# A Novel Numerical Approach for Solving Convection-Diffusion Problem with Boundary Layer Behavior

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## Highlights
• The paper focused on the numerical solution of singularly perturbed boundary value problem.
• To obtain the approach of difference scheme, finite difference method is used.
• Stability and uniform convergence of presented difference method are investigated.
• The examples are solved by using Thomas algorithm.

| Article Info | Abstract |
|---|---|
| | This study deals with a new approach method for solving singularly perturbed boundary value problem of convection-diffusion type. Firstly, bounds on the solution and its derivative of solution to be used later in the article are obtained. This robust method is constructed with fitted difference scheme on a uniform mesh. It is proved that the presented method is first-order convergent with respect to the perturbation parameter $\varepsilon$ in the discrete maximum norm. Two examples are given to illustrate the efficiency of the method. The numerical results are presented in tables and figures. |
| | |

## 1. INTRODUCTION

Here we handle the following convection-diffusion problem with boundary layer behavior:

$$\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), 0 < x < l, \tag{1}$$

$$u(0) = A, u(l) = B, \tag{2}$$

where $A$ and $B$ are given constants and $0 < \varepsilon \ll 1$ is a very small positive parameter. It is supposed that the functions $a(x)$, $b(x)$ and $f(x)$ are continuous differentiable in interval $[0, l]$ and besides, our conditions are

$$a(x) \geq \alpha > 0, \tag{3}$$
$$b(x) \geq \beta > 0, \tag{4}$$

Differential equations with a small parameter $\varepsilon$ multiplying the highest-order derivative terms are called singularly perturbed differential equations. Standard numerical methods for solving singularly perturbed problems are unstable and do not give accurate results due to the perturbation parameter $\varepsilon$. If suitable numerical methods such as finite difference method and finite element method [1-8] for solving these

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problems are developed, then the stable and accurate results are obtained. Therefore, we prefer to apply finite difference method for this problem in this paper.

There are the various approaches to the design and analysis of appropriate numerical methods for singularly perturbed differential equations in [9-17] and the references therein. Singular perturbation problems are located in various fields. For example, chemical-reactor theory, control theory, oceanography, fluid mechanics, quantum mechanics, hydro mechanical problems, meteorology, electrical networks and other physical models [18-25]. There is study of existence and uniqueness of singularly perturbed problems in [26].

The present study is organized as follows: In Section 2, we state some required properties of the exact solution for analysis of numerical method. Then, the finite difference discretization is given in Section 3. In Section 4, we investigate the error evaluation on uniform mesh for the approximate solution of the problem (1)-(2). Furthermore, numerical results supporting the theory are presented. The obtained results are shown via tables and figures.

Throughout the paper, $C, C_0, C_1$ denote generic positive constants independent of $\varepsilon$ and the mesh parameter. For any continuous function $g(x)$ denote norms which $g_\infty \equiv g_{\infty,[0,l]} = \max_{0 \le x \le l} |g(x)|$.

## 2. SOME PROPERTIES OF CONTINUOUS PROBLEM

Here we give asymptotic estimates for the solution of continuous problem (1)-(2). These estimates will be used to analysis of the uniform convergence of difference approximations in the next sections.

**Lemma 2.1.** Let us accept that the function $u(x)$ is the solution of (1)-(2) and $a(x), b(x), f(x) \in C^1[0,l]$. Then the equalities

$$u_\infty \le C_0, \tag{5}$$
$$C_0 = |A| + |B| + \alpha^{-1} f_\infty,$$

and

$$|u'(x)| \le C\left\{1 + \frac{1}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}}\right\}, \tag{6}$$

hold.

**Proof.** We first show the proof of (5). Here we define the maximum principle for the problem (1)-(2). Let $L$ be the differential operator in (1)-(2) and $\kappa \in C^2[0,l]$ if $\kappa(0) \ge 0$, $\kappa(l) \ge 0$ and $L\kappa \le 0$, $0 < x < l$, then $\kappa(x) \ge 0$ for $0 \le x \le l$.
Using the barrier function, we have

$$\Psi^\pm(x) = \pm u(x) + |A| + |B| + \alpha^{-1}(l - x) f_\infty.$$

By applying the maximum principle to the function above, we find

$$\Psi^\pm(0) = \pm A + |A| + |B| + \alpha^{-1}(l - x) f_\infty \ge 0,$$

$$\Psi^\pm(0) = \pm B + |B| + |A| + |A| \ge 0,$$

and

$$L\Psi(x) = \pm f(x) - (|A| + |B|)b(x) + \alpha^{-1}[a(x) - b(x)](l - x)f_\infty$$

$$\le -(|A| + |B|)b(x) \pm |f(x)| - f_\infty \le 0$$

According to maximum principle, we get

$$\Psi(x) \ge 0.$$

From this inequality, we obtain

$$\pm u(x) + |A| + |B| + \frac{(l - x)f_\infty}{\alpha} \ge 0,$$

$$|u(x)| = |A| + |B| + \frac{1}{\alpha}(l - x)f_\infty,$$

which gives the proof of (5).

We now show the proof of (4). We rewrite the problem (1)-(2) as

$$\varepsilon u''(x) + a(x)u'(x) = F(x),\tag{7}$$

where

$$F(x) = f(x) + b(x)u(x).$$

From the relation (7), we have

$$u'(x) = u'(0)e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau} + \frac{1}{\varepsilon}\int_0^x F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi.\tag{8}$$

We need an estimate for $u'(0)$ in (8). Integrating this equality over $(0, l)$, we have

$$B - A = u'(0)\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau}dx + \frac{1}{\varepsilon}\int_0^l\int_0^x F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi dx.$$

From this equality we obtain

$$u'(0) = \frac{B - A - \frac{1}{\varepsilon}\int_0^l\int_0^x F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi dx}{\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau}dx}.\tag{9}$$

We evaluate integral in denominator of (9) as

$$\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau}dx \ge \int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a^* d\tau}dx = \int_0^l e^{-\frac{a^* x}{\varepsilon}}dx = \frac{\varepsilon\left(1 - e^{-\frac{a^* l}{\varepsilon}}\right)}{a^*} = \gamma_0\varepsilon,\tag{10}$$

where $a^* = \max_{[0,l]}\{a(x)\}$.

Applying the mean value theorem to integral in (9), we find

$$\left|\frac{1}{\varepsilon}\int_0^l\left[\int_0^x F(\xi)\exp\left(-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau\right)d\xi\right]dx\right| \le \frac{1}{\varepsilon}\int_0^l\left[\int_0^x |F(\xi)|\exp\left(-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau\right)d\xi\right]dx$$

$$\leq \frac{F_{\infty}}{\varepsilon} \int\limits_{0}^{l}\int\limits_{0}^{x} e^{-\frac{\alpha(x-\xi)}{\varepsilon}} d\xi dx = \alpha^{-1} F_{\infty} \int\limits_{0}^{l}\left[ 1-e^{-\frac{\alpha x}{\varepsilon}} \right] dx \leq \frac{F_{\infty}l}{\alpha} \leq C_{1}. \tag{11}$$

By writing the evaluations (10) and (11) in (9), we get

$$\left| u^{'}(0) \right| \leq \frac{|A|+|B|+C_{1}}{\gamma \varepsilon} = \frac{C}{\varepsilon}. \tag{12}$$

Using the estimate (12) in (8), we have

$$\left| u^{'}(x) \right| \leq \frac{C}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}} + \frac{F_{\infty}}{\alpha}\left( 1-e^{-\frac{\alpha x}{\varepsilon}} \right) \leq \frac{C}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}} + \frac{F_{\infty}}{\alpha},$$

which gives proof of (6). So, we have completed the proof of Lemma 2.1.

## 3. CONSTRUCTION OF DIFFERENCE SCHEMES

Let $\omega_{N}$ be a uniform mesh on $[0,l]$ as follows:

$$\omega_{N} = \left\{ x_{i}, = ih, \ i=1,2,\ldots,N-1; \quad h = \frac{l}{N} \right\},$$

$$\varpi_{N} = \omega_{N} \cup \{x=0, x=l\}.$$

Before describing our numerical method, we introduce some notation for the mesh functions. For any mesh function $g(x)$ defined on $\bar{\omega}_{N}$ we use

$$g_{i} = g(x_{i}), \quad g_{\bar{x},i} = \frac{g_{i}-g_{i-1}}{h}, \quad g_{x,i} = \frac{g_{i+1}-g_{i}}{h}, \quad g_{\overset{o}{x},i} = \frac{g_{i+1}-g_{i-1}}{2h}, \quad u_{\bar{x}x,i} = \frac{g_{i+1}-2g_{i}+g_{i-1}}{h^{2}},$$

$$g_{1} \equiv g_{1,\omega_{N}} = h\sum_{i=1}^{n-1}|g_{i}|.$$

Here we construct difference approximate on a uniform mesh for the problem (1)-(2). To obtain the approach of difference scheme, we will integrate equation (1) over $(x_{i-1}, x_{i+1})$.

$$h^{-1}\int\limits_{x_{i-1}}^{x_{i+1}} Lu\varphi_{i}(x)dx = h^{-1}\int\limits_{x_{i-1}}^{x_{i+1}} f(x)\varphi_{i}(x)dx, \tag{13}$$

where the basis functions $\varphi_{i}(x)$ are as follows:

$$\varphi_{i}(x) = \begin{cases} \varphi_{i}^{(1)}(x) = \dfrac{-1+exp\left(-\dfrac{a_{i}(x-x_{i-1})}{\varepsilon}\right)}{-1+exp\left(-\dfrac{a_{i}h}{\varepsilon}\right)}, & x \in (x_{i-1}, x_{i}), \\[4mm] \varphi_{i}^{(2)}(x) = \dfrac{1-exp\left(-\dfrac{a_{i}(x_{i+1}-x)}{\varepsilon}\right)}{1-exp\left(-\dfrac{a_{i}h}{\varepsilon}\right)}, & x \in (x_{i}, x_{i+1}), \\[4mm] 0, & x \notin (x_{i-1}, x_{i+1}). \end{cases}$$

The functions $\varphi_{i}^{(1)}(x)$ and $\varphi_{i}^{(2)}(x)$ are the solution of the following problems, respectively:

$$\varepsilon\varphi_i'' - a_i\varphi_i' = 0, \quad x \in (x_{i-1}, x_i),$$

$$\varphi_i(x_{i-1}) = 0, \quad \varphi_i(x_i) = 1,$$

$$\varepsilon\varphi_i'' - a_i\varphi_i' = 0, \quad x \in (x_i, x_{i+1}),$$

$$\varphi_i(x_i) = 1, \quad \varphi_i(x_{i+1}) = 0.$$

We can continue from (13) as

$$\frac{1}{h}\int_{x_{i-1}}^{x_{i+1}} \left[\varepsilon u''(x) + a(x)u'(x) - b(x)u(x)\right]\varphi_i(x)dx = \frac{1}{h}\int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx,$$

$$\frac{1}{h}\int_{x_{i-1}}^{x_{i+1}} -\varepsilon u'(x)\varphi_i'(x)dx + \frac{1}{h}\int_{x_{i-1}}^{x_{i+1}} a_i u'(x)\varphi_i(x)dx - \frac{1}{h}\int_{x_{i-1}}^{x_{i+1}} b(x)u(x)\varphi_i(x)dx = f_i + \overline{R}_i,$$

$$\frac{1}{h}\int_{x_{i-1}}^{x_{i+1}} -\varepsilon u'(x)\varphi_i'(x)dx + \frac{1}{h}\int_{x_{i-1}}^{x_{i+1}} a_i u'(x)\varphi_i(x)dx - b_i u_i = f_i + R_i, \tag{14}$$

where we can express the reminder term $R_i$ as

$$R_i = \overline{R}_i + R_{b,i} = R_{a,i} + R_{f,i} + R_{b,i}, \tag{15}$$

with

$$R_{a,i} = \frac{1}{h}\int_{x_{i-1}}^{x_{i+1}} u'(x)[a(x_i) - a(x)]\varphi_i(x)dx, \tag{16}$$

$$R_{b,i} = \frac{1}{h}\int_{x_{i-1}}^{x_{i+1}} [b(x)u(x) - b(x_i)u(x_i)]\varphi_i(x)dx, \tag{17}$$

$$R_{f,i} = \frac{1}{h}\int_{x_{i-1}}^{x_i} [f(x) - f(x_i)]\varphi_i(x)dx. \tag{18}$$

Using the quadrature rules in (2.1) and (2.2) from [1], we obtain the following precise relation:

$$h^{-1}\int_{x_{i-1}}^{x_i} \left[\varepsilon u'\varphi_i^{(1)'}(x) + a_i u'(x)\varphi_i^{(1)}(x)\right]dx + h^{-1}\int_{x_i}^{x_{i+1}} \left[\varepsilon u'\varphi_i^{(2)'}(x) + a_i u'(x)\varphi_i^{(2)}(x)\right]dx$$

$$= -\varepsilon h^{-1}u_{\bar{x},i} + a_i u_{\bar{x},i} \, h^{-1}\int_{x_{i-1}}^{x_i} \varphi_i^{(1)}(x)dx - \varepsilon h^{-1}u_{\bar{x},i} + a_i u_{x,i} \, h^{-1}\int_{x_i}^{x_{i+1}} \varphi_i^{(2)}(x)dx,$$

$$= \varepsilon\frac{u_{x,i} - u_{\bar{x},i}}{h} + a_i u_{\bar{x}} \sigma_i^{(1)} + a_i u_x \sigma_i^{(2)} = \varepsilon u_{\bar{x}x,i} + a_i u_{\bar{x},i}\sigma^{(1)} + a_i u_{x,i}\sigma^{(2)}, \tag{19}$$

where

$$\sigma_i^{(1)} = h^{-1}\int_{x_{i-1}}^{x_i} \varphi_i^{(1)}(x)dx = h^{-1}\int_{x_{i-1}}^{x_i} \frac{-1 + exp\left(-\dfrac{a_i(x - x_{i-1})}{\varepsilon}\right)}{-1 + exp\left(-\dfrac{a_i h}{\varepsilon}\right)}dx = \frac{\varepsilon}{a_i h} + \frac{1}{1 - e^{a_i h/\varepsilon}},$$

$$\sigma_i^{(2)} = h^{-1} \int\limits_{x_i}^{x_{i+1}} \varphi_i^{(2)}(x)\, dx = h^{-1} \int\limits_{x_i}^{x_{i+1}} \frac{1 - exp\left(-\dfrac{a_i(x_{i+1}-x)}{\varepsilon}\right)}{1 - exp\left(-\dfrac{a_i h}{\varepsilon}\right)} = \frac{1}{e^{a_i h/\varepsilon}-1} - \frac{\varepsilon}{a_i h}.$$

Using the relations

$$u_{\substack{0 \\ x,i}} - u_{\bar{x},i} = \frac{h}{2} u_{\bar{x}x,i},$$

and

$$u_{\substack{0 \\ x,i}} - u_{x,i} = \frac{h}{2} u_{x,i},$$

in (19), we obtain

$$\varepsilon u_{\bar{x}x,i} + a_i \sigma_i^{(1)} u_{\substack{0 \\ x,i}}, -a_i \sigma_i^{(1)} \frac{h}{2} u_{\bar{x}x,i} - a_i \sigma_i^{(2)} u_{\substack{0 \\ x,i}} + a_i \sigma_i^{(2)} \frac{h}{2} u_{\bar{x}x,i}$$

$$= \varepsilon \left\{ 1 + \frac{a_i h}{\varepsilon}\left(\sigma_i^{(2)} - \sigma_i^{(1)}\right) \right\} u_{\bar{x}x,i} + a_i\left(\sigma_i^{(1)} + \sigma_i^{(2)}\right) u_{\substack{0 \\ x,i}} = \varepsilon\theta_i u_{\bar{x}x,i} + a_i u_{\substack{0 \\ x,i}}, \tag{20}$$

where

$$\theta_i = \varepsilon \left\{ 1 + \frac{a_i h}{\varepsilon}\left(\sigma_i^{(2)} - \sigma_i^{(1)}\right) \right\} = \gamma_i \coth(\gamma_i), \gamma_i = \frac{a_i h}{2\varepsilon}, \tag{21}$$

$$\sigma_i^{(1)} + \sigma_i^{(2)} = h^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} \varphi_i(x)\, dx = 1.$$

Then, by using the relation (20) in (14), we get

$$ly_i \equiv \varepsilon\theta_i y_{\bar{x}x,i} + a_i y_{\substack{o \\ x,i}} - b_i y_i = f_i + R_i, \quad 1 \le i \le N-1. \tag{22}$$

Neglecting $R_i$ in the relation (22), we propose the following finite difference approximate for the problem (1)-(2):

$$ly_i \equiv \varepsilon\theta_i y_{\bar{x}x,i} + a_i y_{\substack{o \\ x,i}} - b_i y_i = f_i, \tag{23}$$

$$y_0 = A, y_N = B, \tag{24}$$

where $\theta_i$ is defined by (21).

## 4. STABILITY AND $\varepsilon-$ UNIFORM CONVERGENCE

Here we will give stability and uniform convergence of presented difference method.

**Lemma 4.1.** If we accept the smoothness assumptions of Lemma 2.1, then the truncation function $R$ satisfies the following inequality.

$$\|R\|_{\infty,\omega_N} \le Ch. \tag{25}$$

**Proof.** We first show that

$$\|R_f\|_{\infty,\omega_N} \le Ch. \tag{26}$$

Applying the mean value theorem to function in (18), we get

$$\left| \frac{f(x) - f(x_i)}{x - x_i} \right| = |f'(\xi)| \le max|f'(x)||x - x_i| \le C, \ \xi \in (x_i, x).$$

Hence, we obtain

$$\|R_f\|_{\infty, \omega_N} \le Ch.$$

In second case let us indicate that

$$\|R_a\|_{\infty, \omega_N} \le Ch. \tag{27}$$

By applying the mean value theorem to the function in (16), we have

$$\left| \frac{a(x) - a(x_i)}{x - x_i} \right| = |a'(\xi)| \le max|a'(\xi)||x - x_i| \le C_0 h, \ \xi \in (x_i, x).$$

From this, we obtain

$$\|R_a\|_{l_1(\omega_N)} \le C_0 h \sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_{i+1}} C_1 \left( 1 + \frac{1}{\varepsilon} e^{\frac{-\alpha x}{\varepsilon}} \right) d \le C_0 C_1 h \left( 2l + \sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_{i+1}} \frac{1}{\varepsilon} e^{\frac{-\alpha x}{\varepsilon}} dx \right)$$

$$\le C_0 C_1 h \left( 2l + \frac{2}{\varepsilon} \int_0^l e^{\frac{-\alpha x}{\varepsilon}} \right) dx \le Ch.$$

In the same way, we can demonstrate the following inequality as similar to the inequality (27).

$$\|R_b\|_{l_1(\omega_N)} \le Ch. \tag{28}$$

Applying the mean value theorem to the function in the relation (17), we take

$$\left| \frac{b(x)u(x) - b(x_i)u(x_i)}{x - x_i} \right| = |(b(\xi)u(\xi))'| \le max|u(x)b'(x) + u'(x)b(x)||x - x_i|$$

$$\le \left( max|u(x)b'(x)| + max|u'(x)b(x)| \right)|x - x_i| \le c_1 h, \ x_i < \xi < x.$$

From this, we get

$$\|R_b\|_{l_1(\omega_N)} \le c_1 h \sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_{i+1}} c_1 \left( 1 + \frac{1}{\varepsilon} exp \left( -\frac{\alpha x}{\varepsilon} \right) \right) dx \le c_1 h,$$

which leads to (28).

Hence, from the inequalities (26), (27) and (28) we have (25).

**Lemma 4.2.** Error function $z_i$, $0 \le i \le N$ satisfies the following inequality:

$$|z_i| \le CR_{l_1(\omega_N)} = Ch \sum_{i=1}^{n-1} |R_i|. \tag{29}$$

**Proof**. According to maximum principle, we have

$$|z_i| \le \eta_i, \quad i = 1, 2, \cdots, N - 1,$$

where the function $\eta_i$ is the solution of the following difference problem.

$$\varepsilon\theta_i\eta_{\bar{x}x,i} + a_i\eta_{x,i} = -|R_i|, \tag{30}$$

$$\eta_0 = 0, \ \eta_N = 0. \tag{31}$$

Let $\eta_{x,i} = V_i$. Then we rewrite the relation (30) as

$$\varepsilon\theta_i V_{\bar{x},i} + a_i\frac{V_i+V_{i-1}}{2} = -|R_i|.$$

Solving the first order difference with respect to $V_i$ for $1 \le i \le N-1$, we obtain

$$V_i = V_0\prod_{k=1}^{i}\left(\frac{1-\dfrac{ha_k}{2\varepsilon\theta_k}}{1+\dfrac{ha_k}{2\varepsilon\theta_k}}\right) - h\sum_{l=1}^{i}\frac{|R_i|}{\varepsilon\theta_l+\dfrac{ha_i}{2}}. \tag{32}$$

Using the inequality $1-x \le e^{-x}$ and $\dfrac{a_k\rho}{2\theta_k} = \cot h\left(\dfrac{a_k\rho}{2}\right) \le 1$ we get

$$|V_i| = |V_0|\exp\left(-h\sum_{k=1}^{i}\frac{\dfrac{a_k}{\varepsilon\theta_k}}{1+\dfrac{a_k}{2\varepsilon\theta_k}}\right) + \sum_{l=1}^{i}\frac{|R_i|}{\varepsilon\theta_l+\dfrac{ha_i}{2}}\times\left(-h\sum_{j=l+1}^{i}\frac{a_j/\varepsilon\theta_j}{\varepsilon\theta_l+\dfrac{ha_l}{2}}\right). \tag{33}$$

For the second term in the relation (33), we found

$$\max_{1\le l\le N-1}\frac{1}{\varepsilon\theta_l+\dfrac{ha_l}{2}}\exp\left(-h\sum_{j=l+1}^{i}\frac{\dfrac{a_j}{\varepsilon\theta_j}}{1+\dfrac{ha_j}{2\varepsilon\theta_k}}\right) \le C.$$

From this, the second term in the relation (33) is obtained as follows.

$$h\sum_{l=1}^{i}\frac{|R_l|}{\varepsilon\theta_l+\dfrac{a_l h}{2}}\exp\left(-h\sum_{j=l+1}^{i}\frac{a_j/\varepsilon\theta_j}{\varepsilon\theta_l+ha_l/2}\right) \le Ch\sum_{l=1}^{i}|R_l|,$$

and

$$V_0 = \frac{h\sum_{i=0}^{N-1}\left\{h\sum_{l=1}^{i}\dfrac{|R_l|}{\varepsilon\theta_l+\dfrac{a_l h}{2}}\prod_{j=l+1}^{i}\left(\dfrac{1-a_j h/2\varepsilon\theta_j}{1+a_j h/2\varepsilon\theta_j}\right)\right\}}{h\sum_{i=0}^{N-1}\left\{\prod_{k=1}^{i}\left(\dfrac{ha_k/\varepsilon\theta_k}{1+ha_k/2\varepsilon\theta_k}\right)\right\}}. \tag{34}$$

By setting the relation (34) in (32), we take

$$\left| V_0 \prod_{k=1}^{i} \left( \frac{1 - \dfrac{a_k h}{2\varepsilon\theta_k}}{1 + \dfrac{a_k h}{2\varepsilon\theta_k}} \right) \right| \le Ch\sum_{i=1}^{n-1} |R_i|. \tag{35}$$

By using the relation (33) and (35) in (32) we obtain

$$|v_i| \le Ch\sum_{i=1}^{n-1} |R_i|. \tag{36}$$

So, we have

$$|z_i| \le Ch\sum_{i=1}^{n-1} |R_i|.$$

The Lemma 4.2 is proven.

We now give the following uniform convergence result.

**Theorem 4.3**. Suppose that $a, b, f \in C^1[0, l]$. Let $u(x)$ be the solution of the problem (1)-(2) and $y$ be the solution of the problem (23)-(24). Uniform error estimate

$$\| y - u \|_{C(\varpi_N)} \le Ch,$$

holds.

**Proof.** This follows immediately by combining the two previous lemmas.

## 5. NUMERICAL RESULTS

Here we test two examples for applying of the theoretical results.

**Example 5.1.** Our first test problem is as follows [27].

$$\varepsilon u''(x) + \left(1 - \frac{x}{2}\right) u'(x) - 0.5u(x) = 0, \quad x \in (0,1), \tag{37}$$

$$u(0) = 0, \quad u(1) = 1. \tag{38}$$

Its exact solution is given by

$$u(x) = (2 - x)^{-1} - 0.5\exp\left(-\frac{x - \dfrac{x^2}{4}}{\varepsilon}\right).$$

We describe the maximum errors $e_\varepsilon^N$ and the errors $e^N$ as follows:

$$e_\varepsilon^N = \max_{1 < i < N} |u_i^\varepsilon - y_i^\varepsilon|, \qquad e^N = \max_\varepsilon e_\varepsilon^N.$$

We also describe the following convergence rates and the computed parameter uniform convergence rates, respectively:

$$p_\varepsilon^N = \frac{\ln \dfrac{e_\varepsilon^N}{e_\varepsilon^{2N}}}{\ln 2}, \quad p^N = \frac{\ln \dfrac{e^N}{e^{2N}}}{\ln 2}.$$

**Table 1.** *The calculated maximum errors and the orders of convergence for the numerical solution of problem (37)-(38)*

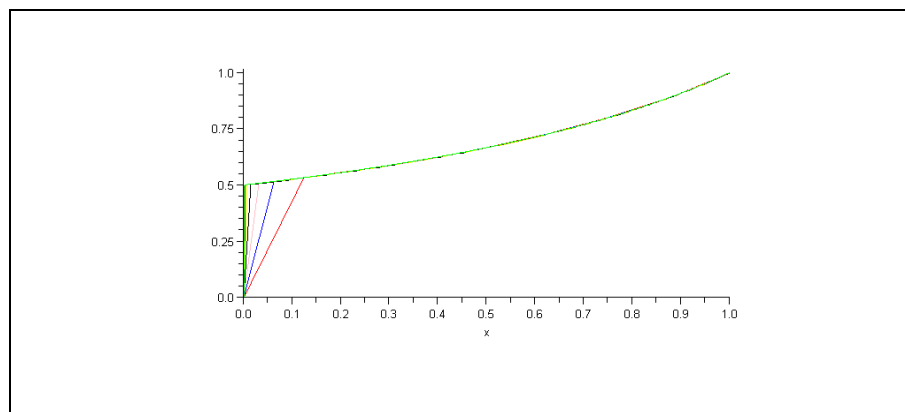| $\varepsilon/N$ | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
|---|---|---|---|---|---|---|---|
| $10^{-3}$ | 0.0291666 | 0.0151209 | 0.0076884 | 0.0038760 | 0.0019725 | 0.0011393 | 0.0006604 |
|  | 0.94 | 0.97 | 0.98 | 0.97 | 0.79 | 0.78 |  |
| $10^{-4}$ | 0.0291666 | 0.0151206 | 0.0076884 | 0.0038754 | 0.0019454 | 0.0010393 | 0.0006304 |
|  | 0.94 | 0.97 | 0.98 | 0.99 | 0.90 | 0.85 |  |
| $10^{-5}$ | 0.0291665 | 0.0151206 | 0.0076884 | 0.0038774 | 0.0019456 | 0.0009754 | 0.0004878 |
|  | 0.95 | 0.98 | 0.99 | 0.99 | 0.99 | 0.99 |  |
| $10^{-6}$ | 0.0291543 | 0.0151203 | 0.0076884 | 0.0038756 | 0.0019123 | 0.0009798 | 0.0004878 |
|  | 95 | 0.98 | 0.99 | 1.00 | 0.97 | 1.00 |  |
| $10^{-7}$ | 0.0290543 | 0.0151204 | 0.0076882 | 0.0038742 | 0.0019419 | 0.0009772 | 0.0004878 |
|  | 0.95 | 0.98 | 0.99 | 0.99 | 0.99 | 1.00 |  |
| $10^{-8}$ | 0.0290342 | 0.0151202 | 0.0076884 | 0.0038750 | 0.0019312 | 0.0009623 | 0.0004878 |
|  | 0.94 | 0.98 | 0.99 | 1.00 | 1.00 | 0.98 |  |
| $10^{-9}$ | 0.0290540 | 0.0151200 | 0.0076884 | 0.0038751 | 0.0019134 | 0.0009601 | 0.0004878 |
|  | 0.94 | 0.98 | 0.99 | 1.01 | 0.99 | 0.98 |  |
| $10^{-10}$ | 0.0290342 | 0.0151201 | 0.0076884 | 0.0038705 | 0.0019017 | 0.0009512 | 0.000487 |
|  | 0.94 | 0.98 | 0.99 | 1.01 | 0.99 | 0.96 |  |



**Figure 1.** *Numerical solution of problem (37)-(38) for different values of $\varepsilon$ and $N$ on the uniform mesh*
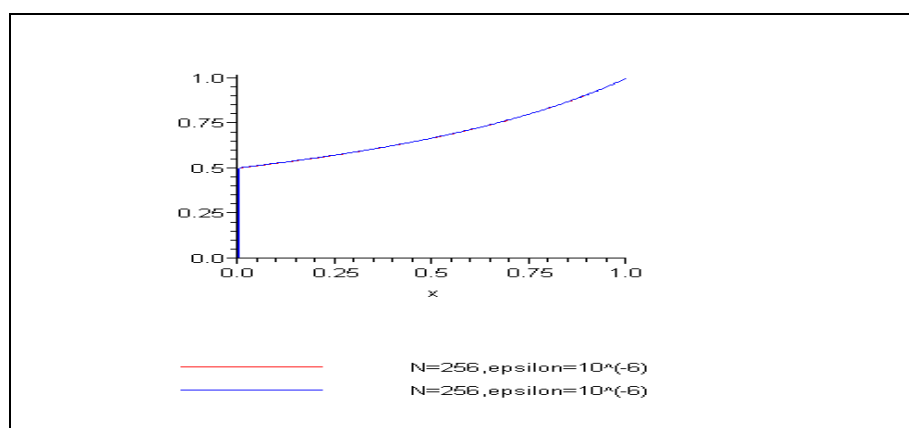


**Figure 2.** *Exact solution and Numerical solution of problem (37)-(38) for varied values of $\varepsilon$ and $N$ on uniform grid*
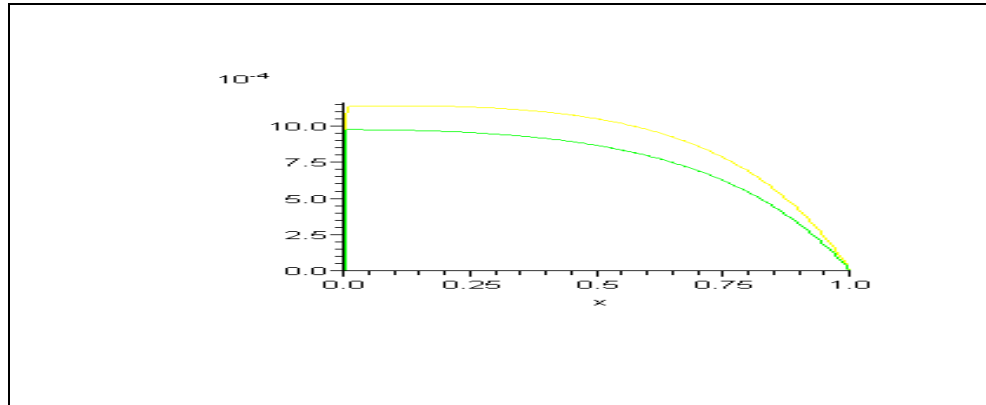
**Figure 3.** *Errors of problem (37)-(38) for values* $\varepsilon = 10^{-4}$ , $10^{-6}$ , $10^{-8}$ , $10^{-10}$ *and* $N = 256$ *on the uniform grid*

**Example 5.2.** We secondly present the following example [27].

$$\varepsilon u^{''}(x) + u^{'}(x) - 1 - 2x = 0, \quad 0 < x < 1, \tag{39}$$

$$u(0) = 0,\ u(1) = 1. \tag{40}$$

Exact solution of example above is as follows.

$$u(x) = x^2 + x - 2x\varepsilon + \frac{(2\varepsilon - 1)\left(1 - \exp\left(-\dfrac{x}{\varepsilon}\right)\right)}{1 - \exp\left(-\dfrac{1}{\varepsilon}\right)} .$$

**Table 2**. *The calculated maximum errors and the orders of convergence for the numerical solution of problem (39)-(40)*

| $\varepsilon/N$ | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
|---|---|---|---|---|---|---|---|
| $10^{-4}$ | 0.1092000 0.90 | 0.0584062 0.95 | 0.0300796 0.98 | 0.0151839 1.00 | 0.0075530 1.03 | 0.0036917 1.07 | 0.0017497 |
| $10^{-5}$ | 0.1093575 0.90 | 0.0585750 0.95 | 0.0302541 0.97 | 0.0153611 099 | 0.0077316 099 | 0.0038710 1.00 | 0.0019293 |
| $10^{-6}$ | 0.1093732 0.90 | 0.0585918 0.95 | 0.0302715 0.97 | 0.0153788 0.98 | 0.0077494 0.99 | 0.0038889 0.99 | 0.0019473 |
| $10^{-7}$ | 0.1093748 090 | 0.0585935 0.95 | 0.0302732 0.97 | 0.0153806 0.98 | 0.0077512 0.99 | 0.0038907 0.99 | 0.0019491 |
| $10^{-8}$ | 0.1093749 0.90 | 0.0585937 0.95 | 0.0302734 0.98 | 0.0153808 0.99 | 0.0077513 0.99 | 0.0038909 1.00 | 0.0019499 |
| $10^{-9}$ | 0.1093749 0.90 | 0.0585937 0.95 | 0.0302734 0.98 | 0.0153808 0.99 | 0.0077514 0.99 | 0.0038909 1.00 | 0.0019493 |
| $10^{-10}$ | 0.1093749 0.89 | 0.0585937 0.95 | 0.0302734 0.98 | 0.0153808 0.99 | 0.0077514 0.99 | 0.0038909 1.00 | 0.0019491 |
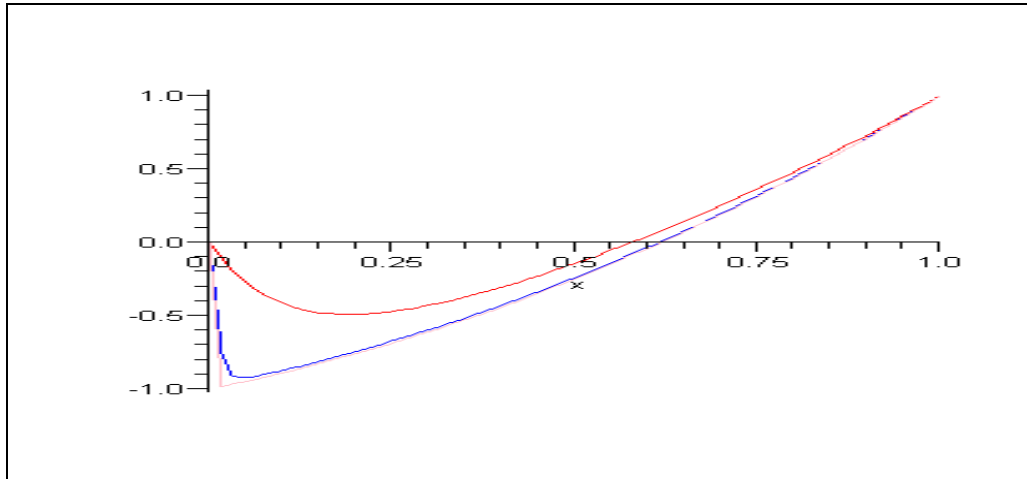
**Figure 4.** *Numerical solution of problem (37)-(38) for values* $\varepsilon = 10^{-1}$, $10^{-5}$, $10^{-9}$ *and* $N = 64$ *on the uniform mesh*
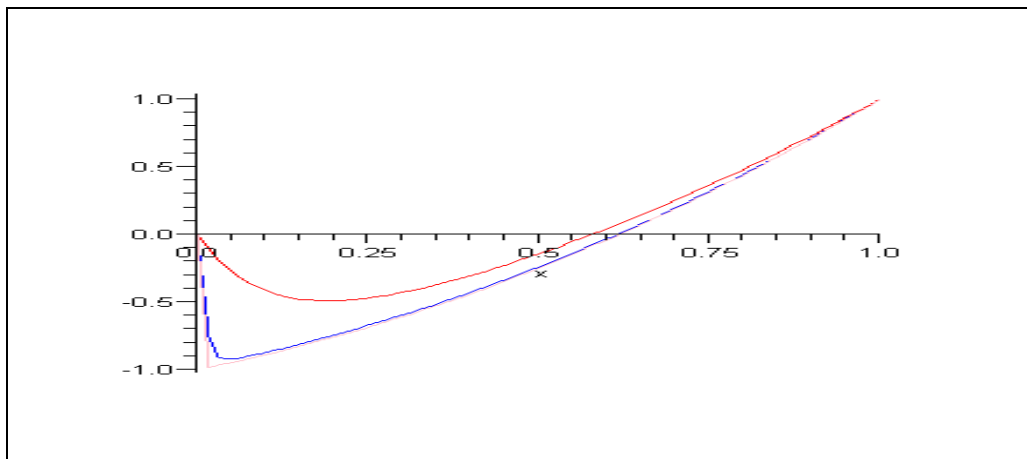


**Figure 5.** *Exact solution and Numerical solution of problem (39)-(40) for values of* $\varepsilon = 10^{-1}$, $10^{-5}$, $10^{-9}$ *and* $N = 64$ *on the uniform mesh*
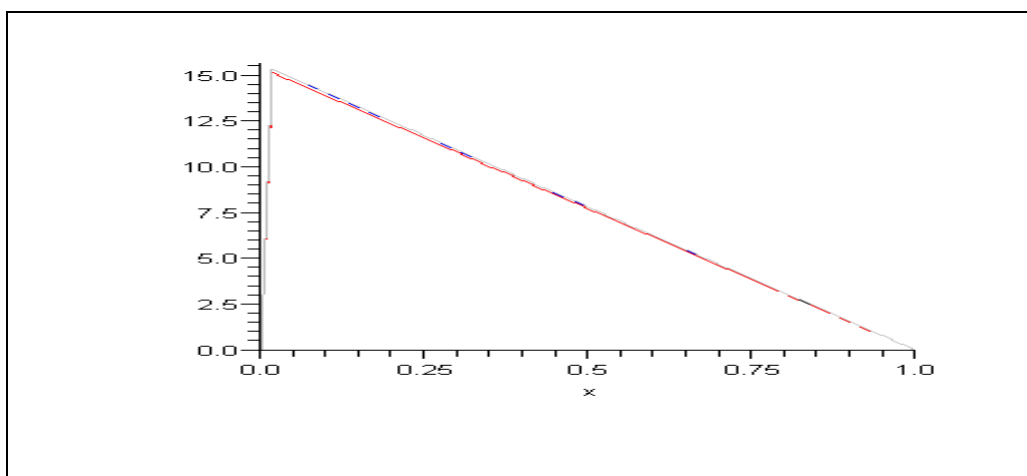


**Figure 6.** *Errors of problem (39)-(40) for values of* $\varepsilon = 10^{-4}$, $10^{-6}$, $10^{-8}$, $10^{-10}$ *and* $N = 64$ *on the uniform mesh*

According to above tables and figures, the curves of exact and approximate solutions are compared in Figures 2 and 5. In Figures 1 and 4, as $\varepsilon$ values decrease, the curves go towards the coordinate axes around $x = 0$. In Figure 3 and 6, the errors in the boundary layer region are maximum for different values $\varepsilon$. The values of $\varepsilon$ and $N$ for which we solve the test problems are $\varepsilon = 2^{-k}$, $k = 4,..,10$; $N = 2^k$, $k = 3,..,9$. The resulting error $e^N$ and the uniform convergence rates $p^N$ are given in Tables 1 and 2.

## 6. CONCLUSION

In this study, we have offered an effective finite difference method for solving second-order linear singularly perturbed boundary value problem. It is given that this method displays uniform convergence with respect to the perturbation parameter $\varepsilon$. We apply two examples to show the advantages and effectiveness of this novel approach. The algorithm, figures and tables are generated in Maple. In table and graphics, when $N$ takes increasing values, it is seen that the convergence rate of the smooth convergence speed $p^N$ is first order. The curves of exact solutions and approximate solutions are almost identical as shown in Figures 2 and 5. In Figures 1 and 4, as ε values decrease, the graph approaches more towards the coordinate axes in the boundary layer region around $x = 0$. In Figure 3 and 6, the errors in these regions are maximum because of the irregularity caused by the sudden and rapid change of solution in the boundary layer region around $x = 0$ for different values ε. As a result, the proposed scheme has worked very well as numerical results show. This method can also be used for solving other more different problems.

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## CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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