to

$$|U_{n-k}| \leq a_{n-k} \sqrt{T_k (U_n, \dots, U_{n-k+1})}.$$

Note that the right side of this inequality does not depend on U_{n-k} , but only on $U_n \dots U_{n-k+1}$. Thus plugging in

$$P_{n-k}^a (s^k) = s_k a_{n-k} \sqrt{T_k (U_n, \dots, P_{n-k+1}^a (s^{n-k+1}))}$$

we obtain

$$\left| s_k a_{n-k} \sqrt{T_k (U_n, \dots, P_{n-k+1}^a (s^{n-k+1}))} \right| \leq a_{n-k} \sqrt{T_k (U_n, \dots, P_{n-k+1}^a (s^{n-k+1}))},$$

which clearly is satisfied for $s_k \in [-1, 1]$. Thus system (3.3) is satisfied. Conversely, if $U^{n-1} \in \mathcal{SDU}_a$ values s_k for $1 \leq k \leq n - 3$ can easily be obtained according to Theorem (3.2). Thus we have that $P^a (I^{n-3}) = \mathcal{SDU}_a$ \square

We will illustrate the assertions of Theorems (3.1) and (3.2) by an example.

Example 3.1 (CKC with five links). We consider the CKC with equal links $a_i = 1$ for $1 \leq i \leq 5$. In this case the inequalities defining \mathcal{SDU}_a are given by

$$-1 \leq U_4 \leq 1 \quad \text{and} \quad -\sqrt{2U_4 + 2} \leq U_3 \leq \sqrt{2U_4 + 2}.$$

Then the map $P^a : [-1, 1]^2 \rightarrow \mathcal{SDU}_a$ is given by

$$s^2 := \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \mapsto \begin{pmatrix} s_1 \\ s_2 \sqrt{2s_1 + 2} \end{pmatrix} = \begin{pmatrix} U_4 (s_1, s_2) \\ U_3 (s_1, s_2) \end{pmatrix}.$$

Note that P^a is injective for $s_1 \neq -1$. Parameters U_3, U_4 are related to C_3, C_4 by the map A_a given by the equations

$$C_4 = U_4 - \frac{1}{2} \quad \text{and} \quad C_3 = U_3 + U_4 + \frac{1}{2},$$

which are derived from (3.5) for $n = 5$. In terms of the parameters $s^2 = (s_1, s_2)$ we have $C_4 = s_1 - \frac{1}{2}$ and $C_3 = s_2 \sqrt{2s_1 + 2} + s_1 + \frac{1}{2}$

The map P^a gives a nice description for parameters U_{n-k} satisfying the system of inequalities (3.3), namely by a cube. Using the affine transform A_a we obtain a map $A_a \circ P^a : I^{n-3} \rightarrow \mathcal{SD}_a$, which gives \mathcal{SDU}_a . Points in $\mathcal{SDU}_a \cap Q_a$ can easily be obtained, when restrictions $C_{n-k}^{\min} \leq C_{n-k} \leq C_{n-k}^{\max}$ are taken into account. Further investigations of P^a like its injectivity or its singularities are an interesting topic for future research.

3.2. Flipping over lines through diagonals of a CKC

Given a $C^{n-1} \in \mathcal{SD}_a \cap Q_a$ the angle β_{n-k} is computed from

$$a_{n-k} \sin(\beta_{n-k} + \Phi(\beta^{n-k-1})) = \frac{C_{n-k+1} - X(\beta^{n-k-1})}{\sqrt{S_{n-k-1} + 2X(\beta^{n-k-1})}}$$

according to Theorem 2.1. Clearly, solving for β_{n-k} is not unique. Each time we solve for β_{n-k} we have to choose which pre image we will take. In the following let $\varepsilon^k = (\varepsilon_2, \dots, \varepsilon_k) \in \{0, 1\}^{k-1}$ be a vector that contains the information, which pre images haven been chosen. We will refer to ε^k as orientation vector. More precisely we set,

$$\beta_{n-k}^{\varepsilon_{n-k}} = \pi \varepsilon_{n-k} + (-1)^{\varepsilon_{n-k}} S_{n,k}(C^{n-1}) - \Phi(\beta^{n-k-1, \varepsilon^{n-k-1}}). \tag{3.6}$$

for $1 \leq k \leq n - 2$. Here the superscript ε_{n-k} in $\beta_{n-k}^{\varepsilon_{n-k}}$ indicates, which preimage is chosen according to equation (3.6). Note, that we used the abbreviation

$$S_{n,k}(C^{n-1}) = \arcsin\left(\frac{C_{n-k+1} - C_{n-k}}{a_{n-k} \sqrt{S_{n-k-1} + 2C_{n-k}}}\right)$$

for $C^{n-1} \in \mathcal{SD}_a \cap Q_a$ and the notation

$$\beta^{n-k, \varepsilon^{n-k}} = (\beta_1, \beta_2^{\varepsilon_2}, \dots, \beta_{n-k}^{\varepsilon_{n-k}}),$$

to indicate the choice of preimages.

Therefore, for each $C^{n-1} \in \mathcal{SD}_a \cap Q_a$ and a $\varepsilon \in \{0, 1\}^{n-2}$ we obtain a circular configuration

$$\beta^{n-1, \varepsilon^{n-1}} = (\beta_1, \beta_2^{\varepsilon_2}, \dots, \beta_{n-1}^{\varepsilon_{n-1}})$$

by formula (3.6). Note that the angle β_1 does not have a superscript since β_1 is chosen arbitrarily in the last step of (2.11) for $k = n - 3$. Each $C^{n-1} \in \mathcal{SD}_a \cap Q_a$ yields 2^{n-2} circular configurations which corresponds to the possible choices for the components of ε^{n-1} . There is a geometric interpretation for the value ε_{n-k} in equation (3.6). It describes how the link a_{n-k} is flipped over the line running through the origin and $f(\beta^{n-k-1, \varepsilon^{n-k-1}})$, when it is attached to $f(\beta^{n-k-1, \varepsilon^{n-k-1}})$, see Figure 3.2. Choosing the value ε_{n-k} corresponds to the choice of

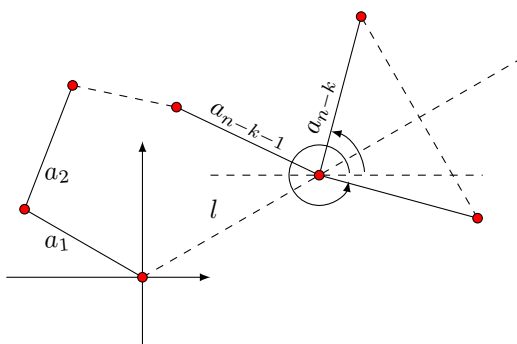


Figure 3. Choosing the value ε_{n-k} in equation (3.6) corresponds to a flipping over the diagonal l .

a triangle orientation in [10] when building up a CKC from its diagonal lengths.

4. Numerical simulations

In this section we provide numerical examples that demonstrate the validity of the methods developed in this work. For illustrative purposes we will consider CKCs with five and six links. For CKCs with five links we will illustrate the spaces \mathcal{SD}_a . We will compute random circular configurations for the CKCs by Theorem 2.1 and depict them in Figures 5 and 6.

4.1. CKCs with five and six links

First we will consider a CKC with $n = 5$ and $n = 6$ links. The domain $SD_a \cap Q_a$ is depicted for two CKCs with different link lengths in Figure 4. The black lines indicate the condition $-a_1 a_2 = C_3^{\min} \leq C_3 \leq C_3^{\max} = a_1 a_2$. The value C_4 lies within $[\max\{C_4^{\min}, C_5^-\}, \min\{C_4^{\max}, C_5^+\}]$. In the depicted cases $\max\{C_4^{\min}, C_5^-\} = C_5^-$ and $\min\{C_4^{\max}, C_5^+\} = C_5^+$ holds. Figure 5 shows 10 random circular configurations for CKCs with five links. Thereby the lengths of the last link is equal to the radius of the depicted circle. The configurations have been obtained for the orientation vector $\varepsilon^4 = (\varepsilon_2, \varepsilon_3, \varepsilon_4) = (0, 0, 0)$.

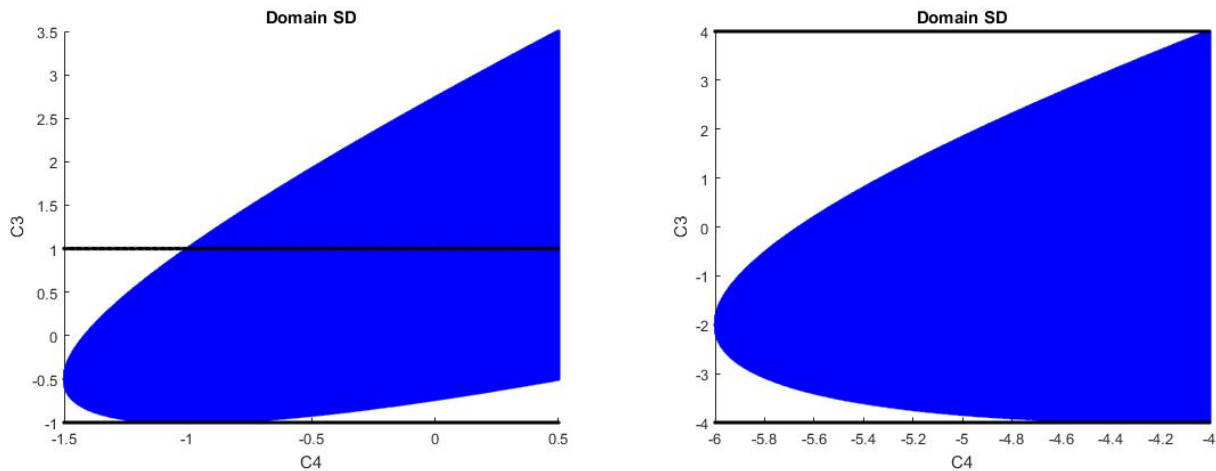


Figure 4. Domain $SD_a \cap Q_a$ is the part of the blue area enclosed within the black lines. Left: CKC with all links equal to one. Right: CKC with link lengths 2, 2, 2, 1, 1.

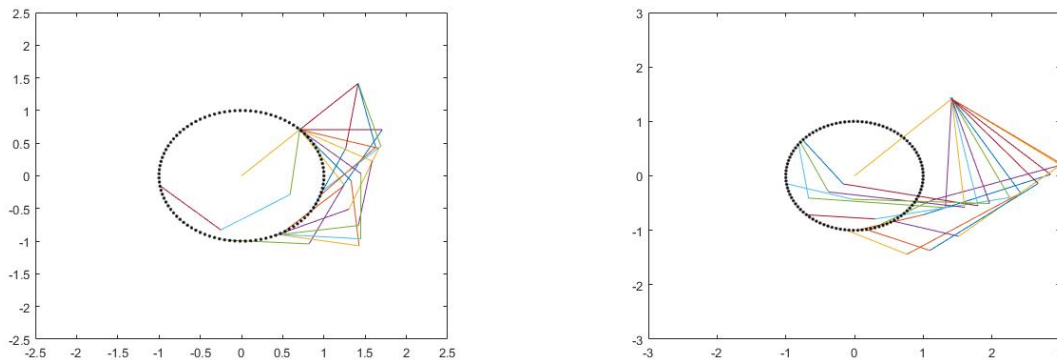


Figure 5. Circular configurations of two CKCs with five links. Left: Ten random circular configurations are depicted for the CKC with link lengths equal to one. Right: Ten random circular configurations are depicted for the CKC with link lengths 2, 2, 2, 1, 1.

Finally, we give examples for CKCs with six links. Figure 6 shows random configurations for CKCs with six links. We consider the orientation vector $\varepsilon^5 = (\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) = (0, 0, 0, 0)$.

5. Conclusion and future work

We have developed a new method to compute configurations in terms of joint angles of a CKC by a systematic procedure. Our approach does not require the solution of a system of linear inequalities by linear programming nor does it rely on probabilist methods. Numerical examples show validity of the proposed work. We expect that the described method can be useful in tasks like motion planning for CKCs. We expect that is an interesting line for future work to investigate the introduced map $P^a : I^{n-3} \rightarrow SDU_a$. We expect that it enables us to use tools from differential geometry, which may lead to further insights and applications. Moreover, it would be interesting to investigate how special designs for CKCs are reflected in the presented computations.

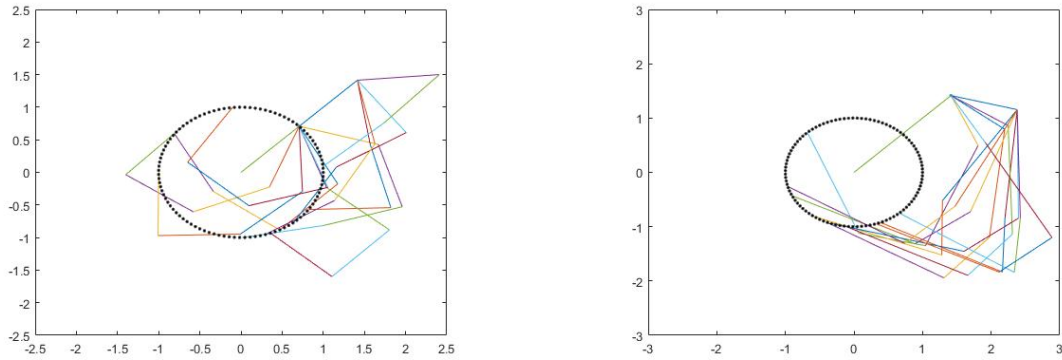


Figure 6. Circular configurations of two CKCs with six links. Left: Ten random circular configurations are depicted for the CKC with link lengths equal to one. Right: Ten random circular configurations are depicted for the CKC with link lengths 2, 1, 2, 1, 2, 1.

A. Appendix

The following Lemma guarantees that values obtained by the procedure described in section 2.2 are indeed real. We state the Lemma after the following definition:

Definition A.1. Furthermore, for $A \in \mathbb{R}$ let $E_k(A)$ be the quadratic equation in the variable C corresponding to the value A given by

$$(A - C)^2 - a_{n-k}^2 (S_{n-k-1} + 2C) = 0$$

and let A^\pm be its solutions.

Lemma A.1. Let C_{n-k+1} be a real number with $2C_{n-k+1} + S_{n-k} \geq 0$, which means that the roots C_{n-k+1}^\pm of the quadratic equation $E_k(C_{n-k+1})$, given by

$$(C_{n-k+1} - C)^2 - a_{n-k}^2 (S_{n-k-1} + 2C) = 0, \tag{A.1}$$

are real. Choosing a value C_{n-k} between the roots of $E_k(C_{n-k+1})$ guarantees that equation $E_{k+1}(C_{n-k})$ also has real solutions. Moreover, the roots of $E_1(C_n)$ are real.

Proof. For $k = 1$ we have that the roots C_n^\pm of the equation $E_1(C_n)$, given by

$$(C_n - C)^2 - a_{n-1}^2 (S_{n-2} + 2C) = 0,$$

are real, since they are computed to be

$$\begin{aligned} C_n^\pm &= C_n + a_{n-1}^2 \pm a_{n-1} \sqrt{2C_n + S_{n-1}} \\ &= \frac{a_n^2 + a_{n-1}^2}{2} - \frac{S_{n-2}}{2} \pm a_{n-1} \sqrt{a_n^2 - S_{n-1} + S_{n-1}} \\ &= \frac{a_n^2 + a_{n-1}^2}{2} - \frac{S_{n-2}}{2} \pm a_{n-1} a_n. \end{aligned}$$

By our assumption the roots of $E_k(C_{n-k+1})$ are real. We have to prove that the roots of equation $E_{k+1}(C_{n-k})$ corresponding to C_{n-k} are still real, whenever $C_{n-k+1}^- \leq C_{n-k} \leq C_{n-k+1}^+$. This means that the roots of

$$(C_{n-k} - C)^2 - a_{n-k-1}^2 (S_{n-k-2} + 2C) = 0$$

are real, which is the case if $2C_{n-k} + S_{n-k-1} \geq 0$. By our assumption C_{n-k} is real and clearly $C_{n-k} \geq C_{n-k+1}^-$. Thus

$$2C_{n-k} + S_{n-k-1} \geq 2C_{n-k+1}^- + S_{n-k-1}.$$

The Lemma follows, if we can show that the right side of the last inequality is greater than zero. We plug in the explicit expression

$$C_{n-k+1}^- = C_{n-k+1} + a_{n-k}^2 - a_{n-k} \sqrt{2C_{n-k+1} + S_{n-k}}$$

into the latter inequality and obtain

$$\begin{aligned} & 2 \left(C_{n-k+1} + a_{n-k}^2 - a_{n-k} \sqrt{2C_{n-k+1} + S_{n-k}} \right) + S_{n-k-1} \geq 0 \Leftrightarrow \\ & 2C_{n-k+1} + \underbrace{(S_{n-k-1} + a_{n-k}^2)}_{=S_{n-k}} - 2a_{n-k} \sqrt{2C_{n-k+1} + S_{n-k}} \geq 0. \end{aligned}$$

By assumption $D := 2C_{n-k+1} + S_{n-k} \geq 0$ and thus the latter inequality is satisfied since

$$D + a_{n-k}^2 - 2a_{n-k} \sqrt{D} = \left(\sqrt{D} - a_{n-k} \right)^2 \geq 0.$$

□

As mentioned in section 2 the diagonal lengths of a CKC are closely related to the parameters C_{n-k} , which justifies the naming for \mathcal{SD}_a . More precisely, in [10] it is shown that $\beta^{n-1} \in \mathcal{CC}_a^{onv}$ is a circular configuration if and only its diagonal lengths satisfy the system inequalities

$$\left(L(\beta^{n-k}) - a_{n-k} \right)^2 \leq L(\beta^{n-k-1})^2 \leq \left(L(\beta^{n-k}) + a_{n-k} \right)^2, \quad (\text{A.2})$$

for $1 \leq k \leq n-2$. Note that here $L(\beta^1) = a_1$ and $L(\beta^{n-1}) = a_n$. Furthermore, it is reasonable here to consider circular configurations, since the diagonal lengths of a CKC are invariant with respect to rotation around the origin. The relation between the diagonal lengths an \mathcal{SD}_a is established by the lemma.

Lemma A.2. *Let $\beta^{n-1} \in (S^1)^{n-1}$ be a vector of joint angles and let $C^{n-1} = (C_3, \dots, C_{n-1})$ be a vector, which entries are given by*

$$C_{n-k} := X(\beta^{n-k-1}) \quad (\text{A.3})$$

for $1 \leq k \leq n-3$. Then $\beta^{n-1} \in \mathcal{CC}_a^{onv}$ if and only if $C^{n-1} \in \mathcal{SD}_a \cap Q_a$.

Proof. Assume $C^{n-1} \in \mathcal{SD}_a \cap Q_a$ and $X(\beta^{n-k-1}) = C_{n-k}$ for a $\beta^{n-1} \in (S^1)^{n-1}$. Then, by (2.9) we have that the inequality $C_{n-k+1}^- \leq C_{n-k}$ is satisfied. Using the explicit expression (2.10) for C_{n-k+1}^- and our assumption we obtain that this is equivalent to

$$C_{n-k+1} + a_{n-k}^2 - a_{n-k} L(\beta^{n-k}) \leq C_{n-k}. \quad (\text{A.4})$$

Multiplying this inequality by two and then adding S_{n-k-1} on both sides gives

$$2C_{n-k+1} + 2a_{n-k}^2 - 2a_{n-k} L(\beta^{n-k}) + S_{n-k-1} \leq 2C_{n-k} + S_{n-k-1} \quad (\text{A.5})$$

and therefore, since $S_{n-k-1} + a_{n-k}^2 = S_{n-k}$, we have that

$$\underbrace{2C_{n-k+1} + S_{n-k}}_{=L(\beta^{n-k})^2} + a_{n-k}^2 - 2a_{n-k} L(\beta^{n-k}) \leq L(\beta^{n-k-1})^2.$$

Completing the square

$$\left(L(\beta^{n-k}) - a_{n-k} \right)^2 \leq L(\beta^{n-k-1})^2,$$

shows that (A.4) is equivalent to the first inequality in (A.2). The second one follows from $C_{n-k} \leq C_{n-k+1}^+$ by analogous computations. Thus the system of inequalities (A.2) is satisfied and $\beta^{n-1} \in \mathcal{CC}_a^{onv}$. Conversely, if $\beta^{n-1} \in \mathcal{CC}_a^{onv}$ is a circular configuration of a CKC its diagonals satisfy inequalities (A.2). Setting $C_{n-k} := X(\beta^{n-k-1})$ and repeating the latter estimates shows that (2.9) are satisfied and thus $C^{n-1} \in \mathcal{SD}_a \cap Q_a$. □

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