

## LEAST SQUARES MULTI-WINDOW EVOLUTIONARY SPECTRAL ESTIMATION

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### ABSTRACT

*We present a multi-window method for obtaining the time-frequency spectrum of non-stationary signals such as speech and music. This method is based on optimal combination of evolutionary spectra that are calculated by using multi-window Gabor expansion. The optimal weights are obtained by using a least square estimation method. An error criterion that is the squared distance between a reference time-frequency distribution and the combination of evolutionary spectra is minimized to determine the weights. Examples are given to illustrate the effectiveness of the proposed method.*

**Keywords:** Time-frequency analysis, Evolutionary spectrum, Multi-window time-frequency analysis.

### I. INTRODUCTION

Time-Frequency (TF) signal analysis is a helpful tool for analyzing the time-varying frequency content of a non-stationary signals such as speech, music, biological signals etc. [1]. For a time-dependent spectral analysis of non-stationary stochastic process, the Wigner-Ville Spectrum (WVS) [2] is given by :

$$S(t, \omega) = E\{W(t, \omega)\} \\ = E\left\{\int_{-\infty}^{\infty} \left[x\left(t - \frac{\tau}{2}\right)x^*\left(t + \frac{\tau}{2}\right)\right] e^{-j\omega\tau} d\tau\right\}$$

where  $W(t, \omega)$  denotes the Wigner Distribution (WD) and the above is the statistical average of

the WDs of the realizations of the process. When we have several observations of the non-stationary process  $x(t)$ , we can use an ensemble average of the individual WDs of these observations to estimate the WVS. However, this is not the case in general; we are only given a single realization of the process. In that case, Time-Frequency Distributions (TFDs) with a smoothing kernel function is used to estimate the WVS [1]. A good amount of research has been done to design kernels with desired properties yielding unbiased and low variance WVS estimates [2,3]. A new estimate of the WVS is proposed as an optimal

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average of multiple-window spectrograms of the process in [4,5] in the least squares sense. In this work we extend this WVS estimate to the weighted average combination of multi-window evolutionary spectra obtained by a Discrete Evolutionary Transform (DET) [6]. The combination weights are determined by minimizing a sum-squared difference (norm squared distance) between the average evolutionary spectrum and a higher order time-frequency representation.

There is a growing interest on higher order time-frequency methods. One example is the class of Time-Varying Higher Order Spectra (TV-HOS) based on Polynomial Wigner-Ville Distributions (PWVD) [7,8,9]. Higher order TF methods are useful in the analysis of non-linear, non-Gaussian signals. Several methods have been presented to estimate a time-varying spectrum using higher order statistics. In [8], it has been shown that PWVD can achieve the delta function concentration for polynomial FM signals (that is signals with the instantaneous phase modelled by a polynomial of possibly order higher than two). TV-HOS have been recently developed in a search for a tool that could perform higher-order spectral analysis of non-stationary random signals. As a general tool for the analysis of non-stationary, non-linear, non-Gaussian signals [10] TV-HOS has grown as a set of hybrid techniques that extends both TF analysis and higher-order spectra (HOS). For a random non-stationary signal  $x(t)$ , the TV-HOS is defined as the expected value of the PWVD by

$$S^{(k)}(t, \omega) = E\{W^{(k)}(t, \omega)\} \quad (1)$$

If we interchange expectation operator  $E$  with integration, TV-HOS becomes

$$S^{(k)}(t, \omega) = \int_{-\infty}^{\infty} \left\{ E \prod_{l=1}^{k/2} x(t + h_l \tau) x^*(t + h_{-l} \tau) \right\} \times e^{-j\omega \tau} d\tau \quad (2)$$

The selection of the  $hl$  coefficients is explained in detail in [9]. The fourth-order member of TVHOS, called the moment Wigner-Ville Trispectrum (M-WVT) [9] and defined as follows:

$$MWVT^{(4)} = \int m_x^{(4)}(t, \tau) e^{-j\omega \tau} d\tau \quad (3)$$

where the fourth-order moment is

$$m_x^{(4)}(t, \tau) = E \left\{ \left[ x\left(t + \frac{\tau}{4}\right) \right]^2 \left[ x^*\left(t - \frac{\tau}{4}\right) \right]^2 \right\}$$

A new estimate of the WVS is proposed as the optimal average of multiple-window spectrograms of the process in [5, 6]. In this work we extend this WVS estimate to the optimal combination of evolutionary spectra obtained by a Discrete Evolutionary Transform (DET) [6]. We present a least-squares, multi-window evolutionary spectral estimation method. The optimal combination coefficients are obtained by minimizing the squared error between a reference TFD and the multi-window spectral estimate.

## 2. THE DISCRETE EVOLUTIONARY TRANSFORM

Given a non-stationary signal,  $x(n)$ ,  $0 \leq n \leq N-1$ , a discrete-time, discrete-frequency spectral representation [11] for it is given by

$$x(n) = \sum_{k=0}^{K-1} X(n, \omega_k) e^{j\omega_k n}, \quad (4)$$

where  $\omega_k = 2\pi k / K$ ,  $K$  is the number of frequency samples, and  $X(n, \omega_k)$  is an evolutionary kernel. The evolutionary spectrum is obtained from this kernel as  $S(n, \omega_k) = |X(n, \omega_k)|^2$ . The sinusoidal Discrete Evolutionary Transformation (DET) is obtained by expressing the kernel in terms of the signal. This is done by using conventional representations such as the Gabor and the Malvar transforms. Thus, for the sinusoidal representation in (4) the DET that provides the evolutionary kernel  $X(n, \omega_k)$ ,  $0 \leq k \leq K-1$ , is given by [6]

$$X(n, \omega_k) = \sum_{\ell=0}^{N-1} x(\ell) W_k(n, \ell) e^{-j\omega_k \ell} \quad (5)$$

where  $W_k(n, \ell)$  is, in general, a time and frequency dependent window. The DET can be

seen as a generalization of the short-time Fourier transform [6], where the windows are constant. The windows  $W_k(n, \ell)$  can be obtained from either the Gabor representation that uses non orthogonal bases, or the Malvar wavelet representation that uses orthogonal bases. Details of how the windows can be obtained for the Gabor and Malvar representations are given in [6]. For example, the multi-window Gabor expansion is given by [11]

$$x(n) = \frac{1}{I} \sum_{i=0}^{I-1} \sum_{m=0}^{M-1} \sum_{k=0}^{K-1} a_{i,m,k} h_i(n-mL) e^{j\omega_k n} \tag{6}$$

$$= \frac{1}{I} \sum_{i=0}^{I-1} \sum_{k=0}^{K-1} X_i(n, \omega_k) e^{j\omega_k n}$$

where  $\{a_{i,m,k}\}$  are the Gabor coefficients,  $\{h_{i,m,k}\}$  are the Gabor basis functions that are obtained by scaling, translating and modulating with a sinusoid a window function:

$$h_{i,m,k}(n) = h_i(n-mL) e^{j\omega_k n} \tag{7}$$

and the synthesis window  $h_i(n)$  is obtained any scaling a unit-energy mother window  $g(n)$  as

$$h_i(n) = 2^{i/2} g(2^i n), \quad i = 0, 1, \dots, I-1$$

The multi-window Gabor coefficients are evaluated by

$$a_{i,m,k} = \sum_{n=0}^{N-1} x(n) \gamma_i^*(n-mL) e^{-j\omega_k n}, \tag{8}$$

where the analysis window  $\gamma_i(n)$  is solved from the bi-orthogonality condition between  $h_i(n)$  and  $\gamma_i(n)$  [11]. The evolutionary kernel is obtained by comparing the spectral and the Gabor representations of the signal (6):

$$X_i(n, \omega_k) = \sum_{m=0}^{M-1} a_{i,m,k} h_i(n-mL) \tag{9}$$

Replacing for the coefficients  $\{a_{i,m,k}\}$ , one can also write

$$X_i(n, \omega_k) = \sum_{\ell=0}^{N-1} x(\ell) W_i(n, \ell) e^{-j\omega_k \ell}, \tag{10}$$

where the time-varying window for scale  $2^i$  is defined as

$$W_i(n, \ell) = \sum_{m=0}^{M-1} \gamma_i^*(\ell-mL) h_i(n-mL).$$

Then the evolutionary spectrum of  $x(n)$  calculated by the window  $W_i(n, \ell)$  is

$$S_i(n, \omega_k) = \frac{1}{K} |X_i(n, \omega_k)|^2,$$

where the factor  $1/K$  is used for proper energy normalization. We should mention that normalizing the  $W_i(n, \ell)$  to unit energy, the total energy of the signal is preserved thus justifying the use of  $S_i(n, \omega_k)$  as a TF representation for  $x(n)$ . Furthermore,  $S_i(n, \omega_k)$  is always non-negative and approximates the marginal conditions [1]; hence, in contrast to many TFDs, interpretable as TF energy density function [11].

### 3. LEAST SQUARES EVOLUTIONARY SPECTRUM

Given a realization of a discrete-time, nonstationary process corrupted by additive noise  $x(n) = s(n) + \eta(n)$  where  $s(n)$  and  $\eta(n)$  denotes the signal and noise processes respectively. We intend to obtain a high resolution evolutionary spectral estimate with good performance in low signal to noise ratio (SNR) conditions. We calculate a weighted average combination of evolutionary spectra  $S_i(n, \omega_k)$  that is closest to a reference TFD in a least squares sense. Given the signal  $x(n)$ , we calculate evolutionary spectra  $S_i(n, \omega_k)$  for  $i=0, 1, \dots, I-1$  as

$$S_i(n, \omega_k) = \frac{1}{K} \left| \sum_{\ell=0}^{N-1} x(\ell) W_i(n, \ell) e^{-j\omega_k \ell} \right|^2. \tag{11}$$

Gauss windows are used as  $h_i(n)$ , for their optimal concentration in the TF plane [12]. Then we estimate the WVS of the process  $x(n)$  as a weighted average of the evolutionary spectra

$$\hat{P}(n, \omega_k) = \sum_{i=0}^{I-1} c_i S_i(n, \omega_k) \tag{12}$$

where the weights  $\{c_i\}$  are obtained by minimizing the error function

$$\varepsilon_i = \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \left| P_R(n, \omega_k) - \sum_{i=0}^{I-1} c_i S_i(n, \omega_k) \right|^2 \quad (13)$$

and  $P_R(n, \omega_k)$  is a reference TFD which is taken here as higher order TF representation [7] of the signal.

By using a matrix notation, the minimization problem in (13) can be rewritten as

$$\min_{c_i} \|\mathbf{P}_R - \mathbf{S}\mathbf{c}\|^2 \quad (14)$$

The solution of this least squares minimization problem is

$$\mathbf{c}^o = (\mathbf{S}^T \mathbf{S})^{-1} \mathbf{S}^T \mathbf{P}_R$$

where the superscript ‘ $o$ ’ stands for optimum. Then a WVS estimate is obtained as optimal weighted average using  $\{c_i^o\}$  as

$$\hat{P}_{ES}(n, \omega_k) = \sum_{i=0}^{I-1} c_i S_i(n, \omega_k). \quad (15)$$

Finally, we mask or threshold our estimate to eliminate any possible negative values as in [5], and result in a non-negative time-varying spectrum, i.e.,

$$\hat{P}_{ES}(n, \omega_k)^+ = \begin{cases} \hat{P}(n, \omega_k), & \hat{P}(n, \omega_k) \geq 0; \\ 0, & \hat{P}(n, \omega_k) < 0. \end{cases} \quad (16)$$

where  $\hat{P}_{ES}(n, \omega_k)^+$  denotes the positive-only part of the evolutionary spectrum.

## 4. EXPERIMENTAL RESULTS

**Example 1.** First we consider a single component signal, which is a linear chirp. We give the Wigner-Ville distribution of this chirp in Fig. 1 and the least squares evolutionary spectrum in Fig. 2. As shown from the figures least squares evolutionary spectrum gives all positive values and a satisfactory TF resolution.

**Example 2.** As a multi-component example, we consider a signal that is composed of two chirps

with different chirp rates. In Fig. 3, we show the 4<sup>th</sup> order TV-HOS or Moment Wigner-Ville Trispectrum [7]. This higher order TFD has the advantage of high concentration, however, it has negative values and interference terms between the two auto components. The above TV-HOS is used as reference TFD in our method, and a least squares evolutionary spectral estimate  $\hat{P}_{ES}(n, \omega_k)^+$  is obtained by  $I = 4$  windows and given in Fig. 4. As shown, it is always positive, displays both components with sufficient resolution, and carries out the advantages of both reference TFD and the multi-window evolutionary spectral estimates. We also compare the performance of our method with multi-window evolutionary spectrum [11] explained in section 2, for same  $I = 4$  windows, and the result is shown in Fig. 5. In Fig. 6, we give Cakrak and Loughlin’s least-squares combination of multiple-window spectrograms [5] with the same 4 windows. As shown in figures, our proposed algorithm provides better localization than these two methods.

**Example 3.** We also analyze a crossing-chirp signal. The reference TFD (Moment Wigner-Ville Trispectrum) is given in Fig. 7. Using this reference TFD, our least squares evolutionary spectrum for  $I = 5$  windows is obtained and given in Fig. 8. The proposed least squares combination provides spectral estimates with sufficient resolution and no interference terms.

**Example 4.** Finally our proposed method is tested on a sinusoidal FM signal. The Wigner-Ville trispectrum which we use as a reference TFD is given in Fig. 9. Notice that the Wigner-Ville trispectrum provides very good TF resolution, but gives negative values in the density. The result of Least Squares evolutionary spectrum is shown in Fig. 10 obtained by  $I=3$  windows yielding positive and high resolution TF spectrum.

## 5. CONCLUSIONS

In this work, we present a new method for estimating the evolutionary spectrum of non-stationary signals using a linear least squares approach. Our method is based on optimal combination of evolutionary spectra that are calculated by using DET. The optimal weights

are obtained by minimizing the squared error between the combination of evolutionary spectra and a reference TFD which is taken here as one of the higher order TFDs. Examples show that the new method combines the advantages of multiple-window evolutionary spectral analysis and higher order TFDs, i.e., it provides non-negative and high resolution time varying spectral estimates with no cross-terms.

## REFERENCES

- [1] Cohen, L., *Time-Frequency Analysis*. Prentice Hall, Englewood Cliffs, NJ, 1995.
- [2] Martin, W., and Flandrin, P., "Wigner-Ville spectral analysis of nonstationary processes," *IEEE Trans. on ASSP*, Vol. 33, No. 6, pp. 1461–1470, Dec. 1985.
- [3] Jones, D.L. and Baraniuk, R.G., "An Adaptive Optimal-Kernel Time-Frequency Representation," *IEEE Trans. on Signal Process.*, Vol. 43, No. 10, pp. 2361–2371, May 1995.
- [4] Bayram, M., and Baraniuk, R.G., "Multiple Window Time-Frequency Analysis," *IEEE SP Intl. Symposium on Time-Frequency and Time-Scale Analysis, TFTS'96*, Paris, France, pp. 173–176, Jun. 1996.
- [5] Cakrak, F., and Loughlin, P., "Multiwindow Time-Varying Spectrum with Instantaneous Bandwidth and Frequency Constraints," *IEEE Trans. on Signal Process.*, Vol. 49, pp. 1656–1666, Aug. 2001.
- [6] Suleesathira, R., Akan, A. and Chaparro, L. F., "Discrete Evolutionary Transform for Time-Frequency Signal Analysis," *J. Franklin Institute, Special Issue on Time-Frequency Signal Analysis and its Applications*, pp. 347–364, Vol. 337, No. 4, Jul. 2000.
- [7] Boashash, B., and Ristic, B., "Polynomial Time-Frequency Distributions and Time-Varying Higher Order Spectra," *Signal Processing*, Vol. 67, pp. 1–23, 1998.
- [8] Boashash, B., and O'Shea, P.J., "Polynomial Wigner-Ville distributions and their relationship to time-varying higher order spectra," *IEEE Trans. Signal Processing* Vol. 42 No. 1, pp. 216–220, January 1994.
- [9] Boashash, B., and Ristic, B., "Analysis of FM signals affected by Gaussian AM using the reduced Wigner-Ville trispectrum," *Proc. Int. Conf. Acoustic, Speech and Signal Processing (ICASSP)*, pp. IV-408–411, Minneapolis, Minnesota, April 1993.
- [10] Boashash, B., and Ristic, B., "A timefrequency perspective of higher order spectra as a tool for non-stationary signal analysis," B. Boashash, E. J. Powers, A. M. Zoubir (Eds.), *Higher Order Statistical Signal Processing*, Chapter 4, Longman, New York, 1996.
- [11] Akan, A., and Chaparro, L.F., "Multi-window Gabor Expansion for Evolutionary Spectral Analysis," *Signal Processing*, Vol. 63, pp. 249–262, Dec. 1997.
- [12] Balian, R., "Un principe d'incertitude fort en theorie du signal on en mecanique quantique," *Comptes Rendus de l'Academie des Sciences*, Paris, Vol. 292, Serie II, pp. 1357–1362, Jun. 1981.