ISTANBUL UNIVERSITY ENGINEERING FACULTY
JOURNAL OF ELECTRICAL & ELECTRONICS
YEAR : 2002 (483-487)
VOLUME : 2
NUMBER : 2

A DISCRETE FRACTIONAL GABOR EXPANSION FOR TIME-- FREQUENCY SIGNAL ANALYSIS

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ABSTRACT

In this work, we present a discrete fractional Gabor representation on a general, non-rectangular time-frequency lattice. The traditional Gabor expansion represents a signal in terms of time and frequency shifted basis functions, called Gabor logons. This constant-bandwidth analysis uses a fixed, and rectangular time-frequency plane tiling. Many of the practical signals require a more flexible, non-rectangular time-frequency lattice for a compact representation. The proposed fractional Gabor method uses a set of basis functions that are related to the fractional Fourier basis and generate a non-rectangular tiling. Simulation results are presented to illustrate the performance of our method.

Keywords: *Spread Time-frequency analysis, Gabor expansion Fractional Fourier Transform.*

1. INTRODUCTION

Time-frequency (TF) analysis provides a characterization of signals in terms of joint time and frequency content [1]. One of the fundamental issues in the TF analysis is obtaining the distribution of signal energy over joint TF plane with a delta function concentration [1]. The discrete Gabor expansion is a TF signal decomposition which represents a signal in terms of time and frequency translated basis functions called TF atoms [2],[3]. Gabor basis functions $g_{m,k}(n)$ are obtained by shifting and modulating with a sinusoid a single window function $g(n)$, which results in a fixed and rectangular TF plane tiling. However, if the signal to be represented is not modeled well by this constant-bandwidth analysis, its Gabor representation displays poor TF localization [4],[5],[6]. Many of the practical signals such as speech, music, biological, and seismic signals have time-varying frequency nature that is not appropriate for sinusoidal analysis [4],[6]. Thus

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Received Date : 29.12.2001
Accepted Date: 28.5.2002

the traditional Gabor expansion of such signals will require large number of coefficients yielding a poor TF localization. The compactness of the Gabor representation is improved if the basis functions match the time-varying frequency behavior of the signal [6],[7],[8]. Here we present a new, fractional Gabor expansion that uses a more flexible, non-rectangular TF lattice. The basis functions of the proposed expansion are related to the fractional Fourier basis.

2. THE DISCRETE GABOR EXPANSION

The traditional Gabor expansion [2],[3] represents a signal in terms of time and frequency shifted basis functions, and has been used in various applications to analyze the time-varying frequency content of a signal [9]. Basis functions of the Gabor representation are obtained by translating and modulating with sinusoids a single window function. The discrete Gabor expansion of a finite-support signal $x(n), 0 \le n \le N-1$ is given by [3]

$$x(n) = \sum_{m=0}^{M-1} \sum_{k=0}^{K-1} c_{m,k}\, \tilde{g}_{m,k}(n) \qquad (1)$$

where the basis function

$$\tilde{g}_{m,k}(n) = \tilde{g}(n - mL) e^{jw_k n} \qquad (2)$$

and $w_k = 2pkL' / N$. The Gabor expansion parameters *M,K,L, and L'* are positive integers constrained by *ML=KL'=N* where *M* and *K* are the number of analysis samples in time and frequency, respectively, and *L* and *L'* are the time and frequency steps, respectively. Existence, uniqueness and numerical stability of the representation depend on the choice of parameters *L* and *L'*. For numerically stable representations, *L* and *L'* must satisfy *L L' N*, or equivalently that *L K*. The case where *L=K*, is called the critical sampling, and the case *L < K* is called the over-sampling. The synthesis window $\tilde{g}(n)$ is a periodic extension (by *N*) of $g(n)$ which is normalized to unit energy for definiteness [3].

In general, the set of time and frequency shifted window functions, i.e., Gabor logons, $\{\tilde{g}_{m,k}(n)\}$ forms a non--orthogonal basis for the square--summable sequences space $\ell^2(\Re)$. Hence the calculation of the Gabor coefficients is not a simple task since projection by the usual inner product cannot be used. One of the methods [3], uses an auxiliary function $g(n)$ called the biorthogonal window or dual function of. $g(n)$ Then the Gabor coefficients $\{c_{m,k}\}$ can be evaluated by

$$c_{m,k} = \sum_{n=0}^{N-1} x(n) \tilde{g}^*_{m,k}(n) \qquad (3)$$

where the analysis functions are

$$\tilde{g}_{m,k}(n) = \tilde{g}(n - mL) e^{jw_k n} \qquad (4)$$

where again $\tilde{g}(n)$ is a periodic version of the dual window $g(n)$. Completeness condition of the basis set is obtained by substituting (3) into (1) to get that

$$\sum_{m=0}^{M-1} \sum_{k=0}^{K-1} \tilde{g}_{m,k}(n) \tilde{g}^*_{m,k}(\ell) = d(n - \ell) \qquad (5)$$

where $d(.)$ denotes the Dirac delta function. The above completeness relation yields equivalent but simpler bi- orthogonality condition between the analysis and synthesis basis sets via the discrete Poisson-sum formula [3]:

$$\sum_{n=0}^{N-1} \tilde{g}(n + mK) e^{-j\frac{2p}{L}kn} \tilde{g}^*(n) = \frac{L}{K} d_m d_k \qquad (6)$$

for *0 m L'-1 , 0 k L-1*. The analysis window $g(n)$ is obtained by solving the equation system of the above biorthogonality condition.

Gabor analysis basis $\{\tilde{g}_{m,k}(n)\}$ with a fixed window and sinusoidal modulation tiles the time--frequency plane in a rectangular fashion causing a constant bandwidth analysis. Constant bandwidth methods, such as spectrogram [1] and the Gabor expansion provide signal representations with poor time-frequency resolution [4]. Recently, representations on a non-rectangular TF grid has attracted a considerable attention [6],[10]. A non--rectangular lattice is more appropriate for the TF analysis of signals with time-varying frequency content. Thus the motivation for a fractional Gabor analysis.

3. A FRACTIONAL GABOR EXPANSION

We define a discrete fractional Gabor expansion for $x(n), 0 \le n \le N-1$, as follows:

$$x(n) = \sum_{m=0}^{M-1}\sum_{k=0}^{K-1} c_{m,k,a}\, \tilde{g}_{m,k,a}(n) \qquad (7)$$

where $c_{m,k,á}$ are the fractional Gabor coefficients, *á* is the order of the fraction, and the basis functions are

$$\tilde{g}_{m,k,a}(n) = \tilde{g}(n - mL) W_{a,k}(n)$$

Here $\tilde{g}(n)$ is a periodic version of a unit energy Gabor window *g(n)* and $W_{a,k}(n)$ is the fractional kernel,

$$W_{a,k}(n) = e^{j\left[-\frac{1}{2}(n^2 + (w_k \sin a)^2)\cot a + w_k n\right]}$$

where $w_k = 2pk / K$. The kernel above is similar to the Fractional Fourier Series basis functions [11]. The expansion in (7) reduces to the traditional Gabor for *á=ð/2*. The parameters *M*, *K*, *L*, and *L'*, are same as in the traditional Gabor expansion. In our derivations, we always consider the general, oversampled case, i.e., *L< K*. The Gabor coefficients can be evaluated as before by

$$c_{m,k,a} = \sum_{n=0}^{N-1} x(n) \tilde{g}^*_{m,k,a}(n) \qquad (8)$$

where the analysis functions are

$$\tilde{g}_{m,k,a}(n) = \tilde{g}(n - mL) W_{a,k}(n)$$

and $\tilde{g}(n)$ is periodic version of a $g(n)$ that is solved from a fractional biorthogonality condition between *g(n)* and $g(n)$

The completeness condition for the fractional Gabor basis, is obtained by substituting (8) in (7),

$$x(n) = \sum_{m=0}^{M-1}\sum_{k=0}^{K-1}\left(\sum_{\ell=0}^{N-1} x(\ell)\tilde{g}^*(\ell - mL) W^*_{a,k}(\ell)\right) \times \tilde{g}(n - mL) W_{a,k}(n)$$

$$= \sum_{l=0}^{N-1}\sum_{m=0}^{M-1}\sum_{k=0}^{K-1} \tilde{g}(n - mL)\tilde{g}^*(\ell - mL) \times e^{j\left[-\frac{1}{2}(n^2 - \ell^2)\cot a + w_k(n-\ell)\right]}$$

Then we obtain that the windows must satisfy the following completeness relation:

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$$\sum_{m=0}^{M-1}\sum_{k=0}^{K-1} \tilde{g}(n - mL)\tilde{g}^*(\ell - mL) e^{j\left[-\frac{1}{2}(n^2 - \ell^2)\cot a\right]} \times e^{jw_k(n-l)} = d(n - \ell) \qquad (9)$$

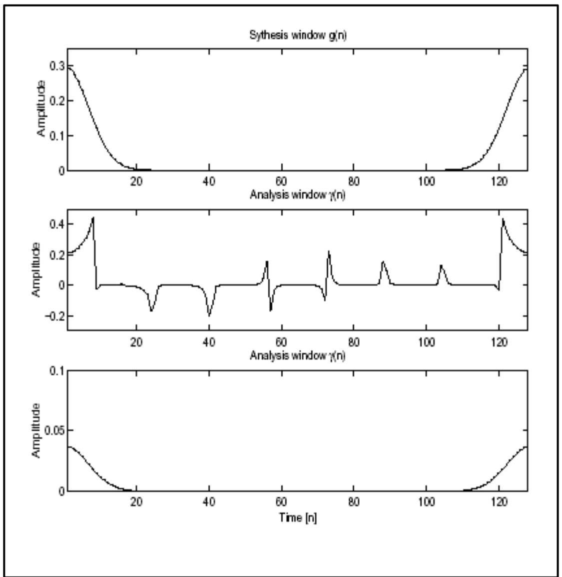

Fig.1 A Gauss synthesis window (top figure), and its biorthogonal windows in critical (middle) and oversampling (bottom) cases.

The fractional biorthogonality condition that we need to solve the analysis or dual function $g(n)$ is obtained from the above completeness relation using discrete Poisson sum formula as

$$\sum_{n=0}^{N-1} \tilde{g}^*(n + mK) e^{jk\frac{2p}{L}(n+mK)} \tilde{g}_{(n)} \times e^{j(nmK + \frac{m^2K^2}{2})\cot a} = \frac{L}{K} d_m d_k$$

0 m L'-1, 0 k L-1

Completeness and biorthogonality conditions given in equations (9) and (10) reduce to the conditions in the traditional case [3] for *á=ð/2*. This indicates that the above fractional expansion is a generalization of the discrete Gabor expansion. In Fig. 1, we show a Gauss window *g(n), n=0,1,… 127* on the top figure, and its biorthogonal $g(n)$ for two different set of sampling parameters obtained by solving equation (10) with *á=ð/4*. The window in the middle is obtained using *L=16, K=16* that is the critical sampling and the window at the bottom is calculated with *L=8, K=64* as an example of the oversampling.

4. SIMULATION RESULTS

We consider a signal composed of two linear chirps. Using our fractional Gabor method, we analyzed the signal with two different fractional orders. Figs. 2 and 3 show the magnitude squared fractional Gabor coefficients, $|C_{m,k,á}|^2$, of this two-chirp signal with *á=ð/4* and *á=3ð/8* respectively. Notice that, the component that is matched by the analysis angle becomes a narrow-band signal and represented with higher resolution.

5. CONCLUSIONS

In this paper, we present a discrete fractional Gabor expansion on a flexible, non--rectangular TF plane for the analysis of non--stationary signals. We give the completeness and biorthogonality conditions of this new expansion. Simulations show that the fractional expansion gives high resolution representations.

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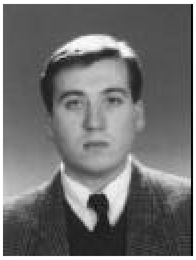
Yalcin CEKIC was born in Istanbul, Turkey in 1971. He received the B.Sc. degree in 1993, and the M.Sc. degree from the University of Istanbul, Istanbul, Turkey in 1996, both in electrical engineering. He served as a Research Assistant at the Department of Electrical and Electronics Engineering, University of Istanbul from 1996 to 2000 where he continues to work on his Ph.D. dissertation. Currently he is a research and teaching fellow at the Computer Engineering Department, Bahcesehir University. His research interest are digital signal processing, time-frequency signal representations, and fractional time-frequency analysis techniques.



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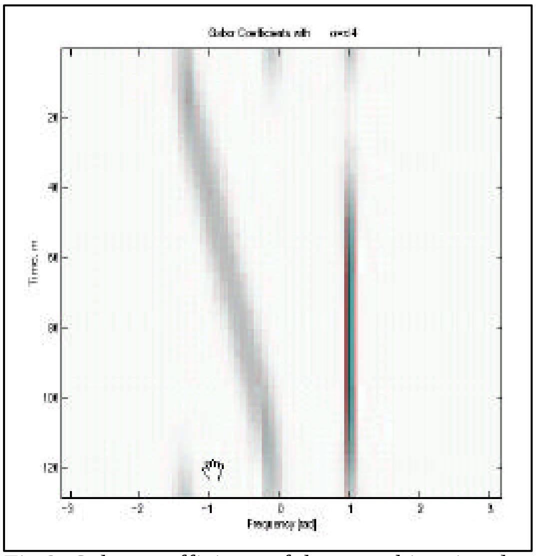

Fig.2. Gabor coefficients of the two-chirp signal using fractional order *á=ð/4*