İstanbul Üniv. Fen Fak.
Mat.Fiz.Astro. Derg. 1 (2004-2005),115-126

SOME RESULTS ON A RIEMANNIAN SUBMERSION

H. Mete TAŞTAN

Abstract

In this paper, we develop some well-known results given by O'Neill [6], Gray [3] and Escobales [1] and obtain a few new results by using them.

1. Introduction

Let M and B be smooth Riemannian manifolds. A Riemannian submersion $\pi : M \to B$ is a mapping of M onto B satisfying the following axioms;

S1. π has maximal rank;
that is, each derivative map π_* of π is onto. Hence, for each $q \in B$, $\pi^{-1}(q)$ is a submanifold of M of dimension $\dim M - \dim B$ where the submanifolds $\pi^{-1}(q)$ are called *fibers* of M. A vector field on M is called *vertical* if it is tangent to a fiber and *horizontal* if orthogonal to fiber.

S2. π_* preserves lengths of horizontal vectors.

Given a Riemannian submersion $\pi : M \to B$ we denote by ν the vector subbundle of TM defined by the foliation of M by the fibers of π. $\hbar$ denote the complementary distribution of ν in TM determined by the metric on M.

Recall that if $p \in M$ where M is any manifold, then T_pM denotes the tangent space of M at p. Following O'Neill [6] we define the tensor **T** of type (1,2) for arbitrary vector fields E and F by

$$\mathrm{T}_E F = \hbar \nabla_{\nu E} \nu F + \nu \nabla_{\nu E} \hbar F$$

where $\nu E, \hbar E$, etc. denote the vertical and horizontal projections of the vector field E. O'Neill has described the following three properties of the tensor T:

(1) T_E is a skew-symmetric linear operator on a tangent space of $'M$ and reversing horizontal and vertical subspaces.

(2) $\mathrm{T}_E = \mathrm{T}_{\nu E}$, that is; T is vertical.

(3) For vertical vector fields V and W, T is symmetric, i.e., $\mathrm{T}_V W = \mathrm{T}_W V$.

In fact, along a fiber, T is the second fundamental form of the fiber provided we restrict ourselves to vertical vector fields.

Now, we simply dualize the definition of T by reversing ν and $\hbar$ define *the integrability tensor* A as follows. For arbitrary vector fields E and F,

$$\mathrm{A}_E F = \hbar \nabla_{\hbar E} \nu F + \nu \nabla_{\hbar E} \hbar F$$

($1'$) A_E is a skew-symmetric operator on TM reversing the horizontal and vertical subspaces.

($2'$) $\mathrm{A}_E = \mathrm{A}_{\hbar E}$, that is; A is horizontal.

($3'$) For X, Y horizontal A is alternating, i.e., $\mathrm{A}_X Y = -\mathrm{A}_Y X$.

2. The properties of vertical and horizontal distributions

Lemma **2.1** The vertical distribution $\nu : TM \to \nu(TM)$ is involutive.

Proof. Let $V,W \in \nu(TM)$, we must show that $[V,W] \in \nu(TM)$ that is,

$\hbar[V,W]=0$.

$\hbar[V,W]=\hbar\nabla_V W-\hbar\nabla_W V$ where ∇ is the Riemannian connection on M. By the definition of T, $\hbar\nabla_V W=\mathrm{T}_V W$ and $\hbar\nabla_W V=\mathrm{T}_W V$. Hence $\hbar[V,W]=\mathrm{T}_V W-\mathrm{T}_W V=0$.

Definition. *A basic vector field* is a horizontal vector field X which is $\pi-$related to a vector field X_* on B, i.e., $\pi_* X_p=X_{*\pi(p)}$ for all $p\in M$.

Lemma 2.2 If X and Y are basic vector fields on M, then

1. $\langle X,Y\rangle=\langle X_*,Y_*\rangle\circ\pi$
2. $\hbar[X,Y]$ is basic and is $\pi-$related $[X_*,Y_*]$
3. $\hbar\nabla_X Y$ is basic and is $\pi-$related $\nabla^*{}_{X_*}Y_*$

where ∇^* is the Riemannian connection on B. The proofs of these results are found in O'Neill [6].

Lemma 2.3 Let Z_i be a basic vector field on M corresponding Z_{i_*} on B. Suppose for a horizontal vector field X, $\langle X,Z_i\rangle_p=\langle X,Z_i\rangle_{p'}$ for all such Z_i and for any $p,p'\in\pi^{-1}(q)$ where $q\in B$. Then $\pi_* X$ is a well-defined vector field on B. In particular X is basic.
See R.H. Escobales [1].

Lemma 2.4 Let X and Y be horizontal vector fields, V and W be vertical vector fields. Then each of the following holds:

(1) $\mathrm{A}_X Y=\frac{1}{2}\nu[X,Y]$

(2) $\nabla_V W = \mathrm{T}_V W + \hat{\nabla}_V W$, where $\hat{\nabla}$ denotes the Riemannian connection along a fiber with respect to the induced metric.

(3) a) $\nabla_V X = \hbar\nabla_V X + \mathrm{T}_V X$

b) If X is basic, $\hbar\nabla_V X = \mathrm{A}_X V$

(4) $\nabla_X V = \mathrm{A}_X V + \nu\nabla_X V$

(5) $\nabla_X Y = \hbar\nabla_X Y + \mathrm{A}_X Y$

The proofs of these results are found in O'Neill[6] and R.H. Escobales[1].

Corollary 2.1 If X and Y are basic vector fields and V is vertical, then

$V\langle X,Y\rangle = 0.$

Proof: $V\langle X,Y\rangle = \langle\nabla_V X,Y\rangle + \langle X,\nabla_V Y\rangle = \langle\hbar\nabla_V X,Y\rangle + \langle X,\hbar\nabla_V X\rangle$

Since X and Y are basic. From Lemma 4 3b) we have

$\langle\hbar\nabla_V X,Y\rangle + \langle X,\hbar\nabla_V X\rangle = \langle\mathrm{A}_X V,Y\rangle + \langle\mathrm{A}_Y X,X\rangle$

Now, if we use $(1')$ we have

$= -\langle\mathrm{A}_X Y,V\rangle - \langle\mathrm{A}_Y X,V\rangle$, by the property $(3')$ of A, $-\mathrm{A}_X Y - \mathrm{A}_Y X = 0$ and it follows.

Corollary 2.2 Horizontal distribution is involutive if and only if $\mathrm{A} \equiv 0$.

Proof. $\Rightarrow$ Suppose horizontal distribution is involutive, that is; for any horizontal X and Y, $[X,Y]$ is horizontal. We must show that $\mathrm{A}_X E = 0$, for any vector field E.

From Lemma 4 -1) $\mathrm{A}_X Y = \frac{1}{2}\nu[X,Y]$=0. Thus, $\mathrm{A}_X \hbar E = 0$.

On the other hand, for any horizontal Z;

$\langle\mathrm{A}_X \nu E,Z\rangle = -\langle\mathrm{A}_X Z,\nu E\rangle = 0 implies \mathrm{A}_X \nu E = 0$, since it is horizontal.

Finally $\mathrm{A}_X E = \mathrm{A}_X(\hbar E + \nu E) = \mathrm{A}_X \hbar E + \mathrm{A}_X \nu E = 0 implies \mathrm{A} \equiv 0$.

$\Leftarrow$ Now, if A is identically zero, for any horizontal X and Y

$0 = \mathrm{A}_X Y = \frac{1}{2} v[X,Y] implies [X,Y]$ is horizontal, that is; horizontal distribution is involutive.

3. Covariant derivatives of T and A

Lemma 3.1 If X and Y are horizontal and V and W are vertical, then

(a) $(\nabla_V \mathrm{A})_W = -\mathrm{A}_{\mathrm{T}_V W}$ (b) $(\nabla_X \mathrm{A})_W = -\mathrm{A}_{\mathrm{A}_X W}$
(c) $(\nabla_X \mathrm{T})_Y = -\mathrm{T}_{\mathrm{A}_X Y}$ (d) $(\nabla_V \mathrm{T})_Y = -\mathrm{T}_{\mathrm{T}_V Y}$.

Proof. We will only prove (c) since the proofs of others are similar. Let E be an arbitrary vector field on $'M$. Then

$(\nabla_X \mathrm{T})_Y E = \nabla_X (\mathrm{T}_Y E) - \mathrm{T}_{\nabla_X Y}(E) - \mathrm{T}_Y(\nabla_X E)$ since **T** is vertical, $\mathrm{T}_Y E = 0, \mathrm{T}_Y(\nabla_X E) = 0,$ and from Lemma 2.4-(5) $\mathrm{T}_{\nabla_X Y}(E) = \mathrm{T}_{v(\nabla_X Y)}(E) = \mathrm{T}_{\mathrm{A}_X Y}(E)$, hence $(\nabla_X \mathrm{T})_Y E = -\mathrm{T}_{\mathrm{A}_X Y}(E)$ and it follows.

Corollary 3.1 a) If **A** is parallel, then **A** is identically zero, i.e., for all $E \in TM$, $(\nabla_E \mathrm{A}) = 0$ implies $\mathbf{A} \equiv 0$.
b) If **T** is parallel, then **T** is identically zero, i.e., $(\nabla_E \mathrm{T}) = 0$ implies $\mathbf{T} \equiv 0$.
The proofs of these results can be found in R.H. Escobales [1]

Lemma 3.2 If X,Y,Z,H are horizontal vector fields and U,V,W,F are vertical, then

(a) $\langle (\nabla_U \mathrm{A})_X V, W \rangle = \langle \mathrm{T}_U V, \mathrm{A}_X W \rangle - \langle \mathrm{T}_U W, \mathrm{A}_X V \rangle$
(b) $\langle (\nabla_X \mathrm{A})_Y V, W \rangle = \langle \mathrm{A}_X V, \mathrm{A}_Y W \rangle - \langle \mathrm{A}_X W, \mathrm{A}_Y V \rangle$
(c) $\langle (\nabla_X \mathrm{A})_Y Z, H \rangle = \langle \mathrm{A}_X Z, \mathrm{A}_Y H \rangle - \langle \mathrm{A}_X H, \mathrm{A}_Y Z \rangle$

(d) $\langle (\nabla_U \mathrm{A})_V W, F\rangle = 0$

Proof. We will only prove b) since the proofs of others are similar.

$$(\nabla_X \mathrm{A})_Y V = \nabla_X (\mathrm{A}_Y V) - \mathrm{A}_{\nabla_Y X}(Y) - \mathrm{A}_Y (\nabla_X V)$$

Hence

$$\langle (\nabla_X \mathrm{A})_Y V, W\rangle = \langle \nabla_X (\mathrm{A}_Y V), W\rangle - \langle \mathrm{A}_{\nabla_X Y}(V), W\rangle - \langle \mathrm{A}_Y (\nabla_X Y), W\rangle$$

$$= \nabla_X \langle \mathrm{A}_Y V, W\rangle - \langle \mathrm{A}_Y V, \nabla_X W\rangle - \langle \mathrm{A}_{\nabla_X Y}(V), W\rangle - \langle \mathrm{A}_Y (\mathrm{A}_X V), W\rangle$$

where, $\langle \mathrm{A}_Y V, W\rangle = 0, \langle \mathrm{A}_{\nabla_X Y}(V), W\rangle = 0$ since **A** reverses the horizontal and vertical subspaces and $\langle \mathrm{A}_Y V, \nabla_X W\rangle = \langle \mathrm{A}_Y V, \mathrm{A}_X W\rangle$ since $\mathrm{A}_Y V$ is horizontal and $\hbar \nabla_X W = \mathrm{A}_X W$, on the other hand $-\langle \mathrm{A}_Y (\mathrm{A}_X V), W\rangle = \langle \mathrm{A}_X V, \mathrm{A}_Y W\rangle$ since **A** is skew-symmetric, thus the result follows.

Lemma 3.3 If X, Y, Z, H are horizontal vector fields and U, V, W, F are vertical, then

(a) $\langle (\nabla_X \mathrm{T})_V Y, Z\rangle = \langle \mathrm{A}_X Y, \mathrm{T}_V Z\rangle - \langle \mathrm{A}_X Z, \mathrm{T}_V Y\rangle$

(b) $\langle (\nabla_U \mathrm{T})_V X, Y\rangle = \langle \mathrm{T}_U X, \mathrm{T}_V Y\rangle - \langle \mathrm{T}_V X, \mathrm{T}_U Y\rangle$

(c) $\langle (\nabla_U \mathrm{T})_V W, F\rangle = \langle \mathrm{T}_U W, \mathrm{T}_V F\rangle - \langle \mathrm{T}_U F, \mathrm{T}_V W\rangle$

(d) $\langle (\nabla_X T)_Y Z, H\rangle = 0$

Proof. We will only prove a) since the proofs of others are similar.

$$(\nabla_X \mathrm{T})_V Y = \nabla_X (\mathrm{T}_V Y) - \mathrm{T}_{\nabla_X V}(Y) - \mathrm{T}_V (\nabla_X Y)$$

Hence

$$\langle (\nabla_X \mathrm{T})_V Y, Z\rangle = \langle \nabla_X (\mathrm{T}_V Y), Z\rangle - \langle \mathrm{T}_{\nabla_X V}(Y), Z\rangle - \langle \mathrm{T}_V (\nabla_X Y), Z\rangle$$

$$= \nabla_X \langle \mathrm{T}_V Y, Z\rangle - \langle \mathrm{T}_V Y, \nabla_X Z\rangle - \langle \mathrm{T}_{\nabla_X V}(Y), Z\rangle - \langle \mathrm{T}_V (\mathrm{A}_X Y), Z\rangle$$

where $\langle T_V Y, Z\rangle = 0, \langle T_{\nabla_X V} Y, Z\rangle = 0$ since **T** reverses the vertical and horizontal subspaces and $\langle T_V Y, \nabla_X Z\rangle = \langle T_V Y, A_X Z\rangle$ since $T_V Y$ is vertical and $\nu(\nabla_X Z) = A_X Z$ from Lemma 2.4-(5). On the other hand $-\langle T_V(A_X Y), Z\rangle = \langle T_V Z, A_X Y\rangle$ since **T** is skew-symmetric, thus it follows.

Lemma 3.4 If X and Y are horizontal, and V and W are vertical, then;

(a) $\langle (\nabla_E A)_X Y, V\rangle$ is alternate in X and Y.
(b) $\langle (\nabla_E T)_V W, X\rangle$ is symmetric in V and W.

Proof. Expand the covariant derivatives and use the properties of **T** and **A.**

Lemma 3.5 If V is vertical and Ω denotes the cyclic sum of over the horizontal vector fields X, Y, Z, then

$$\Omega\langle (\nabla_Z A)_X Y, V\rangle = \Omega\langle A_X Y, T_V Z\rangle$$

Proof. See O'Neill [6].

Corollary 3.2 If U, V and W are vertical and Ω denotes the cyclic sum of over U, V and W, then

$$\Omega\big(\nu(\nabla_U T)_V W\big) = 0$$

Proof. For any vertical vector field F, we compute that

$\langle (\nabla_U T)_V W + (\nabla_W T)_U V + (\nabla_V T)_W U, F\rangle$ and applying Lemma **3.3**-(c) the result follows.

Corollary 3.3 If Ω denotes the cyclic sum of over the horizontal vector fields X, Y, Z and H horizontal, then

$$\Omega\langle(\nabla_X \mathrm{A})_Y Z, H\rangle = 2.\Omega\langle \mathrm{A}_X Z, \mathrm{A}_Y H\rangle$$

Proof. The result easily follows from the from Lemma 3.2 (c).

4. Fundamental Equations

Let $\mathfrak{R}$ denote the curvature tensor of M, and $\mathfrak{R}^*$ the curvature tensor of B. Since there is no danger of ambiguity, we will denote the horizontal lift of $\mathfrak{R}^*$ by $\mathfrak{R}^*$ as well. Following O'Neill [6] we set

$$\langle {\mathfrak{R}^*}_{h_1 h_2} h_3, h_4\rangle = \langle {\mathfrak{R}^*}_{h_1^* h_2^*} h_3^*, h_4^*\rangle$$

where h_i are horizontal vectors such that $\pi_*(h_i) = h_i^*$.

Theorem 4.1 If U, V, W, F are vertical vector fields and X is horizontal, then

(a) $\langle \mathfrak{R}_{UV} W, F\rangle = \langle \hat{\mathfrak{R}}_{UV} W, F\rangle - \langle \mathrm{T}_U W, \mathrm{T}_V F\rangle + \langle \mathrm{T}_V W, \mathrm{T}_U F\rangle$
(b) $\langle \mathfrak{R}_{UV} W, X\rangle = \langle (\nabla_V T)_U W, X\rangle - \langle (\nabla_U \mathrm{T})_V W, X\rangle$

where $\hat{\mathfrak{R}}$ is the curvature tensor of the fiber $\pi^{-1}(\pi(p))$ at p.

Proof. (These equations relate the geometry of M to those of the fibers $\pi^{-1}(q)$; they are clearly the Gauss and Codazzi equations of the fibers.) The proof is the same as that of a single submanifold.

Theorem 4.2 If X, Y, Z, H are horizontal vector fields and V is vertical, then

(a)
$$\langle \Re_{XY}Z,H\rangle = \langle \Re^{*}{}_{XY}Z,H\rangle - 2\langle A_XY, A_ZH\rangle + \langle A_YZ, A_XH\rangle + \langle A_ZX, A_YH\rangle$$

(b)
$$\langle \Re_{XY}Z,V\rangle = \langle (\nabla_Z A)_X Y, V\rangle + \langle A_XY, T_VZ\rangle - \langle A_YZ, T_VX\rangle - \langle A_ZX, T_VY\rangle$$

Proof. See O'Neill [6].

Theorem 4.3 If X and Y are horizontal vector fields, and V and W are vertical, then

(a) $$\langle \Re_{XV}Y,W\rangle = \langle (\nabla_X T)_V W, Y\rangle + \langle (\nabla_V A)_X Y, W\rangle - \langle T_VX, T_WY\rangle + \langle A_XV, A_YW\rangle$$

(b) $$\langle \Re_{VW}X,Y\rangle = \langle (\nabla_V A)_X Y, W\rangle - \langle (\nabla_W A)_X Y, V\rangle + \langle A_XV, A_YW\rangle - \langle A_XW, A_YV\rangle - \langle T_VX, T_WY\rangle + \langle T_WX, T_VY\rangle$$

Proof. See O'Neill [6].

In the case of sectional curvature, the proofs of these theorems become trivial. For the tangent vectors a and b (assumed to be linearly independent), we will denote by P_{ab} the tangent plane which is spanned by them.

Corollary 4.1 Let $\pi : M \to B$ be a submersion and let κ, κ_* and $\hat{\kappa}$ denote the sectional curvatures of M, B and the fibers respectively. If x and y are horizontal vectors at a point of M and v and w are vertical, then;

(1) $$\kappa(P_{vw}) = \hat{\kappa}(P_{vw}) - \frac{\langle T_v v, T_w w\rangle - \langle T_v w, T_v w\rangle}{\langle v\wedge w, v\wedge w\rangle}$$

(2) $\kappa(P_{xv}) = \dfrac{\langle (\nabla_x T)_v v, x\rangle + \langle A_x v, A_x v\rangle - \langle T_v x, T_v x\rangle}{\langle x,x\rangle\langle v,v\rangle}$

(3) $\kappa(P_{xy}) = \kappa_*(P_{x_* y_*}) - \dfrac{3\langle A_x y, A_x y\rangle}{\langle x\wedge y, x\wedge y\rangle}$, where $x_* = \pi_*(x)$.

Proof. (The first equation above is the formulation of Gauss equation for the fibers.) All of them follow from the following well-known equation;

$$\kappa(P_{vw}) = \frac{\langle \Re_{vw} v, w\rangle}{\langle v\wedge w, v\wedge w\rangle}$$

We have obtained in Lemma 3.1, Lemma 3.2 and Lemma 3.3 the covariant derivatives of the fundamental tensors **T** and **A**; that is, we have expressed the tensors $(\nabla_E T)_F$ and $(\nabla_E A)_F$ in **T** and **A**.

Now, we consider $\langle (\nabla_E T)_F G, L\rangle$.

Since it can be written two different types of vector fields, horizontal and vertical, instead of each vector fields *E,F,G,* and *L,* it follows that we can state $\langle (\nabla_E T)_F G, L\rangle$ in exactly sixteen different types; i.e., we can say the covariant derivatives ∇T of T in sixteen different type. The eight of them can be expressed by using Lemma 3.1, and the three one of others can be expressed by using Lemma 3.3 in **T** and **A**. But , the other five types may not be possible to write in **T** and **A** by using the Lemma 3.1 and Lemma 3.3. For instance, we can not state $\langle (\nabla_X T)_V W, Y\rangle$ in **T** and **A**. Similar claims are also valid for the fundamental tensor **A**. In other ways, we can not state $\langle (\nabla_V A)_X Y, W\rangle$ in **T** and **A**. However, we find a relation between $\langle (\nabla_X T)_V W, Y\rangle$ and $\langle (\nabla_V A)_X Y, W\rangle$, and state it in Theorem 4.4 below.

Theorem 4.4 If V and W vertical vector fields and X and Y are horizontal, then

$$\langle(\nabla_X \mathrm{T})_V W,Y\rangle - \langle(\nabla_Y \mathrm{T})_V W,X\rangle = -\langle(\nabla_V \mathrm{A})_X Y,W\rangle - \langle(\nabla_W \mathrm{A})_Y X,V\rangle$$

Proof. From the *first Bianchi Identity* we have that

$$\langle \mathfrak{R}_{XV}Y + \mathfrak{R}_{VY}X + \mathfrak{R}_{YX}V,W\rangle = 0$$

Hence we can write
$\langle \mathfrak{R}_{XV}Y,W\rangle + \langle \mathfrak{R}_{VY}X,W\rangle + \langle \mathfrak{R}_{YX}V,W\rangle = 0$ ...**(1)**

where, we use the Theorem **4.3**-(a). Now we have that

$\langle \mathfrak{R}_{XV}Y,W\rangle = \langle(\nabla_X \mathrm{T})_V W,Y\rangle + \langle(\nabla_V \mathrm{A})_X Y,W\rangle - \langle \mathrm{T}_V X,\mathrm{T}_W Y\rangle + \langle \mathrm{A}_X V,\mathrm{A}_Y W$
...**(2)**,
and by the symmetries of curvature tensor $\mathfrak{R}$, we obtain

$$\langle \mathfrak{R}_{VY}X,W\rangle = -\langle \mathfrak{R}_{YV}X,W\rangle = -\langle(\nabla_Y \mathrm{T})_V W,X\rangle - \langle(\nabla_V \mathrm{A})_Y X,W\rangle + \langle \mathrm{T}_V Y,\mathrm{T}_W X\rangle - \langle \mathrm{A}_Y V,\mathrm{A}_X W\rangle \text{...}\mathbf{(3)}$$

On the other hand, again, by using the symmetries of curvature tensor $\mathfrak{R}$, we have that

$\langle \mathfrak{R}_{YX}V,W\rangle = -\langle \mathfrak{R}_{XY}V,W\rangle = -\langle \mathfrak{R}_{VW}X,Y\rangle$ and use the Theorem **4.3**-(b)

we have

$$\langle \mathfrak{R}_{YX}V,W\rangle = -\langle(\nabla_V \mathrm{A})_X Y,W\rangle + \langle(\nabla_W \mathrm{A})_X Y,V\rangle - \langle \mathrm{A}_X V,\mathrm{A}_Y W\rangle + \langle \mathrm{A}_X W,\mathrm{A}_Y V\rangle + \langle \mathrm{T}_V X,\mathrm{T}_W Y\rangle - \langle \mathrm{T}_W X,\mathrm{T}_V Y\rangle \text{...}\mathbf{(4)}$$

Putting **(2)**,**(3)** and **(4)** in **(1)** the result follows.

F.9

References

[1] R.H. Escobales, Jr., *Riemannian submersions with totally geodesic fibers,* J. Differential Geometry **10** (1975) 253-276.

[2] K.L. Duggal and A. Bejancu, *Lightlike submanifolds of Semi-Riemannian manifolds and applications,* Kluwer Academic Publishers, 1996.

[3] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions,* J. Math. Mech. **16** (1967), 715-737.

[4] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry,* Vol 1, Interscience, Newyork (1963).

[5] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity,* Academic Press, 1983.

[6] B. O'Neill, *The Fundamental Equations of a Submersion,* Michigan Math. J. **13** (1966), 459-469.

H. Mete TAŞTAN
Department of Mathematics
Faculty of Science
Istanbul University
Vezneciler 34459, Istanbul
TURKEY
E-Mail: hakmete@istanbul.edu.tr