Konuralp Journal of Mathematics, 9 (1) (2021) 154-158

Konuralp Journal of Mathematics

Research Paper

https://dergipark.org.tr/en/pub/konuralpjournalmath

e-ISSN: 2147-625X

On The Basic Properties of Linear Graphs - I

Ramazan Sunar[1*] **and İbrahim Günaltılı**[2]

[1] *Institite of Science and Technology, Eskişehir Osmangazi University,Eskişehir,Turkey*
[2] *Professor,Department of Mathematics and Computer, Eskişehir Osmangazi University, Eskişehir, Turkey*
[*] *Corresponding author*

Abstract

A linear graph is a bipartite graph with parts $\mathscr{P}$ and $\mathscr{L}$ that have propertites: LG1: Any two distinct vertices of $\mathscr{P}$ have exactly common neighbour one vertex. LG2: $\delta(G) \geq 2$. In this paper, we determined basic properties of finite linear graph.

Keywords: *bipartite graph; common neighborhood; linear graph.*
2010 Mathematics Subject Classification: *05C07, 05C12, 05C40.*

1. Introduction

In this paper, all graphs are considered finite, undirected without loops or multiple edges. We will mainly deal with the graph properties namely girth, distance, neighborhood, degree, and regularity index of it. For more details,(see [8]). Let G be a graph with vertex sets $V(G) = \mathscr{P} \cup \mathscr{L}$ and edge sets $E(G) = E$, respectively. The degree of a vertex p is number $d(p)$ of edges which are incident with it. The minimum degree among the vertices of G is denoted by $\delta(G)$. The neighborhood of a vertex $u \in V(G)$ is a set $N(u) = \{v \in V(G) : uv \in E(G)\}$. The common neigborhood of vertices $u_1, u_2, ..., u_n$ is a set $CN(u_1, u_2, ..., u_n) = \bigcap_{i=1}^{n} N(u_i)$ and $cn(u_1, u_2, ..., u_n) = |CN(u_1, u_2, ..., u_n)|$ is the number of common neighborhood. The distance is length of the shortest path between two vertices of G and it denote by $d(u, v)$; $u, v \in V(G)$. The diameter of G $diamG = max\{\ d(u, v) : u, v \in V(G)\}$. The girth of G denoted $g(G)$ is the lenght of it's shortest cycle.A connected graph that is 2-regular is called a cycle graph. All definitions and notations may be found in [1],[4],[5],[6],[7]. X $\subseteq$ V(G) is called edge-free if $\{$p,q$\} \notin$ E(G) for all p,q in X. A bipartite graph $G = (\mathscr{P} \cup \mathscr{L}, E)$ is a graph whose vertex set $\mathscr{P} \cup \mathscr{L}$ can be partitioned into subsets $\mathscr{P}$ and $\mathscr{L}$ in which end vertices of each edge of $E(G)$ belongs either $\mathscr{P}$ or $\mathscr{L}$. Also the parts of $\mathscr{P}$ and $\mathscr{L}$ edge-free [3].
In this paper show that elements of $\mathscr{P}$ and $\mathscr{L}$ denoted by $p, q, ...$ and $L, K, ...$, respectively. We give some theorem for useful in the proof of our result.

Theorem 1.1. *[1] G is a bipartite graph if and only if it has not odd cycle.*

2. Main Results

Definition 2.1. *A linear graph is a bipartite graph with parts $\mathscr{P}$ and $\mathscr{L}$ such that satisfies the following conditions:*
LG1: For all $p, q \in \mathscr{P}$ such that $p \neq q$, $cn(p, q) = 1$,
LG2: $\delta(G) \geq 2$.

Example 2.2. *Let S=(P,L) be nontrivial linear space. Then incidence graph of S is a linear graph. (For linear spaces, see [2]).*

Corollary 2.3. *Let $G = (\mathscr{P} \cup \mathscr{L}, E)$ be a linear graph. Then G does not contain isoled vertex and pendant vertex.*

Proof. It is clear from definition of linear graph ($\delta(G) \geq 2$). □

Corollary 2.4. *Let G be a linear graph. Then it does not contain odd cycle.*

Proof. It is clear from Definition 2.1 and Theorem 1.1. □

Lemma 2.5. *Let G be a linear graph. Then $cn(L, K) \leq 1$ for all $K, L \in \mathscr{L}$.*

Email addresses: ramazan_sunar03@hotmail.com (Ramazan Sunar), igunalti@ogu.edu.tr (İbrahim Günaltılı)

Proof. Let G be a linear graph. Suppose that $cn(L,K) \geq n$ for $K, L \in \mathscr{L}$ and $n \geq 2$. Then there exist vertices $p_1, p_2, ..., p_n \in \mathscr{P}$ such that $\{p_1, p_2, ..., p_n\} \subseteq CN(L,K)$.
Then $i \neq j$, for $p_i, p_j \in \{p_1, p_2, ..., p_n\}$, $cn(p_i, p_j) \geq 2$. This case contradicts with $LG1$.
Therefore $cn(K,L) \leq 1$. □

Proposition 2.6. *Let $G = (\mathscr{P} \cup \mathscr{L}, E)$ be a a linear graph. Then $cn(u,v) = 0$ for each $(u,v) \in \mathscr{P} \times \mathscr{L}$.*

Proof. Let G be a linear graph. Suppose that $cn(u,v) = 1$ for $(u,v) \in (\mathscr{P} \times \mathscr{L})$. Then there exist exactly one vertex x in $\mathscr{P} \cup \mathscr{L}$ such that $x \in N(u) \subseteq \mathscr{P}$ and $x \in N(v) \subseteq \mathscr{L}$. So $x \in \mathscr{P} \cap \mathscr{L}$. This contradicts to $\mathscr{P} \cap \mathscr{L} = \emptyset$. Therefore $cn(x,y) = 0$. □

Corollary 2.7. *Let $G = (\mathscr{P} \cup \mathscr{L}, E)$ be a a linear graph. Then $cn(x,y) = \{0,1\}$ for all $x, y \in V(G)$.*

Theorem 2.8. *Let $G = (\mathscr{P} \cup \mathscr{L}, E)$ be a a linear graph. Then G is $C_4 - free$.*

Proof. Let G be a linear graph. Suppose that G is not $C_4 - free$. Then G contains at least one cycle $C_4 : v_1 - v_2 - v_3 - v_4 - v_1$ for $v_1, v_2, v_3, v_4 \in V(G)$. Then $CN(v_1, v_3) = \{v_2, v_4\}$. In this case $cn(v_1, v_3) = 2$. This case contradicts Corollary 2.7. Hence G is $C_4 - free$. □

Proposition 2.9. *Let $G = (\mathscr{P} \cup \mathscr{L}, E)$ be a linear graph. Then*
i) *For all $p, q \in \mathscr{P}$ such that $p \neq q$, $d(p,q) = 2$.*
ii) *For all $L, K \in \mathscr{L}$ such that $L \neq K$, $d(L,K) \in \{2,4\}$.*
iii) *For $p \in \mathscr{P}$ and $L \in \mathscr{L}$, $d(p,L) \in \{1,3\}$.*

Proof. **i)** Let G be a linear graph and let p and q be any distinct vertices of $\mathscr{P}$. Since G bipartite graph, $\{p,q\} \notin E(G)$. So $d(p,q) \neq 1$. From $LG1$, $cn(p,q) = 1$. So there exist exactly one vertex L in $\mathscr{L}$ such that $CN(p,q) = \{L\}$. Therefore a path $P : x - L - y$ is the shortest path between p and q. So $d(p,q) = 2$
ii) For all $L, K \in \mathscr{L}$ such that $L \neq K$, $cn(L,K) \in \{0,1\}$ from Lemma 2.5.
Case 1. Let $cn(L,K) = 1$. There exist exactly one vertex p in $\mathscr{P}$ such that $CN(L,K) = \{p\}$. Therefore a path $P : L - p - K$ is the shortest path L and K. So $d(L,K) = 2$.
Case 2. Let $cn(L,K) = 0$. There exist two vertices p and q in P such that $\{p,L\}$,$\{q,K\} \in E(G)$ from LG2. From $LG1$, $cn(p,q) = 1$. Then there exist exactly one vertex M in $\mathscr{L}$ such that $CN(p,q) = \{M\}$. If $M = L$, then $CN(L,K) = \{q\}$ since $q \in N(K)$ and $q \in N(L)$. This contradict to $cn(L,K) = 0$. So $M \neq L$. Similarly to $M \neq K$. Therefore a path $P : L - p - M - q - K$ is the shortest path between L and K. So $d(L,K) = 4$.
iii) For $p \in \mathscr{P}$ and $L \in \mathscr{L}$.
Case 1. If $\{p,L\} \in E(G)$ then a path $P : p - L$ is the shortest path between p and L. So $d(p,L) = 1$.
Case 2. If $\{p,L\} \notin E(G)$ then there exist a vertex q in P such that $\{q,L\} \in E(G)$ from LG2. Also $cn(p,q) = 1$ from $LG1$. So there exist exactly one vertex $M \in \mathscr{L}$ such that $CN(p,q) = \{M\}$. If $M = L$, this case contradicts $\{p,L\} \notin E(G)$. So M≠L. Therefore a path $P : p - M - q - L$ is the shortest path p and L. So $d(p,L) = 3$. □

Corollary 2.10. *Let G be a linear graph. For any two distinct vertices u and v of $V(G)$, $d(u,v) \in \{1,2,3,4\}$.*

Proof. It is trivial from Proposition 2.9. □

Theorem 2.11. *Let G be linear graph then G is connected.*

Proof. For all $u, v \in V(G)$ such that $u \neq v$, $d(u,v) \in \{1,2,3,4\}$ from Corollary 2.10. So there exist at least a path P between u and v. Therefore G is connected. □

Lemma 2.12. *Let G be a linear graph. Then G contains at least one C_6.*

Proof. Let G be a linear graph. Then there exist a vertex p in $\mathscr{P}$ since $\mathscr{P} \neq \emptyset$. From $LG2$, there exist two vertices L and K in $\mathscr{L}$ such that $L, K \in N(p)$. Also, there exist two vertices q and r in $\mathscr{P}$ such that $q \in N(L)$, $r \in N(K)$ and $q \neq p$, $r \neq p$ from $LG2$.
If $q = r$, this case contradicts Theorem 2.8. So $q \neq r$. Since $cn(q,r) = 1$, there exist exactly one vertex N in $\mathscr{L}$ such that $CN(q,r) = \{N\}$. So G contains at least one $C_6 : p - L - q - N - r - K - p$. □

Proposition 2.13. *Let G be linear graph then $girthG = 6$.*

Proof. The result clearly follows from Definition 2.1, Theorem 1.1 and Lemma 2.12. □

Lemma 2.14. *Let G be a linear graph and let p and L be two vertices in $\mathscr{P}$ and $\mathscr{L}$, respectively such that $p \notin N(L)$. Then $d(p) \geq d(L)$.*

Proof. Let $p \in \mathscr{P}$, $L \in \mathscr{L}$ such that $p \notin N(L)$ and $d(L) = n$. From LG2, $n \geq 2$. Therefore there exist vertices $p_1, p_2, ..., p_n \in \mathscr{P}$ such that $N(L)$=$\{p_1, p_2, ..., p_n\}$. From $LG1$ there exist vertex L_i in $\mathscr{L}$ such that $\{L_i\} \subseteq CN(p, p_i)$ for each $i, 1 \leq i \leq n$. Let $p_i, p_j \in N(L)$ for $i \neq j$. Then we take two vertices $L_i, L_j \in \bigcup_{i=1}^{n} N(p_i)$. If $L_i = L_j$, $\{L, L_i\} \subseteq CN(p_i, p_j)$. Also $L \neq L_i$. In this case contradicts with $LG1$. So $L_i \neq L_j$. Therefore there exist two case.
Case 1. If $q \in N(L)$ for each $q \in \mathscr{P}$ such that $q \neq p$, then $N(p) = \{L_1, L_2, ..., Ln\}$. So

$$d(p) = \mid N(p) \mid = n. \tag{2.1}$$

Case 2. If $q \notin N(L)$ for at least vertex q in $\mathscr{P}$ such that $q \neq p$, there exist exactly one vertex K in $\mathscr{L}$ such that $CN(p,q) = \{K\}$ from $LG1$.
$i)$ If $K = L_i$ for at least $i, 1 \leq i \leq n$. $\{L_1, L_2, ...L_n\} \subseteq N(p)$.

So

$$d(p) \geq \mid N(p) \mid = n. \tag{2.2}$$

ii) If $K \neq L_i$ for each $i, 1 \leq i \leq n$, $K \in N(p)$ and $L_i \in N(p)$.
So

$$d(p) = \mid N(p) \mid \geq n+1. \tag{2.3}$$

From (2.1), (2.2) and (2.3), we can hold

$$d(p) \geq d(L).$$

□

Proposition 2.15. *Let G be a linear graph. For $p \in \mathscr{P}$ and $L_i \in \mathscr{L}$*

$$\mathscr{P} = \bigcup_{L_i \in N(p)} N(L_i).$$

Proof. Let $x \in \bigcup_{L_i \in N(p)} N(L_i)$. Then for $\exists L_i \in N(p)$, $x \in N(L_i)$. In this case $x \in \mathscr{P}$ because of $N(L_i) \subseteq \mathscr{P}$. Therefore $\bigcup_{L_i \in N(p)} N(L_i) \subseteq \mathscr{P}$. From $LG1$, there exist exactly one vertex $q_i \in \mathscr{P}$ such that $CN(p,q_i) = \{L_i\}$, for each $i, 1 \leq i \leq d(p)$. Then $q_i \in N(L_i)$ and $L_i \in N(p)$. So $q_i \in \bigcup_{L_i \in N(p)} N(L_i)$. Therefore $\mathscr{P} \subseteq \bigcup_{L_i \in N(p)} N(L_i)$. In this case, we can hold

$$\mathscr{P} = \bigcup_{L_i \in N(p)} N(L_i).$$

□

Proposition 2.16. *Let G be a linear graph. For $p \in \mathscr{P}$ and $L_i \in N(p)$*

$$\mid \mathscr{P} \mid -1 = \sum_{i=1}^{d(p)} (d(L_i) - 1).$$

Proof. For $p \in \mathscr{P}$, $d(p) \geq 2$ from $LG2$. Then for each $i, 1 \leq i \leq d(p)$ there exist vertex L_i in $\mathscr{P}$ such that $N(p) = \{L_1, L_2, ..., L_{d(p)}\}$. By Proposition 2.15, we get
$\mid \mathscr{P} \mid = \mid \bigcup_{L_i \in N(p)} N(L_i) \mid$. For $L_i \in N(p)$, except p

$$\mid \mathscr{P} \mid -1 = \sum_{i=1}^{d(p)} (d(L_i) - 1).$$

□

Corollary 2.17. *Let G be a linear graph. Then there exist $p \in \mathscr{P}$ and $L_i \in \mathscr{L}$ such that $L_i \in N(p)$ for each $i, 1 \leq i \leq d(p)$. If $d(L_i) = s$ for each $L_i \in N(p)$, then*

$$\mid \mathscr{P} \mid -1 = d(p)(s-1).$$

Proof. For each $L_i \in N(p)$ we can write from Proposition 2.16 and $d(L_i) = s$,

$$\mid \mathscr{P} \mid -1 = \sum_{i=1}^{d(p)} (s-1)$$

then

$$\mid \mathscr{P} \mid -1 = d(p)(s-1).$$

□

Lemma 2.18. *Let G be a linear graph and let $s \geq 2$ be any positive integer. If the part $\mathscr{L}$ is $s-regular$ then G is $(r,s)-biregular$ such that $r = \frac{|\mathscr{P}|-1}{s-1}$.*

Proof. Suppose that $\mathscr{L}$ be regular. Then there exist any positive integer $s \geq 2$ and $d(L) = s$ for all $L \in \mathscr{L}$. From Corollary 2.17, we hold

$$d(p) = \frac{\mid \mathscr{P} \mid -1}{s-1} = r.$$

Therefore G is $(r,s)-biregular$. □

Theorem 2.19. *Let G be a linear graph and let $L_i \in \mathscr{L}$, for each $i, 1 \leq i \leq \mid L \mid$. Then*

$$\mid \mathscr{P} \mid (\mid \mathscr{P} \mid -1) = \sum_{i=1}^{|\mathscr{L}|} (d(L_i)(d(L_i) - 1).$$

Proof. Suppose that G is a linear graph. Then we count the number of pairs of vertices of $\mathscr{P}$ in two different ways. First of all, there are $\binom{|\mathscr{P}|}{2}$.
Second way, from $LG1$, there exist exactly one vertex L in $\mathscr{L}$ such that $CN\{p_i,p_j\}=\{L\}$ for pair of distinct vertices of $\mathscr{P}$. Thus the total number of pairs of vertices of $\mathscr{L}$ is total number of pairs vertices of $N(L)$ for each $L\in\mathscr{L}$. Summed over all vertices of $\mathscr{L}$, $\sum_{i=1}^{|\mathscr{L}|} d(L_i)(d(L_i)-1)$. So

$$\mid \mathscr{P} \mid (\mid \mathscr{P} \mid -1) = \sum_{i=1}^{|\mathscr{L}|}(d(L_i)(d(L_i)-1)$$

□

Theorem 2.20. *Let $G=(\mathscr{P}\cup\mathscr{L},E)$ be a linear graph. For $p_i\in\mathscr{P}$,*

$$\sum_{p_i\in\mathscr{P}} d(p_i)(d(p_i)-1) \leq \mid \mathscr{L} \mid (\mid \mathscr{L} \mid -1)$$

Proof. Let $G=(\mathscr{P}\cup\mathscr{L},E)$ be a linear graph. We introduce the following notation.

$$\Lambda^1_{L_i\neq L_j}(L_i,L_j)=\{(L_i,L_j): cn(L_i,L_j)=1 \text{ ve } L_i,L_j\in\mathscr{L}\}$$

$$\Lambda^0_{L_i\neq L_j}(L_i,L_j)=\{(L_i,L_j): cn(L_i,L_j)=0 \text{ ve } L_i,L_j\in\mathscr{L}\}.$$

Clearly,

$$\Lambda^1_{L_i\neq L_j}(L_i,L_j)\cap\Lambda^0_{L_i\neq L_j}(L_i,L_j)=\emptyset.$$

Also, there are $\binom{|\mathscr{L}|}{2}$ pairs of vertices of $\mathscr{L}$. (Counting $\{L_i,L_j\}$ to be same pair as $\{L_j,L_i\}$.) By Lemma 2.5

$$\mid \Lambda^1_{L_i\neq L_j}(L_i,L_j)\mid + \mid \Lambda^0_{L_i\neq L_j}(L_i,L_j)\mid = \mid \mathscr{L} \mid (\mid \mathscr{L} \mid -1).$$

Let $p_i\in\mathscr{P}$. The number of ordered pairs of vertices in the $\mathscr{L}$ that have a common neighborhood p_i is $d(p_i)(d(p_i)-1)$. Summed over all vertices of $\mathscr{P}$,that is $\mid \Lambda^1_{L_i\neq L_j}(L_i,L_j)\mid = \sum_{p_i\in\mathscr{P}} d(p_i)(d(p_i)-1)$. By Lemma 2.5, $\mid \Lambda^0_{L_i\neq L_j}(L_i,L_j)\mid \geq 0$. So,

$$\sum_{p_i\in\mathscr{P}} (d(p_i)(d(p_i)-1) \leq \mid \mathscr{L} \mid (\mid \mathscr{L} \mid -1).$$

□

Theorem 2.21. *Let G be a cycle graph. G is a linear graph if and only if $G=C_6$.*

Proof. Let G be a cycle graph. Suppose that $G=C_6$. Therefore there exist vertices $v_1,v_2,v_3,v_4,v_5,v_6\in V(G)$ such that $G: v_1-v_2-v_3-v_4-v_5-v_6$. Since G does not contain odd cycle it is a bipartite graph. Also $\delta(G)\geq 2$. In this case without loss of generality, we may assume that $\mathscr{P}=\{v_1,v_3,v_5\}$ and $\mathscr{L}=\{v_2,v_4,v_6\}$ because of $G=C_6$. Therefore $G=(\mathscr{P}\cup\mathscr{L},E)$ is a linear graph.
Conversely let G be a linear graph and suppose that $G\neq C_6$. By Proposition 2.13, $girthG=6$. Also G does not contain odd cycle from Corollary 2.4. Let $G=C_{2k}$ for $k\geq 4$ is any positive integer. Hence we write
$G=u_1-v_2-u_3-v_4-...-u_{2k-1}-v_{2k}-u_1$ for $u_i,u_j\in V(G)$, $i=1,3,...,2k-1$ and $j=2,4,...,2k$. We rewrite G so that it is bipartite graph. $\mathscr{P}=\{u_1,u_3,u_5,...,u_{2k-1}\}$ and $\mathscr{L}=\{v_2,v_4,...,v_{2k}\}$. Also $\mid\mathscr{P}\mid\geq 4$ and $\mid\mathscr{L}\mid\geq 4$ because of $k\geq 4$. So there exist at least four vertices u_a,u_b,u_c,u_d in $\mathscr{P}$.
From $LG1$, there exist vertices v_{ab},v_{ac},v_{ad} in $\mathscr{L}$ such that $CN(u_a,u_b)=\{v_{a_b}\}$, $CN(u_a,u_c)=\{v_{ac}\}$ and $CN(u_a,u_c)=\{v_{ac}\}$. If $v_{ab}=v_{ac}$ then $N(v_{ab})=\{u_a,u_b,u_c\}$. In this case $d(v_{ab})\geq 3$. This case contradicts with being G is a cycle graph. So $v_{ab}\neq v_{ac}$. Similarly $v_{ab}\neq v_{ad}$. However this case $N(u_a)=\{v_{ab},v_{ac},...,v_{ad}\}$ then $d(u_a)\geq 3$ which is a contradiction. Hence our assumption is false. Therefore $G=C_6$. □

Definition 2.22. *The total number of minimum length path between u_i and u_j is called the linked number denoted $c(u_i,u_j)=c_{ij}$.*

Lemma 2.23. *Let G be a linear graph. For $p_i\in\mathscr{P}$ and $L_j\in\mathscr{L}$ if $p_i\in N(L_j)$ then $c_{ij}=1$.*

Proof. It is clear from Proposition 2.9 (*iii*) and linear graphs are without multiple edges. □

Lemma 2.24. *Let G be a linear graph and let $n\geq 2$ be positive integer. For $p_i\in\mathscr{P}$ and $L_j\in\mathscr{L}$ such that $p_i\notin N(L_j)$,*

$$c_{ij}=n \textit{ if and only if } \mid\{K\in N(p_i): cn(K,L_j)=1\}\mid = n.$$

Proof. Suppose that $c_{ij}=n$. From Proposition 2.9, the length of the minimum length paths between p_i and L_j is 3 such that $p_i\notin N(L_j)$. So all minimum length paths between p_i and L_j are form of $p_i-K-q-L_j$ such that $K\in N(p_i)$ and $q\in\mathscr{P}$. Then there exist exactly one vertex $K\in N(p_i)$ for each minimum length paths between p_i and L_j such that $cn(K,L_j)=1$. Therefore $\mid\{K\in N(p_i): cn(K,L_j)=1\}\mid=n$. Conversely the proof can be shown in a similar way. □

Lemma 2.25. *Let G be a linear graph. If $p_i\notin N(L_j)$ then the number neighbour vertices to p_i and don't have common neighbour to L_j is $d(p_i)-c_{ij}$.*

Proof. Let a and b be the following

$$a = \mid \{L \in N(p_i) : cn(L, L_j) = 1\} \mid$$

$$b = \mid \{L \in N(p_i) : cn(L, L_j) = 0\} \mid .$$

From Lemma 2.5 either $cn(K, L_j) = 0$ or $cn(K, L_j) = 1$ for each $K \in N(p_i)$. So $a + b = \mid N(p_i) \mid = d(p_i)$. $c_{ij} = a$ from definition of linked number. Therefore

$$b = d(p_i) - c_{ij}$$

. □

Theorem 2.26. *Let G be a linear graph. For $p_i \in \mathscr{P}$ and $L_j \in \mathscr{L}$ $c_{ij} \leq d(p_i)$.*

Proof. If $p_i \in N(L_j)$ then $c_{ij} = 1 \leq d(p_i)$ from $LG2$. If $p_i \notin N(L_j)$ then $d(p_i) \geq c_{ij}$ from Lemma 2.25. □

Lemma 2.27. *Let $G = (\mathscr{P} \cup \mathscr{L}, E)$ be a linear graph and let $n \geq 2$ be positive integer. For $p_i \in \mathscr{P}$ and $L_j \in \mathscr{L}$ such that $p_i \notin N(L_j)$*

$$d(L_j) = n \text{ if and only if } c_{ij} = n \; .$$

Proof. Suppose that $d(L_j) = n$ for $L_j \in \mathscr{L}$. From LG2, $n \geq 2$. Then there exist vertices $q_1, q_2, ..., q_n \in \mathscr{P}$ such that $N(L_j) = \{q_1, q_2, .., q_n\}$. From $LG1$, there exist $L_k \in \mathscr{L}$ such that $CN(p_i, q_k) = \{L_k\}$ for each k, $1 \leq k \leq n$. If $L_k = L_j$ for at least $k \in \{1, 2, ..., n\}$ then $p_i \in N(L_j)$. However this case contradicts our assumption. So $L_k \neq L_j$.
If $L_k = L_t$ for $k,t \in \{1, 2, ..., n\}$, $CN\{q_k, q_t\} = \{L_j, L_t\}$. However this case conradicts $LG1$. So $L_t \neq L_k$. Therefore $\mid \{K \in N(p_i) : cn(K, L_j) = 1\} \mid = \mid \{L_1, L_2, ..., L_n\} \mid = n$. From Lemma 2.24

$$c_{ij} = n.$$

Conversely suppose that $c_{ij} = n$. By Lemma 2.24,

$$c_{ij} = \mid \{K \in N(p_i) : cn(K, L_j) = 1\} \mid = n.$$

So, for each $K \in N(p_i)$, there exist exacty one $q_j \in N(Lj)$ such that $CN(K, L_j) = \{q_j\}$. In this case $d(L_j) \geq n$. If $d(L_j) \geq n+1$ there exist at least $q' \in N(L_j)$ such that $cn(p_i, q') = 0$ however this case contradicts $LG1$. Therefore $d(L_j) = n$. □

Theorem 2.28. *Let $= G(\mathscr{P} \cup \mathscr{L}, E)$ be a linear graph. For $p_i \in \mathscr{P}$ and $L_j \in \mathscr{L}$ such that $p_i \notin N(L_j)$,*

$$c_{ij} = d(p_i) \text{ if and only if } d(p_i) = d(L_j) \; .$$

Proof. It is trivial from Lemma 2.27.

□

Theorem 2.29. *Let $G = (\mathscr{P} \cup \mathscr{L}, E)$ be a linear graph. For each $p_i \in \mathscr{P}$ and $L_j \in \mathscr{L}$ such that $p_i \notin N(L_j)$,*

$$\text{if } c_{ij} = d(p_i) \text{ then } cn(R_j, S_j) = 1 \text{ for all } R_j, S_j \in \mathscr{L} \text{ such that } R_j \neq S_j \; .$$

Proof. Let $G = (\mathscr{P} \cup \mathscr{L}, E)$ be a linear graph. Assume that for $p_i \in \mathscr{P}$ and $L_j \in \mathscr{L}$ such that $p_i \notin N(L_j)$, $c_{ij} = d(p_i)$. For $R_j, S_j \in \mathscr{L}$ and $R_j \neq S_j$,
Case 1. $R_j, S_j \in N(p_i)$. In this case, $cn(R_j, S_j) = 1$.
Case 2. $R_j \in N(p_i)$ and $S_j \notin N(p_i)$. By assumtion and Lemma 2.25, $cn(R_j, S_j) = 1$.
Case 3. $R_j, S_j \notin N(p_i)$. The proof is complete if $cn(R_j, S_j) = 1$. Suppose that $cn(R_j, S_j) = 0$. Then there exist $q_i \in N(R_j)$ such that $q_i \notin N(S_j)$. By assumption, $c_{ij} = d(q_i)$. In this case the number neighbour vertices to q_i and don't have common neighbour to S_j is $d(q_i) - c_{ij} = 0$, from Lemma 2.25. However we contradict with $cn(R_j, S_j) = 0$. Therefore $cn(R_j, S_j) = 1$.

□

References

[1] A.S. Asratian, T.M.J. Denley and R. Hoggkvist, *Bipartite graphs and their applications*, Cambridge Uni. Press,1998, United Kingdom.
[2] L.M. Batten, *Combinatorics of finite geometries* ,Cambridge University Press, Cambridge-NewYork, 1986.
[3] İ. Günaltılı, A. Ulukan, and Ş. Olgun, *Some Properties Of Fınıte {0,1}-Graphs* , Konualp Journal Of Mathematics,1(1), 2013, 34-39.
[4] İ. Günaltılı,*Classification Of Some {0,1}-Semigraphs*, İnternation Journal of Innovative Research in Computer Science & Technology,4(1),2016,10-12.
[5] F. Harary, D. Hsu and Z. Miller, *The biparticity of a graph*, J. Graph Theory 1, 1977, 131-133.
[6] M. Mulder,*(0, λ)-graph and n-cubes*, Discrete mathematics 28, 1979, 179- 188.
[7] C. Vasudev, *Combinatorics and Graph Theory*, New Age Publications(Academic), India, 2007.
[8] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Englewood Clins, NJ, 1996.