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Research Article

Some inequalities for homogeneous B_n-potential type integrals on $H^p_{\Delta_\nu}$ Hardy spaces

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Abstract

We prove the norm inequalities for potential operators and fractional integrals related to generalized shift operator defined on spaces of homogeneous type. We show that these operators are bounded from $H^p_{\Delta_\nu}$ to $H^q_{\Delta_\nu}$, for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, provided $0 < \alpha < \frac{1}{2}$, and $\alpha < \beta \leq 1$ and $\frac{Q}{Q+\beta} < p \leq \frac{Q}{Q+\alpha}$. By applying atomic-molecular decomposition of $H^p_{\Delta_\nu}$ Hardy space, we obtain the boundedness of homogeneous fractional type integrals which extends the Stein-Weiss and Taibleson-Weiss's results for the boundedness of the B_n-Riesz potential operator on $H^p_{\Delta_\nu}$ Hardy space.

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1. Introduction

The theory of Hardy spaces establish the important part of harmonic analysis. As we know that the atomic-molecular decomposition of Hardy spaces make the singular integral operators acting on this spaces very simple. Thus the decompositions of Hardy spaces are very critical in harmonic analysis. Therefore, many problems in harmonic analysis have natural formulations as questions of boundedness of singular integral operators defined on this spaces or distributions.

As the development of singular integral operators, the fractional type operators and their boundedness theory play important roles in harmonic analysis and other fields. Moving in the same direction, due to its applications to partial differential equations and differentiation theory, the fractional integrals have attracted many attentions. In many applications, a crucial step has been to show that these classical operators of harmonic analysis are bounded on some function spaces. Also, results on weak and strong type inequalities for this operators of this kind in Lebesgue spaces are classical and can be found for example [13].

One of the well-known example of fractional integrals, the Riesz potential I_α of order $\alpha (0 < \alpha < n)$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} f(x-y)|y|^{\alpha-n} dy.$$

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The famous Hardy-Littlewood-Sobolev theorem states that I_α is bounded operator from usual Lebesgue spaces L^p to L^q when $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$ [13,14].

Historically, in 1971, Muckenhoupt and Wheeden showed the weighted (L^p, L^q) boundedness of the homogeneous fractional integral operator $I_{\Omega,\alpha}$ for power weight when $1 < p < n/\alpha$ [12]. In 1988, Ding and Lu obtained the weighted (L^p, L^q) boundedness of $I_{\Omega,\alpha}$ for $A(p,q)$ weight [3]. Moreover, for the other conditions of p, the boundedness of $I_{\Omega,\alpha}$ can also be found in [1,2]. In 1960, Stein and Weiss [15] used the theory of harmonic functions of several variables to prove that I_α is bounded from H^1 to $L^{n/(n-\alpha)}$. The work was later generalized to the H^p spaces by Taibleson and Weiss [16].In 1980, using the molecular characterization of the real Hardy spaces, Taibleson and Weiss proved that I_α is also bounded from H^p to L^q or H^q, where $0 < p < 1$ and $1/q = 1/p - \alpha/n$.

In this paper, we will mainly concerned with the boundedness properties of B_n-Riesz potential with rough kernel $I^\alpha_{\Omega,\nu}$ on $H^p_{\Delta_\nu}(\mathbb{R}^n_+)$ Hardy spaces in the settings of Δ_ν Laplace-Bessel operator. For $0 < p < \infty$, the $H^p_{\Delta_\nu}$ Hardy spaces are defined by

$$H^p_{\Delta_\nu} = \{f \in \mathcal{S}_+ : ||f||_{H^p_{\Delta_\nu}} = ||\sup_{t>0} |\phi_t \otimes f|||_{L^p_\nu} < \infty\}.$$

Here, $\phi \in \mathcal{S}(\mathbb{R}^n_+)$ satisfies $\int_{\mathbb{R}^n_+} \varphi(x) x_n^\nu dx = 1$. Also, B_n-Riesz potential with rough kernel $I^\alpha_{\Omega,\nu}$ is defined by

$$(I^\alpha_{\Omega,\nu} f)(x) = \int_{\mathbb{R}^n_+} T^y f(x)(\Omega(y)|y|^{\alpha-Q}) y_n^\nu dy,$$

where $0 < \alpha < Q$ and T^y is the generalized shift operator [5,9,10]. Here, our investigation are based on the so-called generalized shift operator introduced first by Levitan.

Since the classical Riesz potential operator I_α is essentially the homogeneous fractional integral operators $I_{\Omega,\alpha}$ when $\Omega = 1$, by comparing mapping properties of I_α and $I^\alpha_{\Omega,\nu}$, the problem arises to ask whether the homogeneous B_n-Riesz potential $I^\alpha_{\Omega,\nu}$ has similar boundedness on $H^p_{\Delta_\nu}$ spaces. We would like to point out that our proofs also suit for B_n-Riesz potential operator with homogeneous characteristic type on $H^p_{\Delta_\nu}$ Hardy spaces in terms of atomic-molecular characterization way.

The aim of this paper is to answer this question. Using the atomic-molecular decomposition of $H^p_{\Delta_\nu}$, we showed that $I^\alpha_{\Omega,\nu}$ is bounded from $H^p_{\Delta_\nu}$ to L^p_ν or $H^q_{\Delta_\nu}$ for some $0 < p \leq 1$.Thus, we verify that Stein-Weiss's conclusion for $p = 1$ and Taibleson-Weiss's conclusion for some $0 < p < 1$ hold also for $I^\alpha_{\Omega,\nu}$.

Now let us first recall some necessary notions and notations. Throughout the whole paper, C always means a positive constant independent of the main parameters, it may change from one occurrence to another.

2. Some preliminaries

Let $\mathbb{R}^n_+$ be the part of the Euclidean space $\mathbb{R}^n$ of points $x = (x_1, ..., x_n)$, defined by the inequality $x_n > 0$. We write $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $B(x,r) = \{y \in \mathbb{R}^n_+ \; ; \; |x-y| < r\}$, $B(x,r)^c = \mathbb{R}^n_+ \backslash B(x,r)$. For any measurable set $B \subset \mathbb{R}^n_+$ we define $|B|_\nu = \int_B x_n^\nu dx$, where $\nu > 0$. Then $|B(0,r)|_\nu = \omega(n,\nu) r^Q, Q = n + \nu$, where

$$\omega(n,\nu) = \int_{B(0,1)} x_n^\nu dx = \pi^{\frac{n-1}{2}} \Gamma(\frac{\nu+1}{2})(2\Gamma(\frac{Q-2}{2}))^{-1}.$$

Let $\mathcal{S}_+ = \mathcal{S}(\mathbb{R}^n_+)$ be the space of functions which are the restrictions to $\mathbb{R}^n_+$ of the test functions of the Schwartz that are even with respect to x_n, decreasing sufficiently rapidly at infinity, together with all derivatives of the form

$$D^\gamma_\nu = D^{\gamma'}_{x'} B^{\gamma_n}_n = D^{\gamma_1}_1 ... D^{\gamma_{n-1}}_{n-1} B^{\gamma_n}_n = \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \cdots \frac{\partial^{\gamma_{n-1}}}{\partial x_{n-1}^{\gamma_{n-1}}} B^{\gamma_n}_n,$$

i.e., for all $\varphi \in \mathcal{S}_+$, $\sup_{x\in\mathbb{R}^n_+} |x^\eta D^\gamma_\nu \varphi| < \infty$, where $B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\nu}{x_n}\frac{\partial}{\partial x_n}$ is the Bessel differential expansion, $\gamma = (\gamma_1, ..., \gamma_n)$ and $\eta = (\eta_1, ..., \eta_n)$ are multi-indexes, and $x^\eta = x_1^{\eta_1} \dots x_n^{\eta_n}$. For a fixed parameter $\nu > 0$, let $L^p_\nu = L^p_\nu(\mathbb{R}^n_+)$ be the space of measurable functions with a finite norm

$$\|f\|_{L^p_\nu} \equiv \left(\int_{\mathbb{R}^n_+} |f(x)|^p x_n^\nu dx \right)^{1/p}$$

is denoted by $L^p_\nu \equiv L^p_\nu(\mathbb{R}^n_+)$, $1 \le p < \infty$. The space of the essentially bounded measurable function on $\mathbb{R}^n_+$ is denoted by $L^\infty_\nu(\mathbb{R}^n_+)$. The space $\mathcal{S}_+$ equipped with the usual topology. We denote by $\mathcal{S}'_+ \equiv \mathcal{S}'_+(\mathbb{R}^n_+)$ the topological dual of $\mathcal{S}_+$ is the collection of all tempered distributions on $\mathbb{R}^n_+$ equipped with the strong topology.

The mixed Fourier-Bessel transform on $\mathcal{S}_+$ has the form

$$F_\nu f(x) = \int_{\mathbb{R}^n_+} f(y)\, e^{-i(x',y')} j_{\frac{\nu-1}{2}}(x_n y_n)\; y_n^\nu dy, \tag{2.1}$$

where $(x', y') = x_1 y_1 + \ldots + x_{n-1} y_{n-1}$, j_ν, $\nu > -1/2$, is the normalized Bessel function, and $C_{n,\nu} = (2\pi)^{n-1} 2^{\nu-1} \Gamma^2((\nu+1)/2) = \frac{2}{\pi}\omega(2,\nu)$. This transform is associated to the Laplace-Bessel differential operator

$$\Delta_\nu = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\nu}{x_n}\frac{\partial}{\partial x_n} = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + B_n, \quad \nu > 0,$$

where $B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\nu}{x_n}\frac{\partial}{\partial x_n}$.

The Fourier-Bessel transform is invertible on $\mathcal{S}_+$ and the inverse transform is given by the relation

$$F_\nu^{-1} f(x) = C_{n,\nu} F_\nu f(-x', x_n). \tag{2.2}$$

The generalized shift operator is defined as follows:

$$T^y f(x) = C_\nu \int_0^\pi f\left(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos\theta + y_n^2}\right) \sin^{\nu-1}\theta d\theta, \tag{2.3}$$

where $C_\nu = \pi^{-\frac{1}{2}} \Gamma\left(\frac{\nu+1}{2}\right) [\Gamma\left(\frac{\nu}{2}\right)]^{-1}$ (see [9, 10]). Following [9, 10], let us introduce the generalized convolution generated by shift (2.3) according to the formula

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_+} f(y)\; T^y g(x)\, y_n^\nu dy.$$

The integrals of the B_n-fractional type with homogeneous characteristic $\Omega(x)$ of degree zero on $\mathbb{R}^n_+$ have the following form:

$$(I^\alpha_{\Omega,\nu} f)(x) = \int_{\mathbb{R}^n_+} f(y) T^y \left(\frac{\Omega(x)}{|x|^{Q-\alpha}} \right) y_n^\nu dy, \qquad 0 < \alpha < Q. \tag{2.4}$$

It is clear that when $\Omega = 1$, $I^\alpha_{\Omega,\nu}$ is the usual B_n-Riesz potential I^α_ν ([5–8, 11]).

For the B_n-Riesz potentials the following theorem is valid.

Theorem 2.1 ([7], Corollary 1). *Let $0 < \alpha < Q$ and $\Omega \in L^r_\nu(S^{n-1}_+)$ with $r > \frac{Q}{Q-\alpha}$ be homogeneous characteristic of degree zero on $\mathbb{R}^n_+$.*

i) *If $1 < p < \frac{Q}{\alpha}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$ is necessary and sufficient for the boundedness of $I^\alpha_{\Omega,\nu}$ from $L^p_\nu(\mathbb{R}^n_+)$ to $L^q_\nu(\mathbb{R}^n_+)$.*

ii) *If $p = 1$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$ is necessary and sufficient for the boundedness of $I^\alpha_{\Omega,\nu}$ from $L^1_\nu(\mathbb{R}^n_+)$ to $WL^q_\nu(\mathbb{R}^n_+)$.*

Definition 2.2. Let $0 < p \le 1 \le q \le \infty$ with $p \ne q$. A (p,q,s)-atom $a(x)$ is a function in $L^q_\nu(\mathbb{R}^n_+)$ which satisfies the following properties:

i) $\operatorname{supp} a \subset B$,

ii) $||a(x)||_{L^q_\nu} \le |B|_\nu^{\frac{1}{q}-\frac{1}{p}}$,

iii) $\int_B a(x) x^\lambda x_n^\nu dx = 0$ for all s with $|\lambda| \le s,\ s = [Q(\frac{1}{p} - 1)]$.

Now we are in a position to state our main results as follows.

Theorem 2.3. *Let $0 < \alpha < Q$, and let $\Omega \in L^r_\nu(S^{n-1}_+)$ for $r > \frac{Q}{Q-\alpha}$ be homogeneous characteristic of degree zero on $\mathbb{R}^n_+$. Then there is a constant $C > 0$ such that*

$$||I^\alpha_{\Omega,\nu} f||_{L_\nu^{\frac{Q}{Q-\alpha}}} \le C ||f||_{H^1_{\Delta_\nu}}.$$

Theorem 2.4. *Let $0 < \alpha < 1$, $\frac{Q}{Q+\alpha} \le p < 1$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ and $\Omega \in L^r_\nu(S^{n-1}_+)$ with $r > \frac{Q}{Q-\alpha}$ be homogeneous characteristic of degree zero on $\mathbb{R}^n_+$. Then there is a constant $C > 0$ such that*

$$||I^\alpha_{\Omega,\nu} f||_{L^q_\nu} \le C ||f||_{H^p_{\Delta_\nu}}.$$

Theorem 2.3 and 2.4 give the $(H^p_{\Delta_\nu}, L^q_\nu)$ boundedness of $I^\alpha_{\Omega,\nu}$. The following theorem will give the $(H^p_{\Delta_\nu}, H^q_{\Delta_\nu})$ boundedness of $I^\alpha_{\Omega,\nu}$.

Theorem 2.5. *Let $0 < \alpha < \frac{1}{2}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ and let $\Omega \in L^r_\nu(S^{n-1}_+)$ with $r > \frac{1}{1-2\alpha}$ be homogeneous characteristic of degree zero on $\mathbb{R}^n_+$. Then for $\alpha < \beta \le 1$ and $\frac{Q}{Q+\beta} < p \le \frac{Q}{Q+\alpha}$, there is a constant $C > 0$ such that*

$$||I^\alpha_{\Omega,\nu} f||_{H^q_{\Delta_\nu}} \le C ||f||_{H^p_{\Delta_\nu}}.$$

3. The proof of main results

This section is devoted to the proofs of the theorems. For an operator, to prove the boundedness from $H^1_{\Delta_\nu}$ to L^1_ν or $H^p_{\Delta_\nu}$ to L^p_ν, a common method is to take one atom at a time. It isn't hard to verify (p,q,s)-atoms are mapped into L^p spaces, uniformly. However, to study the problem of boundedness of B_n-Riesz potential operator on $H^p_{\Delta_\nu}$ Hardy spaces, we need a modification. The method we adopted is similar to the same in [4].

Before we prove our main results, we need to give some necessary facts.

Theorem 3.1 ([11], Theorem 1.1). *Let $1 \le r \le \infty$, $0 < \alpha < Q$ and $K(x)$ be a kernel of B-fractional type with homogeneous characteristic of degree zero on $\mathbb{R}^n_+$. Then there exists $A, C > 0$ such that for all $t > 0$ $(t = 2^j)$ and $x \in \mathbb{R}^n_+$*

$$\left(\int\limits_{|x|>A} |T^y K(tx) - K(tx)|^r x_n^\nu dx \right)^{\frac{1}{r}} \le C t^{\frac{Q-\alpha}{r}}, \qquad |y| < \frac{1}{A}. \tag{3.1}$$

Proof of Theorem 2.3 and 2.4. Let us first start to give the proof of Theorem 2.3. By the atomic decomposition theory of Hardy spaces, it is sufficient to prove that there is constant C such that for any $(1,\ell,0)$-atom $a(x)$, the inequality

$$||(I^\alpha_{\Omega,\nu} a)(x)||_{L^q_\nu} \le C \tag{3.2}$$

holds, where $\ell > 1$ and $q = \frac{Q}{Q-\alpha}$. We now take $1 < \ell_1 < \ell_2 < \infty$, such that $\frac{1}{\ell_1} - \frac{1}{\ell_2} = \frac{\alpha}{Q}$. For the present investigation of the proof, we consider the function $a(x)$ is $(1,\ell_1,0)$-atom supported in a ball $B = B(0,d)$ with center at zero and radius d. So we can write

$$\begin{aligned}||(I^\alpha_{\Omega,\nu}a)(x)||_{L^q_\nu} &\le \left(\int_{2B}|(I^\alpha_{\Omega,\nu}a)(x)|^q x_n^\nu dx\right)^{\frac{1}{q}} + \left(\int_{(2B)^c}|(I^\alpha_{\Omega,\nu}a)(x)|^q x_n^\nu dx\right)^{\frac{1}{q}}\\ &:= I_1 + I_2.\end{aligned}$$

By applying Hölder's inequality and Theorem 2.1, we may estimate I_1 as follows:

$$I_1 \le C||I^\alpha_{\Omega,\nu}a||_{L^{\ell_2}_\nu}|B|_\nu^{\frac{1}{q}-\frac{1}{\ell_2}} \le C||a||_{L^{\ell_1}_\nu}|B|_\nu^{\frac{1}{q}-\frac{1}{\ell_2}} \le C.$$

For I_2, by the vanishing condition (iii) of $a(x)$, we obtain

$$\begin{aligned}I_2 &= \left(\int_{(2B)^c}|(I^\alpha_{\Omega,\nu}a)(x)|^q x_n^\nu dx\right)^{\frac{1}{q}} = \left(\int_{(2B)^c}\left|\int_{\mathbb{R}^n_+}T^y(K_\alpha(x))a(y)y_n^\nu dy\right|^q x_n^\nu dx\right)^{\frac{1}{q}}\\ &= \int_{(2B)^c}|a(y)|\left(\int_{\mathbb{R}^n_+}|T^y(K_\alpha(x)) - K_\alpha(x)|^q x_n^\nu dx\right)^{\frac{1}{q}} y_n^\nu dy \qquad (3.3)\\ &\le \int_{(2B)^c}|a(y)|\left(\sum_{j=1}^\infty \int\limits_{2^jd\le|x|<2^{j+1}d}|T^y(K_\alpha(x)) - K_\alpha(x)|^q x_n^\nu dx\right)^{\frac{1}{q}} y_n^\nu dy\end{aligned}$$

where $K_\alpha(x) = \Omega(x)|x|^{\alpha-Q}$. Since $r > \frac{Q}{Q-\alpha} = q$, by Hölder's inequality, we obtain

$$\begin{aligned}&\left(\int\limits_{2^jd\le|x|<2^{j+1}d}|T^y(K_\alpha(x)) - K_\alpha(x)|^q x_n^\nu dx\right)^{\frac{1}{q}}\\ &\quad\le C(2^jd)^{Q(\frac{1}{q}-\frac{1}{r})}\left(\int\limits_{2^jd\le|x|<2^{j+1}d}|T^y(K_\alpha(x)) - K_\alpha(x)|^r x_n^\nu dx\right)^{\frac{1}{r}}.\end{aligned} \qquad (3.4)$$

Applying Theorem 3.1, we have

$$\left(\int\limits_{2^jd\le|x|<2^{j+1}d}|T^y(K_\alpha(x)) - K_\alpha(x)|^r x_n^\nu dx\right)^{\frac{1}{r}} \le C(2^jd)^{\frac{Q-\alpha}{r}}. \qquad (3.5)$$

By the inequalities (3.4) and (3.5), we get

$$\begin{aligned}&\sum_{j=1}^\infty\left(\int\limits_{2^jd\le|x|<2^{j+1}d}|T^y(K_\alpha(x)) - K_\alpha(x)|^q x_n^\nu dx\right)^{\frac{1}{q}}\\ &\quad\le C\sum_{j=1}^\infty(2^jd)^{Q(\frac{1}{q}-\frac{1}{r})}(2^jd)^{\frac{Q-\alpha}{r}} < \infty.\end{aligned} \qquad (3.6)$$

Therefore, by (3.3) and (3.6) we obtain

$$I_2 \le C\int_B|a(y)|y_n^\nu dy \le C||a||_{L^{\ell_1}_\nu}|B|_\nu^{\frac{1}{\ell_1'}} \le C.$$

The proof of Theorem 2.3 is finished.

The proof of Theorem 2.4 is similar to Theorem 2.3. Then, we only give the main steps of the proof by choosing $1 < \ell_1 < \ell_2 < \infty$ such that $\frac{1}{\ell_1} - \frac{1}{\ell_2} = \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$. Let $a(x)$ be $(p,\ell_1,0)$-atom supported in the ball $B(0,d)$. Here we still need to verify the validity of (3.2) for the atom $a(x)$. As in the previous proof, we give the similar estimates for I_1 and I_2, respectively. We estimate I_1 again by using Hölder's inequality and Theorem 2.1. However, using the conditions of Theorem 2.4, if $p \geq \frac{Q}{Q+\alpha}$, then we obtain $(\alpha+Q)-(Q/p) \leq 0$. In this case, by the Theorem 2.1, we have

$$\sum_{j=1}^{\infty}\Big(\int\limits_{2^jd\leq|x|<2^{j+1}d}|T^y(K_\alpha(x))-K_\alpha(x)|^q x_n^\nu dx\Big)^{\frac{1}{q}} \leq C\sum_{j=1}^{\infty}(2^jd)^{-\frac{Q}{q}}$$
$$\leq |B|_\nu^{-1/q} < \infty.$$

Finally, from the discussion above and (3.3) we have

$$I_2 \leq C|B|_\nu^{-1/q}\int\limits_B |a(y)|y_n^\nu dy \leq C|B|_\nu^{-1/q}||a||_{L_\nu^{\ell_1}}|B|_\nu^{1/\ell_1'} \leq C.$$

This completes the proof of Theorem 2.4. □

Proof of Theorem 2.5. First, let us state $r > \frac{Q}{Q-\alpha}$. We can select $1 < \ell_1 < \ell_2$ such that $\frac{1}{\ell_1}-\frac{1}{\ell_2} = \frac{1}{p}-\frac{1}{q} = \frac{\alpha}{Q}$ and $\frac{Q}{Q-\alpha} < \ell_2 < r$. Take ϵ so that $\frac{1}{q}-1 < \epsilon < \frac{\beta-\alpha}{Q} \leq \frac{1-\alpha}{Q}$. Denote $a_0 = 1-\frac{1}{q}+\epsilon$, $b_0 = 1-\frac{1}{\ell_2}+\epsilon$ and let $a(x)$ be a $(p,\ell_1,0)$-atom supported in the ball $B(0,d)$. By the atomic-molecular decomposition theory of real Hardy spaces [14], it suffices to show that $I_{\Omega,\nu}^\alpha a$ is a $(q,\ell_2,0,\epsilon)$-molecule for proving Theorem 2.5. To prove this, we still need to verify that $(I_{\Omega,\nu}^\alpha a)(x)$ satisfies the following conditions:

i) $|x|^{Qb_0}(I_{\Omega,\nu}^\alpha a)(x) \in L_\nu^{\ell_2}$,

ii) $\mathcal{N}_\nu^{\ell_2}(I_{\Omega,\nu}^\alpha a) := ||I_{\Omega,\nu}^\alpha a||_{L_\nu^{\ell_2}}^{a_0/b_0} |||.|^{Qb_0}(I_{\Omega,\nu}^\alpha a)(.)||_{L_\nu^{\ell_2}}^{1-a_0/b_0} < \infty$,

iii) $\int_B (I_{\Omega,\nu}^\alpha a)(x)x_n^\nu dx = 0$.

Moreover, we also need to prove that there is a constant $C > 0$, independent of $a(x)$, such that

$$\mathcal{N}_\nu^{\ell_2}(I_{\Omega,\nu}^\alpha a) \leq C.$$

Let us estimate every part. For (i), write

$$\begin{aligned}|||.|^{Qb_0}(I_{\Omega,\nu}^\alpha a)(.)||_{L_\nu^{\ell_2}} &\leq |||.|^{Qb_0}(I_{\Omega,\nu}^\alpha a)(.)\chi_{2B}(.)||_{L_\nu^{\ell_2}} + \\ &+|||.|^{Qb_0}(I_{\Omega,\nu}^\alpha a)(.)\chi_{(2B)^c(.)}||_{L_\nu^{\ell_2}} \\ &:= J_1 + J_2.\end{aligned}$$

Observe that $\frac{Q}{Q-\alpha} < \ell_2 < r$ and $\frac{1}{\ell_1}-\frac{1}{\ell_2} = \frac{\alpha}{Q}$, by Theorem 2.1, we have

$$J_1 \leq C|B|_\nu^{b_0}||I_{\Omega,\nu}^\alpha a||_{L_\nu^{\ell_2}} \leq C|B|_\nu^{b_0}||a||_{L_\nu^{\ell_1}}. \tag{3.7}$$

For J_2, by the moment condition of $a(x)$ we obtain

$$J_2 \leq \int\limits_B |a(y)|\Big(\sum_{j=1}^{\infty}\int\limits_{2^jd\leq|x|<2^{j+1}d}|T^y(K_\alpha(x))-K_\alpha(x)|^{\ell_2}|x|^{Qb_0\ell_2}x_n^\nu dx\Big)^{\frac{1}{\ell_2}} y_n^\nu dy. \tag{3.8}$$

If we apply the Hölder's inequality and Theorem 3.1, we get

$$\begin{aligned}\Big(\int_{2^jd\leq|x|<2^{j+1}d} & |T^y(K_\alpha(x))-K_\alpha(x)|^{\ell_2}|x|^{Qb_0\ell_2}x_n^\nu dx\Big)^{\frac{1}{\ell_2}} \\ &\leq \Big(\int_{2^jd\leq|x|<2^{j+1}d}|T^y(K_\alpha(x))-K_\alpha(x)|^r x_n^\nu dx\Big)^{\frac{1}{r}} \\ &\times\Big(\int_{2^jd\leq|x|<2^{j+1}d}|x|^{Qb_0\ell_2(r/\ell_2)'}x_n^\nu dx\Big)^{\frac{1}{\ell_2(r/\ell_2)'}} \\ &\leq C(2^jd)^{\frac{Q}{r}}(2^jd)^{Qb_0}(2^jd)^{Q(\frac{1}{\ell_2}-\frac{1}{r})} = C(2^jd)^{Q+Q\epsilon} \leq |B|_\nu^{1+\epsilon}.\end{aligned}$$

Thus, by the inequality above (3.8), we have

$$J_2 \le C|B|_\nu^{\epsilon+\frac{\alpha}{Q}} \int\limits_B |a(y)| y_n^\nu dy \le C|B|_\nu^{\epsilon+\frac{\alpha}{Q}} ||a||_{L_\nu^{\ell_1}} |B|_\nu^{1/\ell_1'}. \tag{3.9}$$

By (3.7) and (3.9), we know that (i) holds and

$$\begin{aligned} \mathcal{N}_\nu^{\ell_2}(I^\alpha_{\Omega,\nu}a) &= ||I^\alpha_{\Omega,\nu}a||^{a_0/b_0}_{L_\nu^{\ell_2}} |||.|^{(Q)b_0}(I^\alpha_{\Omega,\nu}a)(.)||^{1-a_0/b_0}_{L_\nu^{\ell_2}} \\ &\le C||a||^{a_0/b_0}_{L_\nu^{\ell_1}} |B|_\nu^{\epsilon+\frac{\alpha}{Q}(1-a_0/b_0)} ||a||^{1-a_0/b_0}_{L_\nu^{\ell_1}} |B|_\nu^{1-a_0/b_0(1/\ell_1')} \le C. \end{aligned}$$

Finally, we need to verify (iii) to complete the proof of Theorem 2.5. To this end, we first show that $(I^\alpha_{\Omega,\nu}a)(x) \in L^1_\nu(\mathbb{R}^n_+)$. So, we may write

$$\begin{aligned} \int\limits_{\mathbb{R}^n_+} |(I^\alpha_{\Omega,\nu}a)(x)| x_n^\nu dx &= \int\limits_{|x|<1} |(I^\alpha_{\Omega,\nu}a)(x)| x_n^\nu dx + \int\limits_{|x|\ge 1} |(I^\alpha_{\Omega,\nu}a)(x)| x_n^\nu dx \\ &:= E_1 + E_2. \end{aligned}$$

Clearly $E_1 \le C$ since $I^\alpha_{\Omega,\nu}a(x) \in L_\nu^{\ell_2}$. On the other hand, by $b_0 - 1/\ell_2' = \epsilon > 0$ and $|x|^{Qb_0}(I^\alpha_{\Omega,\nu}a)(.) \in L_\nu^{\ell_2}$, again using Hölder's inequality we obtain

$$E_2 \le |||.|^{Qb_0}(I^\alpha_{\Omega,\nu}a)(.)||_{L_\nu^{\ell_2}} \left(\int\limits_{|x|\ge 1} |x|^{-Qb_0\ell_2'} x_n^\nu dx \right) < \infty.$$

Therefore, $F_\nu(I^\alpha_{\Omega,\nu}a) \in \mathcal{C}(\mathbb{R}^n_+)$. In order to check

$$\int (I^\alpha_{\Omega,\nu}a)(x) x_n^\nu dx = F_\nu[I^\alpha_{\Omega,\nu}a](0) = 0,$$

it is sufficient to show

$$\lim_{|\xi|\to 0} F_\nu[I^\alpha_{\Omega,\nu}a](\xi) = 0. \tag{3.10}$$

It is well known that $F_\nu[I^\alpha_{\Omega,\nu}a](\xi) = F_\nu[a](\xi) F_\nu[K_\alpha(x)](\xi)$, and

$$\begin{aligned} F_\nu(K_\alpha(\xi)) &= \int\limits_{|x|<1} K_\alpha(x) e^{-i(x',\xi')} j_{\frac{\nu-1}{2}}(x_n\xi_n)\, x_n^\nu dx \\ &+ \sum_{j=1}^\infty \int_{2^{j-1}\le|x|<2^j} K_\alpha(x) e^{-i(x',\xi')} j_{\frac{\nu-1}{2}}(x_n\xi_n)\, x_n^\nu dx, \end{aligned}$$

where $K_\alpha(x) = \Omega(x)|x|^{\alpha-Q}$. Thus, we obtain

$$\Big| F_\nu(K_\alpha(\xi)) \Big| \le C + \sum_{j=1}^\infty |F_\nu[K_{\alpha,\chi_j}(\xi)]|,$$

where $K_{\alpha,\chi_j}(\xi) = \Omega(\xi)|\xi|^{\alpha-Q}\chi_{[2^{j-1},2^j)}(|\xi|)$. Here we give an estimate of $|F_\nu[K_{\alpha,\chi_j}(\xi)]|$ for any $j \ge 1$ in the study of this problem.

Lemma 3.2. *Suppose that $0 < \alpha < \frac{1}{2}$, and $\Omega \in L^r_\nu(S^{n-1})$ with $r > \frac{1}{1-2\alpha}$ is homogeneous characteristic on $\mathbb{R}^n_+$. Then there exists C and $\sigma > 0$, such that $2\alpha < \sigma < 1/r' \le 1$ and for $j \ge 1$*

$$|F_\nu[K_{\alpha,\chi_j}(\xi)]| \le C_{n,\nu,\alpha} 2^{(\alpha-Q/2)j} |\xi|^{-\sigma/2},$$

where $C_{n,\nu,\alpha} = \dfrac{\Gamma\big((n+\nu-\alpha)/2\big)}{\Gamma\big(\alpha/2\big)}$.

Proof. First, we shall need the Fourier-Bessel transforms of the function

$$F_\nu(e^{-r|x|^2})(\xi) = e^{-\frac{|\xi|^2}{4r}}(2r)^{\frac{-2\nu-n}{2}} \qquad r>0, x, \xi \in \mathbb{R}^n_+.$$

By the property $< F_\nu K_\alpha, \varphi > = < K_\alpha, F_\nu \varphi >$ of generalized functions, we may write

$$\int_{\mathbb{R}^n_+} e^{-r|x|^2} F_\nu \varphi(x) x_n^\nu dx \quad = \int_{\mathbb{R}^n_+} \varphi(x) F_\nu e^{-\frac{|x|^2}{4r}} x_n^\nu dx.$$

We now integrate both sides of the above with respect to r from 0 to ∞ having multiplied the equation by $r^{(Q-\alpha)/2-1}$. We obtain

$$\int_{\mathbb{R}^n_+} F_\nu\varphi(x)\left(\int_0^\infty r^{(Q-\alpha)/2-1}e^{-r|x|^2}dr\right)x_n^\nu dx = \int_{\mathbb{R}^n_+}\varphi(x)\left(\int_0^\infty r^{(Q-\alpha)/2-1}(2r)^{\frac{-2\nu-n}{2}}e^{-\frac{|x|^2}{4r}}\right)x_n^\nu dx.$$

If we calculate the inner integrals, we have

$$\Gamma((n+\nu-\alpha)/2)\int_{\mathbb{R}^n_+} F_\nu\varphi(x)\Omega(x)|x|^{\alpha-Q}x_n^\nu dx = 2^{\alpha-Q/2}\Gamma(\alpha/2)\int_{\mathbb{R}^n_+}\varphi(x)\Omega(x)|x|^{-\alpha}x_n^\nu dx.$$

Taking the inverse Fourier-Bessel transform and the modulus property, the required inequality is obtained. □

Now let us return to the proof of Theorem 2.5. Applying the conclusion of Lemma 3.2, we obtain

$$\begin{aligned} F_\nu[K_\alpha(\xi)] &\le C_{n,\nu,\alpha} + \sum_{j=1}^{\infty} |F_\nu[K_{\alpha,\chi_j}(\xi)]| \\ &\le C_{n,\nu,\alpha} + C_{n,\nu,\alpha}\sum_{j=1}^{\infty} 2^{(\alpha-Q/2)j}|\xi|^{-\sigma/2} \\ &\le C_{n,\nu,\alpha}(1+|\xi|^{-\sigma/2}). \end{aligned} \tag{3.11}$$

On the other hand, for $F_\nu[a](\xi)$ we have

$$\begin{aligned} \left|\int_{\mathbb{R}^n_+} a(x)e^{-i(x',\xi')}j_{\frac{\nu-1}{2}}(x_n\xi_n)\, x_n^\nu dx\right| &= \left|\int_B a(x)[e^{-i(x',\xi')}-1]j_{\frac{\nu-1}{2}}(x_n\xi_n)\, x_n^\nu dx\right| \\ &\le C_{n,\nu,\alpha}\int_B |a(x)||\xi||x|x_n^\nu dx \le C_{n,\nu,\alpha}|\xi|. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) we obtain

$$|F_\nu(I^\alpha_{\Omega,\nu}a)(\xi)| \le |F_\nu(a(\xi))||F_\nu(K_\alpha(\xi))| \le C_{n,\nu,\alpha}(|\xi|+|\xi|^{1-\sigma/2}). \tag{3.13}$$

By the choice of σ it is known that $1-\sigma/2>0$. So, the required equality (3.10) holds by (3.13). Hence $(I^\alpha_{\Omega,\nu}a)(x)$ satisfies the condition (iii) and Theorem 2.5 follows. □

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